Temperature fluctuations in isotropic turbulence

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NCTR4, Les Houches, March 2016

General motivation

1. Mixing is everywhere
2. Mixing is important

Overview

- Introduction
  - Advection of a passive scalar
  - multiscale description of turbulence
- Viscous heating
  - Modeling
  - Numerical simulations
- Anisotropic scalar mixing
  - Isotropy/anisotropy and correlations
  - Mean scalar gradient

Temperature fluctuations in a turbulent flow

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \nabla \theta = D \Delta \theta + f_\theta$$

where f_θ is a source or sink (chemical reactions, local injection, viscous friction).

Statistics of the temperature field?

$$\frac{\partial\theta}{\partial t} + \boldsymbol{u}\cdot\nabla\theta = D\Delta\theta + f_\theta$$

Introduce: $\theta = \langle\theta\rangle + \theta'$, $\boldsymbol{u}(\boldsymbol{x},t) = \langle\boldsymbol{u}\rangle + \boldsymbol{u}'$.

$$\frac{\partial\langle\theta\rangle}{\partial t} + \langle\boldsymbol{u}\rangle\cdot\nabla\langle\theta\rangle + \left\langle\boldsymbol{u}'\cdot\nabla\theta'\right\rangle = D\Delta\langle\theta\rangle + \langle f_\theta\rangle$$

and

$$\frac{\partial\theta'}{\partial t} + \langle\boldsymbol{u}\rangle\cdot\nabla\theta' + \boldsymbol{u}'\cdot\nabla\langle\theta\rangle + \boldsymbol{u}'\cdot\nabla\theta' - \left\langle\boldsymbol{u}'\cdot\nabla\theta'\right\rangle = D\Delta\theta' + f'_\theta$$

Statistics of the temperature field?

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and

$$\frac{\partial \theta'}{\partial t} + \langle \boldsymbol{u} \rangle \cdot \nabla \theta' + \boldsymbol{u}' \cdot \nabla \langle \theta \rangle + \boldsymbol{u}' \cdot \nabla \theta' - \left\langle \boldsymbol{u}' \cdot \nabla \theta' \right\rangle = D\Delta\theta' + f'_\theta$$

No mean flow $\langle \boldsymbol{u} \rangle = 0$

Incompressibility, homogeneous fluctuations: $\langle \boldsymbol{u}' \cdot \nabla \theta' \rangle = \nabla \cdot \langle \boldsymbol{u}' \theta' \rangle = 0$,

$$\frac{\partial \langle \theta \rangle}{\partial t} = D\Delta \langle \theta \rangle + \langle f_\theta \rangle$$

and

$$\frac{\partial \theta'}{\partial t} + \boldsymbol{u}' \cdot \nabla \theta' = D\Delta\theta' + f'_\theta - \boldsymbol{u}' \cdot \nabla \langle \theta \rangle$$

Evolution of the variance of temperature fluctuations

$$\left\langle \theta' \left[\frac{\partial \theta'}{\partial t} + \boldsymbol{u}' \cdot \nabla \theta' = D \Delta \theta' + f'_\theta - \boldsymbol{u}' \cdot \nabla \langle \theta \rangle \right] \right\rangle$$

yields

$$\frac{\partial \frac{1}{2} \left\langle \theta'^2 \right\rangle}{\partial t} = -D \left\langle (\nabla \theta')^2 \right\rangle + \left\langle f'_\theta \theta' \right\rangle - \left\langle \boldsymbol{u}' \theta' \right\rangle \cdot \nabla \langle \theta \rangle$$

Two possible production mechanisms:

1. By a mean scalar gradient
2. By a local injection process.

K41, spectra etc.: A short reminder on the spectral description of turbulence.

Consider a homogeneous flow.

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\frac{1}{\rho} \nabla p + \nu \Delta \boldsymbol{u} + \boldsymbol{f}, \quad \nabla \cdot \boldsymbol{u} = 0.$$

Introducing:

$$\mathcal{K} = \tfrac{1}{2} \left\langle (\boldsymbol{u})^2 \right\rangle,$$
$$P = \langle \boldsymbol{f} \cdot \boldsymbol{u} \rangle,$$
$$\langle \epsilon \rangle = \nu \left\langle (\nabla \boldsymbol{u})^2 \right\rangle$$

we derive

$$\dot{\mathcal{K}} = P - \langle \epsilon \rangle$$

K41, spectra etc.: A short reminder on the spectral description of turbulence.

The averaged energy budget is $\dot{\mathcal{K}} = P - \langle\epsilon\rangle$.

$$\int E(k)dk = \mathcal{K}$$

and the evolution-equation,

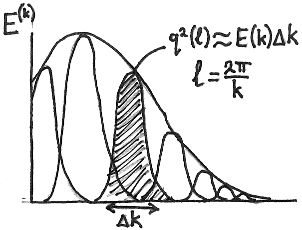
$$\frac{dE(k)}{dt} = T(k) + P(k) - D(k)$$

with

$$\int P(k)dk = P = \langle \boldsymbol{f} \cdot \boldsymbol{u} \rangle \,, \qquad (1)$$

$$\int D(k)dk = \int 2\nu k^2 E(k)dk = \langle\epsilon\rangle$$

Importance of $T(k)$?

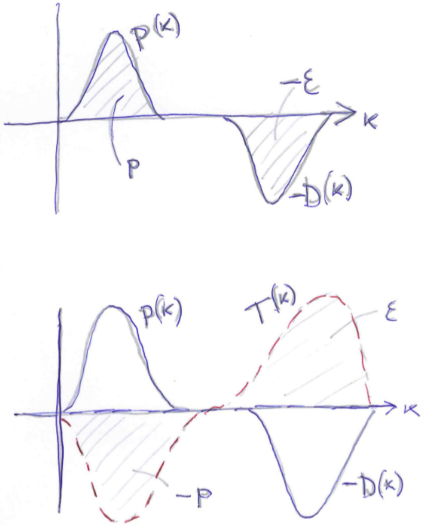


$$\int \left[\dot{E}(k) = T(k) + P(k) - D(k)\right] dk \qquad (2)$$

$$\dot{\mathcal{K}} = P - \langle\epsilon\rangle + \int T(k)dk \qquad (3)$$

$$\rightarrow \int T(k)dk = 0.$$

Spectral balance



$$0 = T(k) + P(k) - D(k)$$

Energy flux and transfer

unclosed term. How to model?
Since $\int T(k)dk = 0$
we can write

$$T(k) = -\partial_k \Pi(k)$$

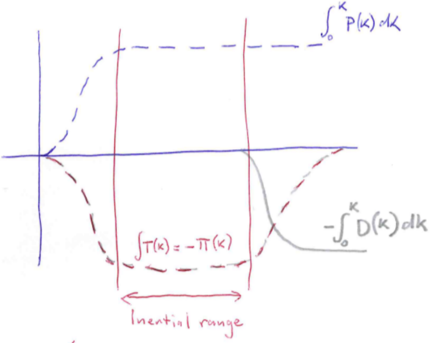
where $\Pi(k=0) = \Pi(k=\infty) = 0$.

Simplest model: $\Pi(k) = F(k, E(k))$. Dimensional analysis[1] $\rightarrow$

$$\Pi(k) = C_{Kov} E(k)^{3/2} k^{5/2}$$

[1] Kovaznay J. Aeronaut. Sci. 1948

Spectral flux



$$\cancel{\frac{\partial E(k)}{\partial t}} = T(k) + P(k) - D(k)$$

Inertial range:

$$\int_0^K P(k)\,dk \simeq -\int_0^K T(k)\,dK = \Pi(k)$$

K41 and Kovaznay's closure

In the inertial range $\int T(k)dk \approx -\int_0^k P(k)dk$

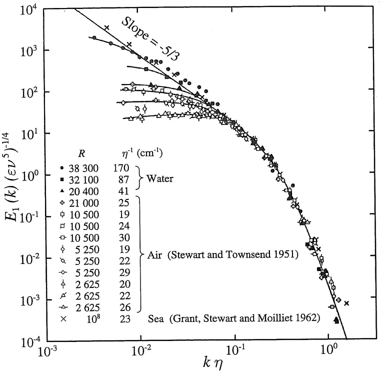
$$\int_0^k P(k)dk = C_{Kov} E(k)^{3/2} k^{5/2}$$

we find

$$E(k) = C_{Kov}^{-2/3} \left[\int_0^k P(k)dk \right]^{2/3} k^{-5/3}.$$

If $\int_0^k P(k)dk = P = \langle \epsilon \rangle$,

$$E(k) \sim \langle \epsilon \rangle^{2/3} k^{-5/3}.$$



K41 or O41?

ИЗВЕСТИЯ АКАДЕМИИ НАУК СССР. 1941
BULLETIN DE L'ACADÉMIE DES SCIENCES DE L'URSS
Серия географическая и геофизическая
Série géographique et géophysique

А. М. ОБУХОВ

О РАСПРЕДЕЛЕНИИ ЭНЕРГИИ В СПЕКТРЕ ТУРБУЛЕНТНОГО ПОТОКА

(*Представлено академиком* А. Н. Колмогоровым)

В работе исследуется однородный турбулентный поток методом спектральных разложений. Вводится понятие спектра потока и распределения энергии в спектре. Функция распределения энергии определяется из соображений баланса энергии и стационарности турбулентного поля. Полученные результаты позволяют теоретически объяснить указанную Ричардсоном зависимость „коэффициента обмена" от масштаба наблюдения.

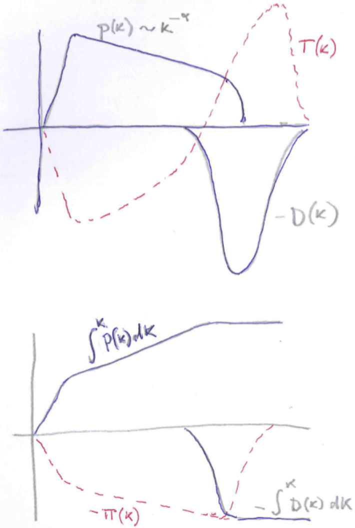
В настоящей работе сделана попытка построения спектральной теори
турбулентности. Положенная в основу физическая схема была указан

K41 or O41?

* Принятый нами в дальнейшем изложении подход к математической обработке схемы Ричардсона был предложен А.Н. Колмогоровым в докладе, прочитанном в Институте теоретической геофизики в конце 1939 г. В этом докладе было указано, что для средних частот вводимая ниже функция $E(p)$ должна иметь вид: $E(p) = Cp^{\alpha}$. А. Н. Колмогорову не удалось, однако, в 1939 г. определить значение показателя α. Мы показываем ниже, основываясь на уравнении спектрального баланса энергии, что $\alpha = -\frac{2}{3}$.

Тот же вывод можно получить из соображений подобия, развитых в работе А. Н. Колмогорова [2], опубликованной после получения нами излагаемых ниже результатов. Вывод, основанный на уравнении спектрального баланса энергии, более полно освещает, по нашему мнению, физическую картину передачи энергии от возмущений крупного масштаба к возмущениям мелкого масштаба.

Broadband forcing: no K41



$$\int_{k_0 \sim 1/L}^{k} P(k)\,dk \simeq \Pi(k)$$

Assume $P(k) = (\varepsilon_{in} L)(kL)^{-\alpha}$

$$\int_{k_0}^{k} P(k)\,dk = \frac{\varepsilon_{in}}{\alpha-1}\left[1-(kL)^{-\alpha+1}\right]$$

$\alpha > 1$: $\Pi(k) \simeq \varepsilon_{in}$

$\alpha < 1$: $\Pi(k) \simeq \varepsilon_{in}\left[kL\right]^{1-\alpha}$

Broadband forcing and Kovaznay's closure

Since

$$E(k) = C_{Kov}^{-2/3} \left[\int_0^k P(k) dk \right]^{2/3} k^{-5/3}.$$

and

$$\int_0^k P(k) dk = \epsilon_{in} (kL)^{1-\alpha}$$

we find

$$\begin{aligned} E(k) &\sim \left[\epsilon_{in} (kL)^{1-\alpha} \right]^{2/3} k^{-5/3} \\ &\sim k^{-1-\frac{2}{3}\alpha} \end{aligned} \quad (4)$$

e.g. Mazzi and Vassilicos 2004

When do we not find K41?

1. Low Reynolds number (no scale separation)
2. Broadband forcing with k^α, $\alpha > -1$.
3. Transfer is not local in k-space
4. Internal intermittency

Passive scalar

$$\frac{d\frac{1}{2}\left\langle\theta^2\right\rangle}{dt} = \langle f_\theta \theta\rangle - \langle \epsilon_\theta\rangle$$

$$\int E_\theta(k)dk = \langle\theta^2\rangle$$

and the evolution-equation,

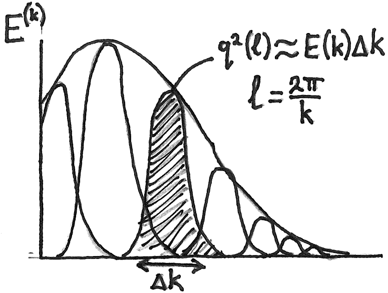
$$\frac{dE_\theta(k)}{dt} = T_\theta(k) + P_\theta(k) - D_\theta(k)$$

with

$$\int P_\theta(k)dk = \langle f_\theta \theta\rangle\,, \qquad (5)$$

$$\int D_\theta(k)dk = \int 2Dk^2 E_\theta(k)dk = \langle\epsilon_\theta\rangle \qquad (6)$$

$$\rightarrow \int T_\theta(k)dk = 0.$$



Need to model $T_\theta(k)$ and $P_\theta(k)$

Corrsin-Obukhov scaling

Extending the ideas of Kovaznay to the passive scalar [2]

$$T_\theta(k) = -\partial_k \Pi_\theta(k)$$

with $\Pi_\theta(k) = F(E(k), E_\theta(k), k)$,

$$\Pi_\theta(k) = C_\theta E_\theta(k) E(k)^{1/2} k^{5/2}$$

Leading to

$$E_\theta(k) \sim \left[\int^k P_\theta(k) dk\right] \langle\epsilon\rangle^{-1/3} k^{-5/3}$$

[2] Rubinstein & Clark, J. Turbul. 2013

Viscous heating

$$\frac{\partial \frac{1}{2}\left\langle \theta^2 \right\rangle}{\partial t} = -D(\nabla\theta)^2 + \left\langle f'_\theta \theta' \right\rangle - \left\langle \boldsymbol{u}'\theta \right\rangle \cdot \nabla \left\langle \theta \right\rangle$$

Motivation

Every viscous flow converts kinetic energy into heat.

- What does the temperature field look like in a turbulent flow?
- How large are the fluctuations?

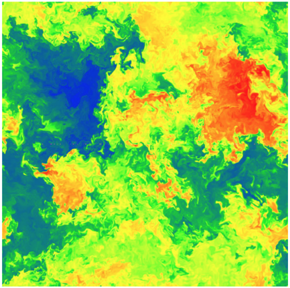


Figure : from Watanabe et al., *New Journal of Physics (2004)*

Introduction: turbulence and injected energy

$$\partial_t \mathcal{K} = P - \langle \epsilon \rangle \, .$$

In a steady state, we constantly inject energy: $\mathcal{K}_{inj} = \int_0^t P(t)dt$

How much heat?

Using Taylor's zeroth order law[a]

$$\langle \epsilon \rangle \sim U^3/L$$

Consider an air-experiment
$L = 1m,\ U = 1m/s \rightarrow \langle \epsilon \rangle \approx 1m^2/s^3.$
This is the kinetic energy we need to inject per second, per m^3 of fluid to keep the flow going.
In a closed system this will heat the fluid,

$$\frac{d\langle \theta \rangle}{dt} = \frac{\langle \epsilon \rangle}{c_p}$$

with $c_p = 10^3 J/kg/K$ the specific heat.
After one hour: $\Delta \langle \theta \rangle = 3600 * 1/10^3 = 3.6K$.

[a] Taylor, Proc. R. Soc. Lond 1935; Vassilicos ARFM 2015.

How large are the fluctuations? What is their size?

The average heat is easily estimated, but the fluctuations? Only few relevant studies[3].

Introduce: $\theta = \langle\theta\rangle + \theta'$, $\epsilon(\boldsymbol{x}, t) = \langle\epsilon\rangle + \epsilon'$

$$\frac{\partial\theta'}{\partial t} + \boldsymbol{u}\cdot\nabla\theta' = D\Delta\theta' + \frac{\epsilon'}{c_p}$$

[3] De Marinis et al. JFM 2013, Cadot & Plaza APS 2005

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$$\frac{\partial \theta'}{\partial t} + \boldsymbol{u} \cdot \nabla \theta' = D \Delta \theta' + \frac{\epsilon'}{c_p}$$

so that

$$\frac{d \frac{1}{2} \left\langle \theta'^2 \right\rangle}{dt} = \frac{\langle \epsilon' \theta' \rangle}{c_p} - \langle \epsilon_\theta \rangle$$

where $\langle \epsilon_\theta \rangle = D \left\langle (\nabla \theta')^2 \right\rangle$.

Unclosed equation $\rightarrow$ we need Simulation/Theory/Experiment.
First try: spectral model.

[3] De Marinis et al. JFM 2013, Cadot & Plaza APS 2005

How to model the viscous heat production?

How to model the viscous heat production?

$$P_\theta(k) = F(E_\theta(k), E(k), \nu, c_p, k)$$

A suggestion

430 D. De Marinis, S. Chibbaro, M. Meldi and P. Sagaut

Accounting for Joule internal energy production introduces a new nonlinear term in the temperature spectrum (2.15). The numerical closure of this new term can be performed by a number of different approaches. Among these, the formulation of a new EDQNM equation for the source term or convolution closures are possible approaches to model the Joule heating term. In the present work, we choose the approach proposed by Corrsin (1964), who derived a model for the destruction term in the equation for the transport of a reactive scalar. The formulation is explicitly local in the spectral space, and dimensional analysis leads to:

$$k^*\hat{P}^*(k) \simeq \frac{\nu}{c}\sqrt{\left(\theta^*\frac{\partial u_i^*}{\partial x_k^*}\frac{\partial u_i^*}{\partial x_k^*}\right)} \simeq 2\frac{\nu}{c}k^{*3}E^*(k)\sqrt{k^*E_\theta^*(k)} \tag{2.18}$$

and therefore

$$\hat{P}^*(k) \simeq 2\frac{\nu}{c}k^{*5/2}E^*(k)E_\theta^{*1/2}(k). \tag{2.19}$$

Problem with this model

$$P_\theta \sim \frac{\nu}{c_p} k^{5/2} E(k) E_\theta(k)^{1/2}$$

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$$P_\theta \sim \frac{\nu}{c_p} k^{5/2} E(k) E_\theta(k)^{1/2}$$

Consider $E_\theta(t=0) = 0$ what happens?

Turbulence and analytics

Whenever we try to do anything analytical in turbulence we start by assuming

1. isotropy
2. Gaussianity

Or at least expansions around isotropy/Gaussianity

Derivation EDQNM

Analytical closures, derived from Navier-Stokes. QN [4], DIA, LHDIA, TFM [5], EDQNM [6], LMFA[7]

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] \widehat{\overline{uu}} = f(\widehat{\overline{uuu}})$$

$$\left[\frac{\partial}{\partial t} + \nu k^2\right] \widehat{\overline{uuu}} = f_2(\widehat{\overline{uuuu}})$$

QN: Quasi Normal $\widehat{\overline{uuuu}} \approx \sum \widehat{\overline{uu}}\ \widehat{\overline{uu}}$

[4] Millionschikov, Dokl. Akad. Nauk SSSR 1941, Proudman & Reid, Phil. Trans. R. Soc. Lond. A 1954

[5] Kraichnan JFM 1959; PoF 1965; JFM 1970

[6] Orszag JFM 1970, Vignon & Cambon PoF 1970

[7] Bos & Bertoglio JoT 2013

Analytical closures

Ian Proudman

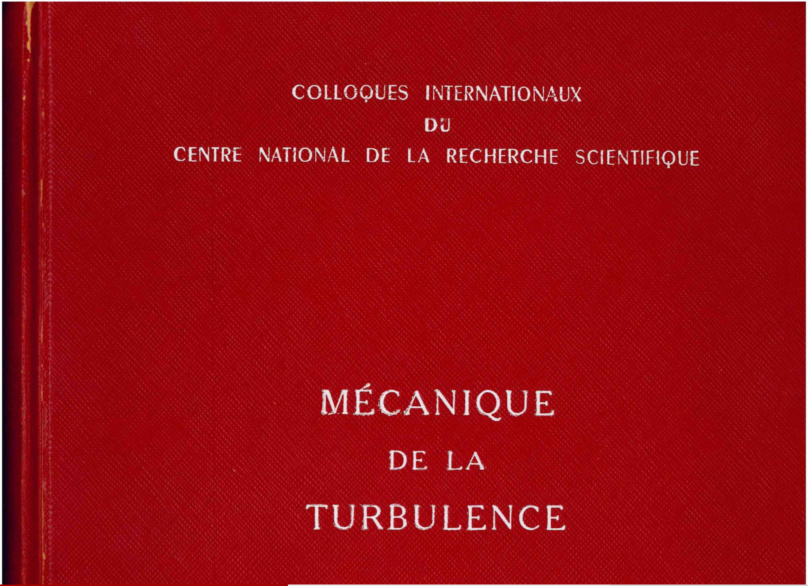
In other words, the zero-fourth-cumulant theory implies that the triple moment is non-zero only on account of interaction between its own three wavenumbers. Such a theory may be termed a "direct interaction theory." Kraichnan's theory is of this kind, and down at this conceptual level, therefore, it is closely related to zero-fourth-cumulant Theories. Indeed both theories tend to have the same very general properties and to stand or fall by similar criteria.

The differences in the assumptions underlying the two approximations have been mentioned in the text. The quasi-normality approximation involves neglect of fourth-order cumulants. The direct-interaction is a dynamical approximation and says nothing directly about what cumulants survive.

(*Left to right*) M. D. Millionshchikov, A. N. Kolmogorov, A. M. Yaglom, and R. Kraichnan at meeting at the Institut de Mécanique Statistique de la Turbulence, Marseille, 1961. (Photo courtesy J. L. Lumley.)

Robert Kraichnan

Analytical closures



Derivation EDQNM

Analytical closures, derived from Navier-Stokes. QN [8], DIA, LHDIA, TFM [9], EDQNM [10], LMFA[11]

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QN: Quasi Normal

$$\hat{\overline{uuuu}} \approx \sum \hat{\overline{uu}}\ \hat{\overline{uu}}$$

ED: Eddy Damping

$$\left[\frac{\partial}{\partial t} + \nu k^2 + \mu(k)\right] \hat{\overline{uuu}} = \sum \hat{\overline{uu}}\ \hat{\overline{uu}} \qquad \mu(k) = \tau(k)^{-1}$$

$$\mu(k) \sim \langle \epsilon \rangle^{1/3} k^{2/3}$$

M: Markovianization

[8] Millionschikov, Dokl. Akad. Nauk SSSR 1941, Proudman & Reid, Phil. Trans. R. Soc. Lond. A 1954

[9] Kraichnan JFM 1959; PoF 1965; JFM 1970

[10] Orszag JFM 1970, Vignon & Cambon PoF 1970

[11] Bos & Bertoglio JoT 2013

EDQNM

Resulting expression,

$$P_\theta(k) = 16\pi k^2 \left(\frac{\nu}{c_p}\right)^2 \iint \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}) \int_0^t G_\theta(k,t,s) \left[(p_m q_m)^2 \Phi_{ij}(\boldsymbol{p},t,s)\Phi_{ij}(\boldsymbol{q},t,s) \right.$$
$$+ 2 p_m q_m p_i q_j \Phi_{aj}(\boldsymbol{p},t,s)\Phi_{ia}(\boldsymbol{q},t,s)$$
$$\left. + p_i p_j q_m q_n \Phi_{mn}(\boldsymbol{p},t,s)\Phi_{ij}(\boldsymbol{q},t,s) \right] ds\, d\boldsymbol{p}\, d\boldsymbol{q}$$

where $\Phi_{ij}(\boldsymbol{k}) = FT[\langle u_i(\boldsymbol{x}) u_j(\boldsymbol{x}+\boldsymbol{r})\rangle]$

Structure of this expression:

$$P_\theta \sim \left(\frac{\nu}{c_p}\right)^2 k^5 E(k)^2 \tau(k)$$

instead of

$$P_\theta \sim \frac{\nu}{c_p} k^{5/2} E(k) E_\theta(k).$$

This seems more plausible.

Numerical integration

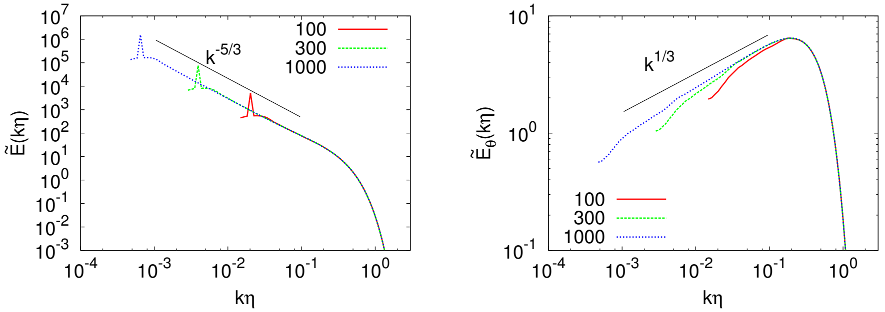

Figure : Left: Energy spectrum, normalized by Kolmogorov variables, for three different Reynolds numbers. Right: corresponding temperature fluctuation spectrum, generated by frictional heating ($Pr = 1$).

Analytical analysis of these equations

$$P_\theta(k) \sim \left(\frac{\nu}{c_p}\right)^2 k^5 E(k)^2 \tau(k) \quad \text{and} \quad \tau(k) \sim \langle\epsilon\rangle^{-1/3} k^{-2/3}$$

$$E_\theta(k) \sim \left[\int_{k_0}^{k} P_\theta(k) dk\right] \langle\epsilon\rangle^{-1/3} k^{-5/3},$$

leading to,

$$\boxed{E_\theta(k) = \left(\frac{\nu}{c_p}\right)^2 \langle\epsilon\rangle^{2/3} k^{1/3}.}$$

How large are the heat fluctuations?

$$\langle \theta^2 \rangle = \int_0^{k_\eta} E_\theta(k) dk = \frac{\langle \epsilon \rangle \nu}{c_p^2}$$

In the air experiment $\nu = 10^{-5} m^2 s^{-1}, \langle \epsilon \rangle = 1 m^2 s^{-3}, c_p = 10^3 J kg^{-1} K^{-1}$,

How large are the heat fluctuations?

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In the air experiment $\nu = 10^{-5} m^2 s^{-1}, \langle \epsilon \rangle = 1 m^2 s^{-3}, c_p = 10^3 J kg^{-1} K^{-1}$,

$$\langle \theta^2 \rangle = \int_0^{k_\eta} E_\theta(k) dk \sim 10^{-11} K^2$$

That is not very large since $\langle \theta^2 \rangle \sim Re^{-1}$ (or $\sim R_\lambda^{-2}$).

Bad news for the experimentalist: $\theta' \approx 3 \mu K$

Validation by Direct numerical simulations

DNS

Carried out by Robert Chahine

PhD student LMFA, Ecole Centrale de Lyon

Andrey Pushkarev

PostDoc LEGI Grenoble

Basic equations

- Navier-Stokes for an incompressible flow:

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}.\nabla)\boldsymbol{u} = -\frac{\nabla p}{\rho} + \nu \Delta \boldsymbol{u} + \boldsymbol{f}$$

$$\nabla.\boldsymbol{u} = 0.$$

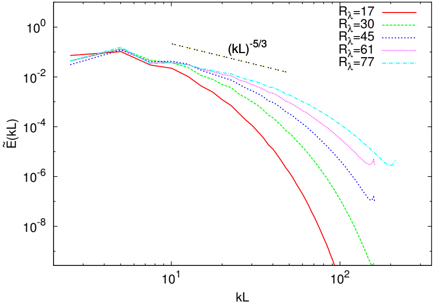
- Advection-diffusion equation + viscous friction term:

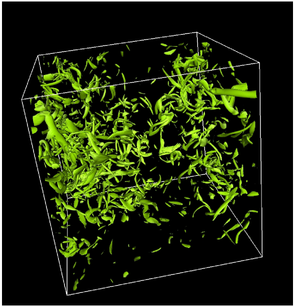
$$\frac{\partial \theta}{\partial t} + u_i \frac{\partial \theta}{\partial x_i} = D \frac{\partial^2 \theta}{\partial x_i^2} + \frac{\epsilon}{c_p}.$$

Pseudo-spectral code 256^3, periodic boundary conditions, large-scale forcing.

Scaling of the energy spectrum

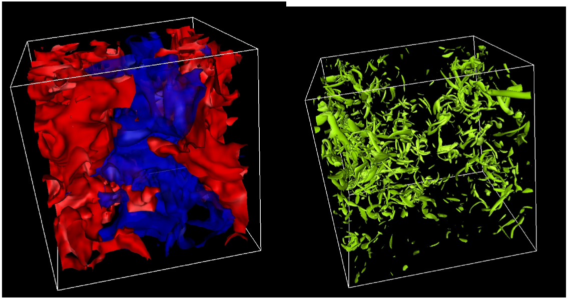
The normalized energy spectrum using quantities $\langle\epsilon\rangle$, L is
$\tilde{E}(k) = \frac{E(k)}{\langle\epsilon\rangle^{2/3} L^{5/3}} \sim (kL)^{-5/3}$





The enstrophy-isosurfaces show small-scale correlation, in agreement with $k^2E(k) \sim k^{1/3}$

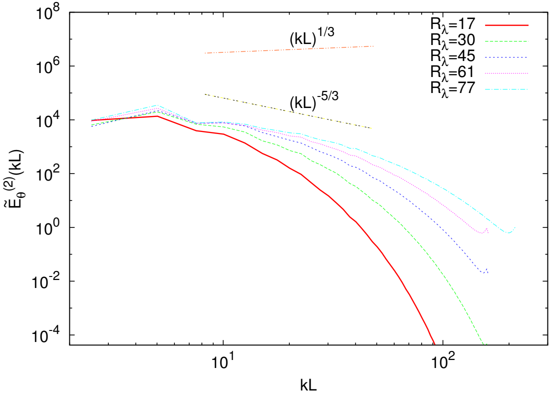
DNS Results, visualizations



- Temperature fluctuations are correlated at large scales,
- Positive fluctuations exist in the zones of where many worms are clustered.

This does not seem in agreement with the previous results!

Scaling of the temperature fluctuations spectrum



What is wrong?

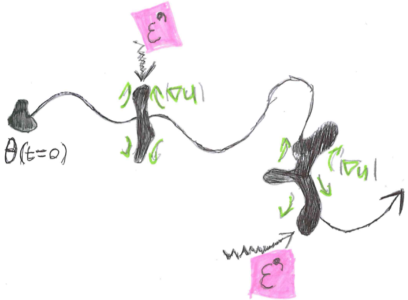
Lagrangian picture

Recall the equation for θ:

$$\frac{\partial\theta}{\partial t} + u_i\frac{\partial\theta}{\partial x_i} = D\frac{\partial^2\theta}{\partial x_i^2} + \frac{\epsilon'}{c_p}.$$

In inertial range:

$$\frac{D\theta}{Dt} = \frac{\epsilon'}{c_p}.$$



A fluid partice is heated by ϵ' and deformed by $|\nabla u|$

Evolution of the spectrum of the heat fluctuations

Formal expression for the temperature fluctuation:

$$\theta(\boldsymbol{x},t) = \theta(\boldsymbol{x},t|0) + \frac{1}{c_p}\int_0^t \int g_\theta(\boldsymbol{x},t|\boldsymbol{y},s)\epsilon'(\boldsymbol{y},s)d\boldsymbol{y}ds,$$

$$\frac{\partial E_\theta(k)}{\partial t} = T_\theta(k) + P_\theta(k) - 2Dk^2 E_\theta(k),$$

The production term is proportional to

$$P_\theta(k) \sim FT\left[\left\langle \epsilon'(\boldsymbol{x},t)\theta(\boldsymbol{x}+\boldsymbol{r},t)\right\rangle / c_p\right]$$

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$$\left\langle \epsilon'(\boldsymbol{x},t)\theta(\boldsymbol{x}+\boldsymbol{r},t)\right\rangle / c_p = \frac{1}{c_p^2}\int_0^t \int \left\langle g_\theta(\boldsymbol{x}+\boldsymbol{r},t|\boldsymbol{y},s)\epsilon'(\boldsymbol{x},t)\epsilon'(\boldsymbol{y},s)\right\rangle d\boldsymbol{y}ds.$$

And therefore $P_\theta(k) \sim \frac{\tau(k)}{c_p^2}E_\epsilon(k)$

EDQNM prediction for $E_\epsilon(k)$

Comparing

$$P_\theta(k) \sim \frac{\tau(k)}{c_p^2} E_\epsilon(k),$$

with

$$P_\theta \sim \frac{\tau(k)}{c_p^2} \nu^2 \langle \epsilon \rangle^{4/3} k^{5/3},$$

we can deduce that EDQNM (DIA) predicts

$$E_\epsilon(k) \sim \nu^2 \langle \epsilon \rangle^{4/3} k^{5/3}.$$

Possible problems

The expression $P_\theta(k) \sim \frac{\tau(k)}{c_p^2} E_\epsilon(k)$ seems to be consistent with the previous results.

- The timescale is wrong?
- The dissipation-rate spectrum $E_\epsilon(k)$ is wrong?

Experimental results: where to search?

Monin & Yaglom pp. 605-608

Atmospheric Measurements of $E_\epsilon(k)$

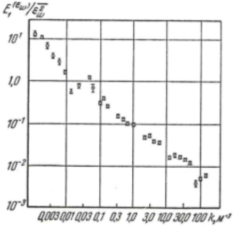


Fig. 102 The spectrum $E_1^{(\epsilon_w)}(k)$, according to Gurvich and Zubkovskii (1963).

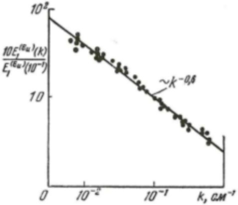


Fig. 103 The spectrum $E_1^{(\epsilon_u)}(k)$, according to Pond and Stewart (1965).

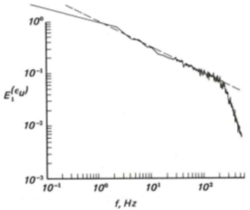
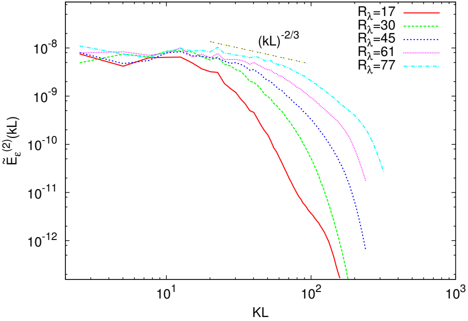


Fig. B Energy spectrum $(\partial u/\partial x)^2$ for $z = 31$ m, $\overline{u} = 11.3$ m/s after Van Atta and Chen (1970). The dashed line has a slope of -0.5.

Gurvich & Zubkovskii 1963; Pond & Stewart 1965; Van Atta & Chen 1970: $E_\epsilon(k) \sim k^{-\alpha}$, $0.5 < \alpha < 0.7$.

DNS prediction for $E_\epsilon(k)$

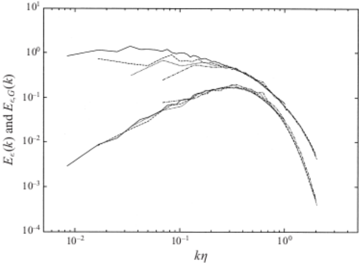
Dissipation rate spectra at $R_\lambda \leq 77$:



Is this a low Reynolds number effect?

DNS prediction for $E_\epsilon(k)$ (higher Reynolds number, $R_\lambda = 172$)

T. Gotoh and R. S. Rogallo



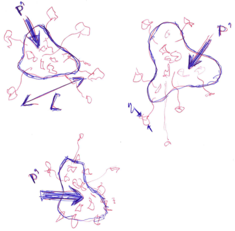
IGURE 8. Comparison of dissipation spectra $E_\epsilon(k)$ and $E_{\epsilon G}(k)$. Upper curves: $E_\epsilon(k)$. Lower curves: $E_{\epsilon G}(k)$. Line styles as in figure 4.

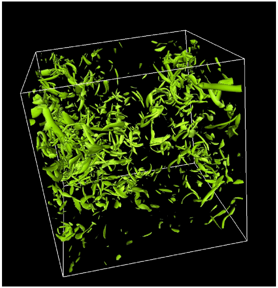
Why is the dissipation rate spectrum long-range correlated?

Read the intermittency literature starting with the famous Marseille Turbulence Conference 1961.

Why is the dissipation rate spectrum long-range correlated?

Consider the energy-injection fluctuations $\left\langle (P - \langle P \rangle)^2 \right\rangle$.





Because the injection fluctuations are !

Novikov-Yaglom estimate of the dissipation rate fluctuations

Yaglom-Novikov estimate:

$$E_\epsilon(k) \sim \langle\epsilon\rangle^2 L(kL)^{-1+\mu}$$

with $0.3 < \mu < 0.5$, intermittency parameter, characterizing the space filling of the dissipation rate.

Phenomenological theory

Using Yaglom's expression,

$$E_\epsilon(k) \sim \langle\epsilon\rangle^2 L(kL)^{-1+\mu},$$

with $P_\theta(k) \sim \frac{\tau(k)}{c_p^2} E_\epsilon(k)$ and $\tau(k) \sim \epsilon^{-1/3} k^{-2/3}$ we have

$$\int_{k_0}^{k} P_\theta(k) dk = \frac{\langle\epsilon\rangle^{5/3} L^{2/3}}{c_p^2},$$

so that

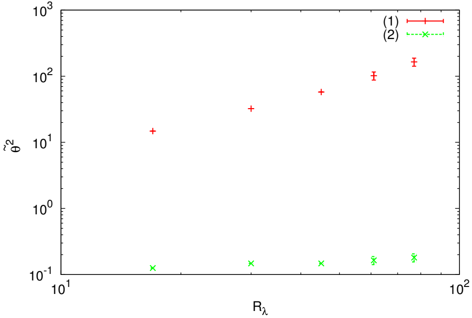
$$E_\theta(k) \sim \left[\int_{k_0}^{k} P_\theta(k) dk\right] \langle\epsilon\rangle^{-1/3} k^{-5/3}$$

$$\Longrightarrow \boxed{E_\theta(k) \sim \frac{\langle\epsilon\rangle^{4/3} L^{2/3} k^{-5/3}}{c_p^2}}$$

Reynolds number dependence of $\overline{\theta^2}$

$$\langle\theta^2\rangle = \int E_\theta(k)dk \sim \frac{(\langle\epsilon\rangle L)^{4/3}}{c_p^2}$$

- EDQNM: $\overline{\theta^2}^{(1)} \sim \frac{\langle\epsilon\rangle\nu}{c_p^2}$
  $\rightarrow \tilde{\theta^2}^{(1)} = \overline{\theta^2}\frac{c_p^2}{\langle\epsilon\rangle\nu}$
- Yaglom: $\overline{\theta^2}^{(2)} \sim \frac{(\langle\epsilon\rangle L)^{4/3}}{c_p^2}$
  $\rightarrow \tilde{\theta^2}^{(2)} = \overline{\theta^2}\frac{c_p^2}{(\langle\epsilon\rangle L)^{4/3}}$



Summarize

Implications for the value

$$\theta_{rms} \sim \frac{\langle\epsilon\rangle^{1/2} L^{2/3}}{c_p} \sim 10^{-3}K$$

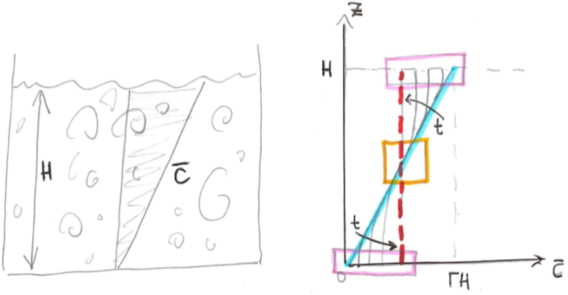
Small but measurable!

P.S. Also the timescale was wrong

Scalar injection by a uniform scalar gradient

$$\frac{\partial \theta'}{\partial t} + \boldsymbol{u}' \cdot \nabla \theta' = D \Delta \theta' + f'_\theta - \boldsymbol{u}' \cdot \nabla \langle \theta \rangle$$

Scalar injection by a uniform scalar gradient



Away from the walls

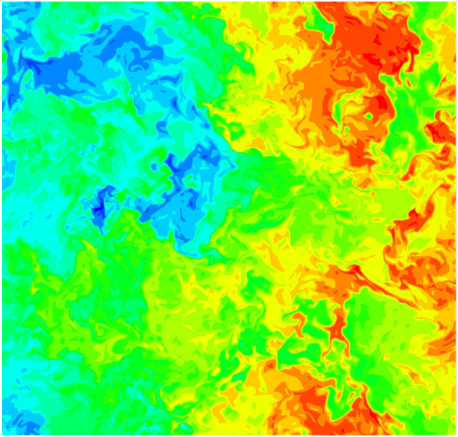
$$\frac{\partial \langle \theta \rangle}{\partial t} = 0$$

and

$$\frac{\partial \theta'}{\partial t} + \boldsymbol{u}' \cdot \nabla \theta' = D \Delta \theta' + {\color{red}-u'_z \partial_z \langle \theta \rangle}$$

Anisotropy induced by the mean gradient

Small scale isotropy



O'Gorman and Pullin 2005

Isotropy and statistical correlations

One point correlations in isotropic, mirror symmetric, turbulence
are a function of δ_{ij}!

Isotropy and statistical correlations

One point correlations in isotropic, mirror symmetric, turbulence

Vectors

$$\langle A_i \rangle = 0$$

For example $\langle u_i \rangle = 0$, $\langle \partial_i \theta \rangle = 0$

Rank 2 tensors

$$\langle A_{ij} \rangle = a\delta_{ij}$$

Examples: $\langle u_i u_j \rangle = \frac{2}{3}\mathcal{K}\delta_{ij}$,
$$\left\langle \frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j} \right\rangle = \frac{1}{3}(\epsilon_\theta/D)\delta_{ij}.$$

Isotropy and statistical correlations

Rank 3 tensors

$$\langle A_{ijm} \rangle = 0$$

Example $\left\langle \frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}\frac{\partial\theta}{\partial x_m} \right\rangle$

Rank 4 tensors

$$\langle A_{ijmn} \rangle = a\delta_{ij}\delta_{mn} + b\delta_{im}\delta_{jn} + c\delta_{in}\delta_{jm}$$

Example:

$$\left\langle \frac{\partial\theta}{\partial x_i}\frac{\partial\theta}{\partial x_j}\frac{\partial u_m}{\partial x_n} \right\rangle = -\frac{1}{30}\left(2\delta_{ij}\delta_{mn} - 3\delta_{im}\delta_{jn} - 3\delta_{in}\delta_{jm}\right)\langle \gamma_i\gamma_j s_{ij} \rangle .$$

Isotropy and statistical correlations

Rank 5 tensors

$$\langle A_{ijmnl} \rangle = 0 \tag{7}$$

Rank 6 tensors

$$\begin{aligned}\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_k}{\partial x_l}\frac{\partial u_m}{\partial x_n}} = [&\delta_{ij}\delta_{kl}\delta_{mn} - \tfrac{4}{3}(\delta_{ij}\delta_{km}\delta_{ln} + \delta_{ik}\delta_{lj}\delta_{mn} \\ &+ \delta_{im}\delta_{jn}\delta_{kl}) - \tfrac{1}{6}(\delta_{ij}\delta_{kn}\delta_{lm} + \delta_{il}\delta_{kj}\delta_{mn} + \delta_{in}\delta_{kl}\delta_{jm}) \\ &- \tfrac{3}{4}(\delta_{il}\delta_{kn}\delta_{jm} + \delta_{in}\delta_{kj}\delta_{lm}) + \delta_{il}\delta_{km}\delta_{jn} + \delta_{in}\delta_{km}\delta_{lj} \\ &+ \delta_{ik}\delta_{lm}\delta_{jn} + \delta_{ik}\delta_{ln}\delta_{jm} + \delta_{im}\delta_{jl}\delta_{kn} + \delta_{im}\delta_{jk}\delta_{ln}]\,\overline{\left(\frac{\partial u_1}{\partial x_1}\right)^3},\end{aligned}$$

From Champagne JFM 1978

Isotropy and statistical correlations

Rank 8 tensors

$$
\begin{aligned}
\langle u_{i,j}u_{k,l}u_{m,n}u_{p,q}\rangle = {} & a_4\delta_{ij}\delta_{kl}\delta_{mn}\delta_{pq} + b_4(\delta_{ij}\delta_{kl}\delta_{mp}\delta_{nq} + \delta_{ij}\delta_{kp}\delta_{lq}\delta_{mn} + \delta_{ij}\delta_{km}\delta_{ln}\delta_{pq} + \delta_{ip}\delta_{jq}\delta_{kl}\delta_{mn} + \delta_{im}\delta_{jn}\delta_{kl}\delta_{pq} \\
& + \delta_{ik}\delta_{jl}\delta_{mn}\delta_{pq}) + c_4(\delta_{ij}\delta_{kl}\delta_{mq}\delta_{np} + \delta_{ij}\delta_{kq}\delta_{lp}\delta_{mn} + \delta_{ij}\delta_{kn}\delta_{lm}\delta_{pq} + \delta_{iq}\delta_{jp}\delta_{kl}\delta_{mn} + \delta_{in}\delta_{jm}\delta_{kl}\delta_{pq} \\
& + \delta_{il}\delta_{jk}\delta_{mn}\delta_{pq}) + d_4(\delta_{ij}\delta_{kn}\delta_{lp}\delta_{mq} + \delta_{ij}\delta_{kq}\delta_{lm}\delta_{np} + \delta_{in}\delta_{jp}\delta_{kl}\delta_{mq} + \delta_{iq}\delta_{jm}\delta_{kl}\delta_{np} + \delta_{il}\delta_{jp}\delta_{kq}\delta_{mn} \\
& + \delta_{iq}\delta_{jk}\delta_{lp}\delta_{mn} + \delta_{il}\delta_{jm}\delta_{kn}\delta_{pq} + \delta_{in}\delta_{jk}\delta_{lm}\delta_{pq}) + e_4(\delta_{ij}\delta_{kn}\delta_{lq}\delta_{mp} + \delta_{ij}\delta_{kq}\delta_{ln}\delta_{mp} + \delta_{ij}\delta_{km}\delta_{lq}\delta_{np} \\
& + \delta_{ij}\delta_{km}\delta_{lp}\delta_{nq} + \delta_{ij}\delta_{kp}\delta_{ln}\delta_{mq} + \delta_{ij}\delta_{kp}\delta_{lm}\delta_{nq} + \delta_{in}\delta_{jq}\delta_{kl}\delta_{mp} + \delta_{iq}\delta_{jn}\delta_{kl}\delta_{mp} + \delta_{im}\delta_{jp}\delta_{kl}\delta_{nq} \\
& + \delta_{im}\delta_{jq}\delta_{kl}\delta_{np} + \delta_{ip}\delta_{jm}\delta_{kl}\delta_{nq} + \delta_{ip}\delta_{jn}\delta_{kl}\delta_{mq} + \delta_{il}\delta_{jq}\delta_{kp}\delta_{mn} + \delta_{iq}\delta_{jl}\delta_{kp}\delta_{mn} + \delta_{ik}\delta_{jq}\delta_{lp}\delta_{mn} \\
& + \delta_{ik}\delta_{jp}\delta_{lq}\delta_{mn} + \delta_{ip}\delta_{jl}\delta_{kq}\delta_{mn} + \delta_{ip}\delta_{jk}\delta_{lq}\delta_{mn} + \delta_{il}\delta_{jn}\delta_{km}\delta_{pq} + \delta_{in}\delta_{jl}\delta_{km}\delta_{pq} + \delta_{ik}\delta_{jn}\delta_{lm}\delta_{pq} \\
& + \delta_{ik}\delta_{jm}\delta_{ln}\delta_{pq} + \delta_{im}\delta_{jl}\delta_{kn}\delta_{pq} + \delta_{im}\delta_{jk}\delta_{ln}\delta_{pq}) + f_4(\delta_{ik}\delta_{jl}\delta_{mp}\delta_{nq} + \delta_{im}\delta_{jn}\delta_{kp}\delta_{lq} + \delta_{ip}\delta_{jq}\delta_{km}\delta_{ln}) \\
& + g_4(\delta_{ik}\delta_{jl}\delta_{mq}\delta_{np} + \delta_{im}\delta_{jn}\delta_{kq}\delta_{lp} + \delta_{ip}\delta_{jq}\delta_{kn}\delta_{lm} + \delta_{iq}\delta_{jp}\delta_{km}\delta_{ln} + \delta_{in}\delta_{jm}\delta_{kp}\delta_{lq} + \delta_{il}\delta_{jk}\delta_{mp}\delta_{nq}) \\
& + h_4(\delta_{il}\delta_{jk}\delta_{mq}\delta_{np} + \delta_{in}\delta_{jm}\delta_{kq}\delta_{lp} + \delta_{iq}\delta_{jp}\delta_{kn}\delta_{lm}) + i_4(\delta_{ik}\delta_{jq}\delta_{lm}\delta_{np} + \delta_{ik}\delta_{jn}\delta_{lp}\delta_{mq} + \delta_{ik}\delta_{jm}\delta_{lq}\delta_{np} \\
& + \delta_{ik}\delta_{jp}\delta_{ln}\delta_{mq} + \delta_{im}\delta_{jl}\delta_{kq}\delta_{np} + \delta_{im}\delta_{jq}\delta_{kn}\delta_{lp} + \delta_{im}\delta_{jp}\delta_{kq}\delta_{ln} + \delta_{im}\delta_{jk}\delta_{lp}\delta_{nq} + \delta_{ip}\delta_{jl}\delta_{kn}\delta_{mq} \\
& + \delta_{ip}\delta_{jn}\delta_{kq}\delta_{lm} + \delta_{ip}\delta_{jm}\delta_{kn}\delta_{lq} + \delta_{ip}\delta_{jk}\delta_{lm}\delta_{nq} + \delta_{il}\delta_{jm}\delta_{kp}\delta_{nq} + \delta_{il}\delta_{jp}\delta_{km}\delta_{nq} + \delta_{il}\delta_{jn}\delta_{kq}\delta_{mp} \\
& + \delta_{il}\delta_{jq}\delta_{kn}\delta_{mp} + \delta_{in}\delta_{jp}\delta_{km}\delta_{lq} + \delta_{in}\delta_{jk}\delta_{lq}\delta_{mp} + \delta_{in}\delta_{jl}\delta_{kp}\delta_{mq} + \delta_{in}\delta_{jq}\delta_{kp}\delta_{lm} + \delta_{iq}\delta_{jm}\delta_{kp}\delta_{ln} \\
& + \delta_{iq}\delta_{jk}\delta_{ln}\delta_{mp} + \delta_{iq}\delta_{jl}\delta_{km}\delta_{np} + \delta_{iq}\delta_{jn}\delta_{km}\delta_{lp}) + j_4(\delta_{ik}\delta_{jn}\delta_{lq}\delta_{mp} + \delta_{ik}\delta_{jq}\delta_{ln}\delta_{mp} + \delta_{im}\delta_{jq}\delta_{kp}\delta_{ln} \\
& + \delta_{im}\delta_{jl}\delta_{kp}\delta_{nq} + \delta_{ip}\delta_{jn}\delta_{km}\delta_{lq} + \delta_{ip}\delta_{jl}\delta_{km}\delta_{nq}) + k_4(\delta_{ik}\delta_{jm}\delta_{lp}\delta_{nq} + \delta_{ik}\delta_{jp}\delta_{lm}\delta_{nq} + \delta_{im}\delta_{jp}\delta_{kn}\delta_{lq} \\
& + \delta_{im}\delta_{jk}\delta_{lq}\delta_{np} + \delta_{ip}\delta_{jm}\delta_{kq}\delta_{ln} + \delta_{ip}\delta_{jk}\delta_{ln}\delta_{mq} + \delta_{il}\delta_{jq}\delta_{km}\delta_{np} + \delta_{il}\delta_{jn}\delta_{kp}\delta_{mq} + \delta_{in}\delta_{jl}\delta_{kq}\delta_{mp} \\
& + \delta_{in}\delta_{jq}\delta_{km}\delta_{lp} + \delta_{iq}\delta_{jl}\delta_{kn}\delta_{mp} + \delta_{iq}\delta_{jn}\delta_{kp}\delta_{lm}) + l_4(\delta_{il}\delta_{jm}\delta_{kq}\delta_{np} + \delta_{il}\delta_{jp}\delta_{kn}\delta_{mq} + \delta_{in}\delta_{jk}\delta_{lp}\delta_{mq} \\
& + \delta_{in}\delta_{jp}\delta_{kq}\delta_{lm} + \delta_{iq}\delta_{jk}\delta_{lm}\delta_{np} + \delta_{iq}\delta_{jm}\delta_{kn}\delta_{lp}).
\end{aligned}
$$

Hierro and Dopazo PoF 2003

Anisotropy, axisymmetric

Everything is function of δ_{ij} and the axis of symmetry e_i

vectors

$$\langle A_i \rangle = a e_i$$

Example $\left\langle \frac{\partial \theta}{\partial x_i} \right\rangle = \Gamma e_i$, $\langle u_i \theta \rangle = \langle w\theta \rangle e_i$

Rank 2 tensors

$$\langle A_{ij} \rangle = a\delta_{ij} + b e_i e_j$$

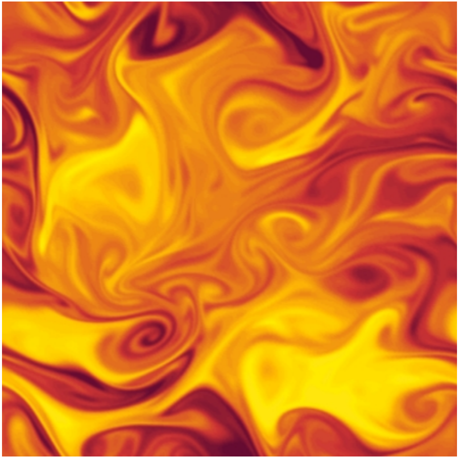
Rank 3 tensors

$$\langle A_{ijm} \rangle = a\delta_{ij} e_m + b\delta_{im} e_j + c\delta_{jm} e_i$$

Example

$$\langle \gamma_i \gamma_j \gamma_m \rangle = \frac{1}{3} \left\langle \gamma_3^3 \right\rangle (e_i \delta_{jm} + e_j \delta_{im} + e_m \delta_{ij}) \,.$$

Small scale anisotropy



Brethouwer 2000

How anisotropic are the small scales?

We will consider two quantities $\langle w\theta \rangle$ and $\left\langle \frac{\partial\theta}{\partial x_3}\frac{\partial\theta}{\partial x_3}\frac{\partial\theta}{\partial x_3}\right\rangle$.
Introduce

$$\int F_{w\theta}(k)dk = \langle w\theta \rangle$$

A measure for isotropy:

$$\rho_{w\theta}(k) = \frac{F_{w\theta}(k)}{\sqrt{E(k)E_\theta(k)}}$$

How anisotropic are the small scales?

Experiments/closure/numerics[12]

$$F_{w\theta}(k) \sim \Gamma \epsilon^{1/3} k^{-7/3}$$

And

$$E(k) \sim \epsilon^{2/3} k^{-5/3}, \qquad E_\theta(k) \sim \epsilon_\theta \epsilon^{-1/3} k^{-5/3}$$

we find

$$\rho_{w\theta}(k) = \frac{F_{w\theta}(k)}{\sqrt{E(k)E_\theta(k)}} \sim (kL)^{-2/3}$$

[12] Lumley 1964,'67, Mydlarski and Warhaft 1998, WB et al. 2005, Gotoh and Watanabe 2007

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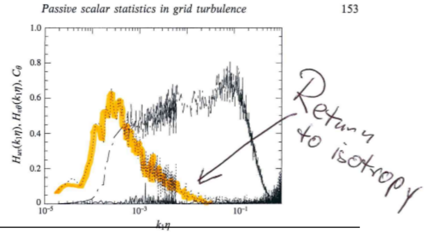


[12] Lumley 1964,'67, Mydlarski and Warhaft 1998, WB et al. 2005, Gotoh and Watanabe 2007

Another measure for isotropy

In an isotropic scalar field

$$\left\langle \left(\frac{\partial \theta}{\partial z} \right)^3 \right\rangle = 0$$

in an anisotropic field: not.
Small scale anisotropy:

$$\frac{\left\langle \left(\frac{\partial \theta}{\partial z} \right)^3 \right\rangle}{\left\langle \left(\frac{\partial \theta}{\partial z} \right)^2 \right\rangle^{3/2}} \neq 0$$

How evolves the anisotropy

Introduce $\frac{\partial\theta}{\partial x_i} \equiv \gamma_i$ and $\frac{\partial u_i}{\partial x_j} \equiv s_{ij}$

$$\frac{\partial}{\partial t}\left\langle \gamma_3^3 \right\rangle + \frac{\partial}{\partial x_j}\left\langle u_j \gamma_3^3 \right\rangle = -3\left\langle \gamma_3\gamma_3\gamma_j s_{3j} \right\rangle - \left\langle \gamma_3\gamma_3 s_{33} \right\rangle \Gamma + D...$$

In an isotropic flow $\left\langle \gamma_3\gamma_3 s_{33} \right\rangle \neq 0$.

Another measure for isotropy

Exact relation:

$$\langle \gamma_3 \gamma_3 s_{33} \rangle \sim \int k^2 T_\theta(k) dk$$

Good approximation[13]

$$\int k^2 T_\theta(k) dk \approx \int 2Dk^4 E_\theta(k) dk \approx \epsilon_\theta \epsilon^{1/2} \nu^{-3/2}$$

[13] Batchelor, CUP, 1953

Another measure for isotropy

Exact relation:

$$\langle \gamma_3 \gamma_3 s_{33} \rangle \sim \int k^2 T_\theta(k) dk$$

Good approximation[13]

$$\int k^2 T_\theta(k) dk \approx \int 2Dk^4 E_\theta(k) dk \approx \epsilon_\theta \epsilon^{1/2} \nu^{-3/2}$$

Starting from isotropy we have

$$\frac{\partial}{\partial t} \left\langle \gamma_3^3 \right\rangle = -\langle \gamma_3 \gamma_3 s_{3j} \rangle \, \Gamma$$

We can thus estimate $\left\langle \gamma_3^3 \right\rangle \approx -\mathcal{T} \epsilon_\theta \epsilon^{1/2} \nu^{-3/2}$.

[13] Batchelor, CUP, 1953

Another measure for isotropy

Exact relation:

$$\langle \gamma_3 \gamma_3 s_{33} \rangle \sim \int k^2 T_\theta(k) dk$$

Good approximation[13]

$$\int k^2 T_\theta(k) dk \approx \int 2Dk^4 E_\theta(k) dk \approx \epsilon_\theta \epsilon^{1/2} \nu^{-3/2}$$

Starting from isotropy we have

$$\frac{\partial}{\partial t} \langle \gamma_3^3 \rangle = -\langle \gamma_3 \gamma_3 s_{3j} \rangle \Gamma$$

We can thus estimate $\langle \gamma_3^3 \rangle \approx -\mathcal{T} \epsilon_\theta \epsilon^{1/2} \nu^{-3/2}$.

$$\frac{\langle \gamma_3^3 \rangle}{\langle \gamma_3^2 \rangle^{3/2}} \sim \frac{\Gamma \epsilon_\theta \epsilon^{1/2} \nu^{-3/2} \mathcal{T}}{(\epsilon_\theta / \nu)^{3/2}} \sim \frac{\Gamma \epsilon^{1/2}}{\epsilon_\theta^{1/2}} \mathcal{T}$$

But what is $\mathcal{T}$

[13] Batchelor, CUP, 1953

Timescale

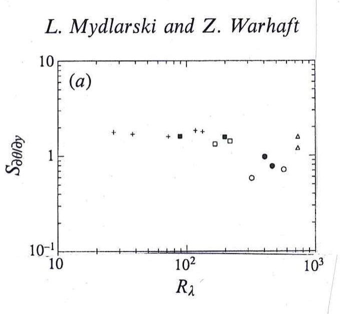
In general, gradient-statistics are correlated over a Kolmogorov-time, so that $\mathcal{T} \sim (\epsilon/\nu)^{1/2}$
Implication:

$$\frac{\left\langle \gamma_3^3 \right\rangle}{\left\langle \gamma_3^2 \right\rangle^{3/2}} \sim Re^{-1/2}$$

$\rightarrow$ for large Reynolds numbers small scale isotropy.

Experimental observations

Experimental results and DNS



Why?

Gibson et al. 1970, Pumir, PoF 1994, Mydlarski and Warhaft 1998

A look at the equations

On the anisotropy of the turbulent passive scalar 61

Appendix C. A model for the displacement triple correlation

The closed expressions for $\Xi^{I}(k)$, $\Xi^{II}(k)$ and $\Xi^{III}(k)$ are obtained by the same procedure as outlined in the previous section, leading to

$$
\begin{aligned}
D_{\parallel}^{(3)}(r) &= -6\frac{r_i r_j r_m}{r^3}\int \Xi_{ijm}(\boldsymbol{k})[1-\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}]\,\mathrm{d}\boldsymbol{k} \\
&= -3\mathrm{i}r\int \Theta^{\Xi}(k,p,q)j(kr)[-y(x+yz)p^3F(k)E^s(q) \\
&\quad + y(x+yz)p^3F(k)E^c(q) - (z+xy)p^3F(q)E^s(k) \\
&\quad + \tfrac{1}{2}(1-2xyz-3y^2z^2+y^2-z^2)k^3F(p)E^s(q) \\
&\quad + \tfrac{1}{2}(1+2xyz+3y^2z^2-y^2-z^2)k^3F(p)E^c(q)] \\
&\quad + \Theta^{\Xi}(k,p,q)l(kr)[-(z+xy)p^3F(q)E^c(k) \\
&\quad + (1-z^2)(1-y^2)k^3F(p)E^s(q) \\
&\quad + (1-z^2)y^2k^3F(p)E^c(q)]\,\frac{\mathrm{d}p}{p}\,\frac{\mathrm{d}q}{q}\,\mathrm{d}k. \qquad \text{(C 1)}
\end{aligned}
$$

Needs the definition of a timescale.

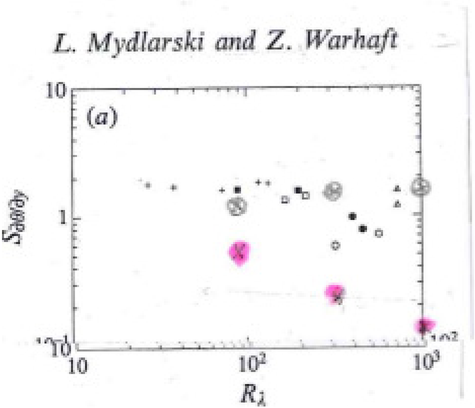
Two choices:

$\mathcal{T} = L/U$

or

$\mathcal{T} = \epsilon^{-1/3}k^{-2/3} \rightarrow \mathcal{T}(k_\eta) = (\nu/\epsilon)^{-1/2}$

Try !



Pink: $\mathcal{T} = \epsilon^{-1/3} k^{-2/3}$; White: $\mathcal{T} = L/U$
We seem to need the integral timescale (unlike the velocity gradient).
Why?

Mydlarski, Pumir et al. PRL 1998

Analogy with Burgulence

We know that in forced Burgulence,

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{1}{T} \boldsymbol{u}$$

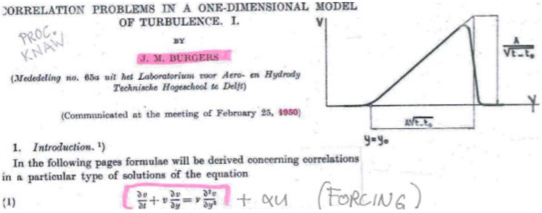
The Lagrangian correlation time is $\Theta(r) \sim T$. Since fluid particles decorrelate only due to the forcing mechanism or in the presence of shocks (Kraichnan 1968).
T is the time it takes for a fluid particle to encounter a shock.

Analogy?

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \frac{1}{T} \boldsymbol{u}$$

compared to

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = -u_3 \Gamma$$



CORRELATION PROBLEMS IN A ONE-DIMENSIONAL MODEL OF TURBULENCE. I.

PROC. KNAW

BY

J. M. BURGERS

(Mededeling no. 65a uit het Laboratorium voor Aero- en Hydrodynamica der Technische Hogeschool te Delft)

(Communicated at the meeting of February 25, 1950)

1. *Introduction.* [1])

In the following pages formulae will be derived concerning correlations in a particular type of solutions of the equation

(1) $\boxed{\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial y^2}} + \alpha u$ (FORCING)

Conclusions

On scalar anisotropy

- The Lagrangian time-scale of the scalar is the integral time-scale (cf. Burgulence)

On viscous heating

- Viscous temperature fluctuations are a large-scale quantity
- For $R_\lambda = 1000$ closure theory will misestimate $\overline{\theta^2}$ by an error of the order of 10^6!

To the experimentalist: you can be the first to measure these fluctuations !

References:
WB, *The temperature spectrum generated by frictional heating in isotropic turbulence*, J. Fluid Mech. (2014)
WB, Chahine, Pushkarev, *On the scaling of temperature fluctuations induced by frictional heating*, Phys. Fluids (2015)
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