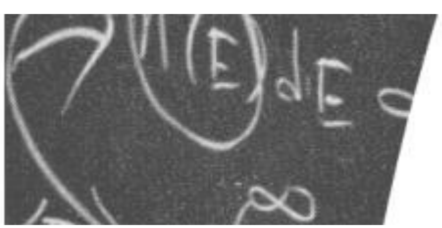
# Phenomenological model of turbulent instationary cascade in the framework of scaling symmetry approach.

**Mikhael Gorokhovski** (Ecole Centrale de Lyon, LMFA)

Joint work with

**Vladimir Saveliev** (Institute of Ionosphere, Almaty/Tohoku University, Sendai)



Nouveaux défis en turbulence IV, 21-25, March, 2016

ÉCOLE DE PHYSIQUE des HOUCHES

UNIVERSITE JOSEPH FOURIER

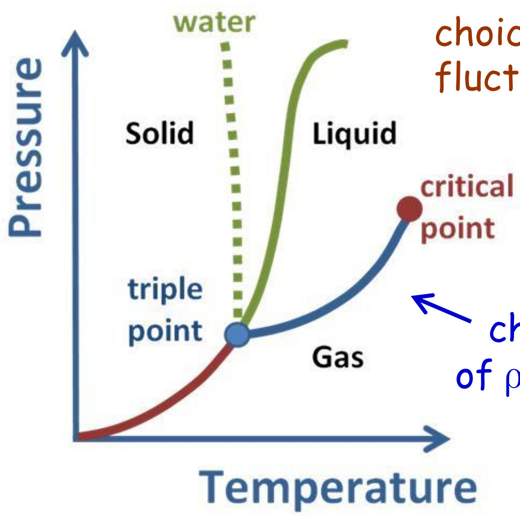
# Outline

- ✓ universality, renormalization and scaling symmetry;
  our motivations in the new formulation of turbulent cascade
- ✓ fragmentation under scaling symmetry in terms of the energy density
- ✓ renormalization of the fragmentation equation
- ✓ auto-similar solutions (intermediate asymptotics)
- ✓ applications to decaying isotropic turbulence and surprising scenario
- ✓ comparison with Kolmogorov-Oboukhov, 41
- ✓ preliminary DNS (by Alexis Barge, LMFA)

- ✓ universality, renormalization, scaling symmetry;

  our motivations in description of turbulent cascade

# Second-order phase transition (critical phenomena)




choice ceases; large-scale fluctuations of density

$(T_c; P_c)$

choice of $\rho_L$ or $\rho_G$

Observations at $T \to T_C$ :

$$\rho_L - \rho_C \sim (T_C - T)^{\beta}$$

$$\rho_C - \rho_G \sim (T_C - T)^{\beta}$$

$$\rho - \rho_C \sim (P - P_C)^{1/\sigma}$$

$$\text{compressibility} \sim |T - T_C|^{-\gamma}$$

**Striking observations near the critical point:**

✓ $\beta, \sigma, \gamma$ : don't depend upon the particular fluid system studied

(idem for magnetization $m \sim (T_C - T)^{\beta}$, $m \sim h^{1/\sigma}$) $\Rightarrow$ details of interactions between particles (such as the force laws between the molecules) are irrelevant.

This characteristic of critical phenomena is called **universality**.

✓ large-scale fluctuations are responsible for critical behavior i.e. **long-range correlations**

$\Rightarrow$ all particles contribute to the force on every particle

✓ unsymmetrical state smoothly connects to the ordered state: **scaling symmetry** appears.

# Definition of a phenomena possessing the scaling symmetry property

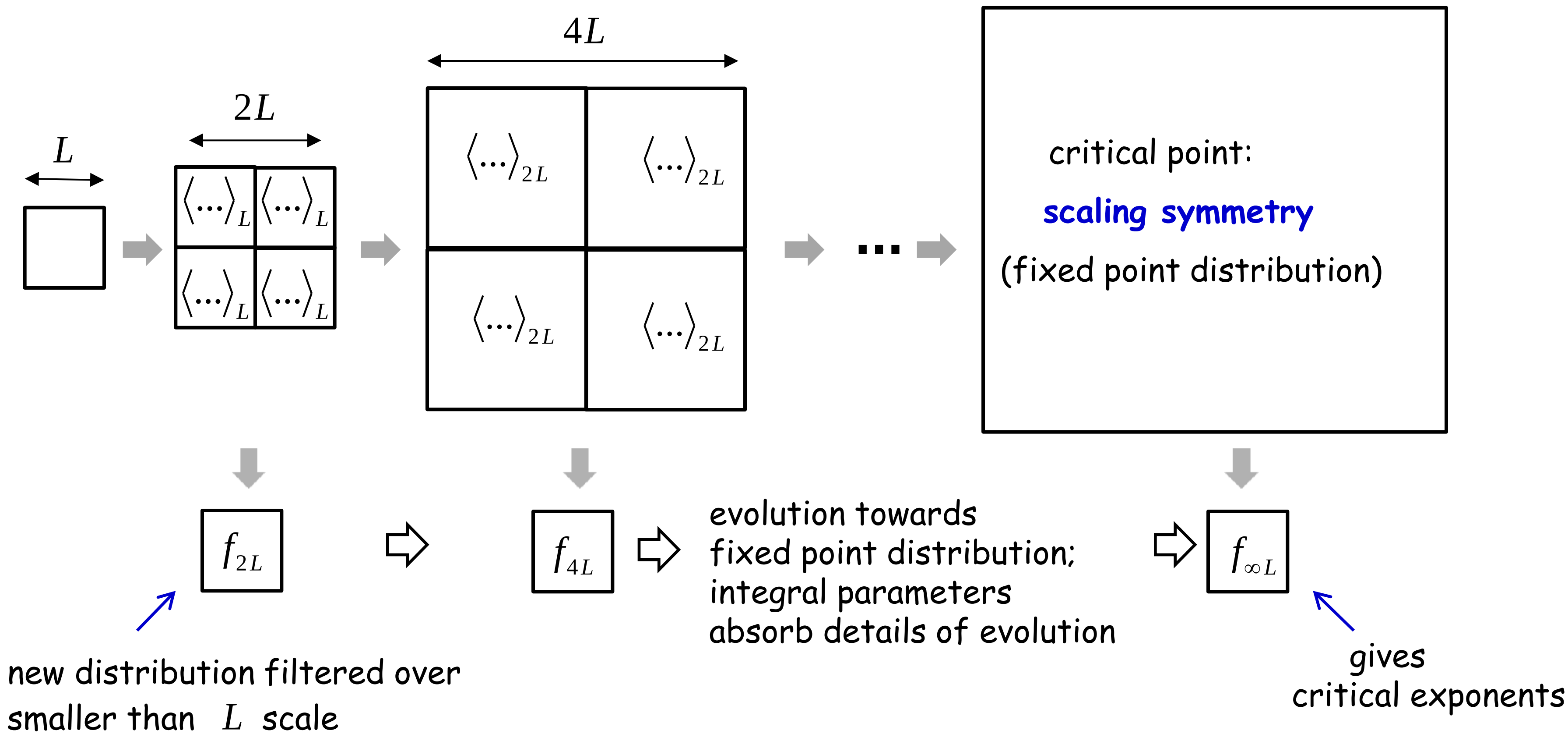
✓ with respect to change of the scale of observation, a system exhibits an invariant statistical behavior, insensitively to details of multiply short-range interactions

✓ details are effectively absorbed into integral parameters (critical exponents, for example)

Examples in different areas of physics:

- physics of high-energy elementary particles
- phase transitions in fluids and magnets
- astrophysics
- physics of polymer solutions

## Idea of renormalization transformation (Wilson – Kadanoff – Bogolioubov)





Goldenfeld, Lectures on Phase Transitions and the Renormalization Group, Addison-Wesley, 1992
Kadanoff, Statistical Physics: Statics, Dynamics, and Renormalization, World Scientific, 2000.
Ma, Modern Theory of Critical Phenomena, Westview Press, 2000

## Kolmogorov-Oboukhov "minus one-third law" and scaling symmetry

✓ a continuity equation for the energy density in the inertial range of turbulent length-scales:

$$\frac{\partial E(r,t)}{\partial t} = -\frac{\partial}{\partial r} J(E, \mathrm{v})$$

✓ hypotheses of the local statistical equilibrium of small-scale motions (statistical stationarity):

$$\frac{\partial E}{\partial t} = 0 \Rightarrow J(E, \mathrm{v}) = const$$

✓ scale-invariance of $J(E, \mathrm{v})$ : only possibility is permitted $\quad j = E \cdot \mathrm{v} \Rightarrow \frac{\mathrm{v}^2}{r}\mathrm{v} = const$

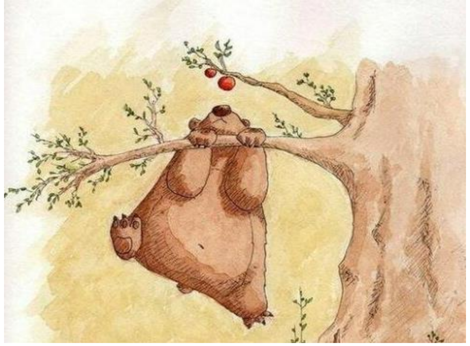
$\Rightarrow$ the "minus one-third law": $E(r) \sim const^{2/3}\, r^{-1/3}$

✓ Kolmogorov required scaling symmetry of the constant ( $\varepsilon$ );

along with hypotheses of the local statistical equilibrium this gives the "minus one-third law"

**Our motivation:**

- **to avoid the hypotheses of the local statistical equilibrium**
- **to exploit scaling symmetry idea to its fullest extent**



✓ a continuity model-equation with flux as a functional:

$$\frac{\partial E(r,t)}{\partial t}=-\frac{\partial}{\partial r}\int D(r,r')E(r',t)\,d\,r'$$

✓ with scale-invariant kernel (evolution towards fixed point distribution) :

$$\frac{\partial E(r,t)}{\partial t}=-\frac{\partial}{\partial r}\int D\left(\frac{r}{r'}\right)E(r',t)\,d\,r'$$

✓ There are exist two approaches in order to construct such an equation:

- on the basis of scaling symmetry of hydrodynamic equations, and of their corresponding solutions
- on the basis of phenomenological equation where scaling symmetry is already embedded

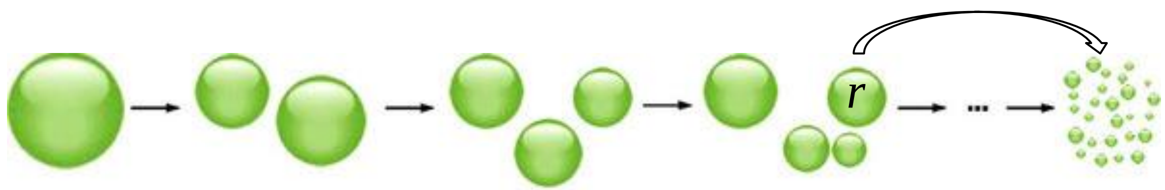
✓ here the latter approach is shown: we used the equation of fragmentation under scaling symmetry

✓ starting point: fragmentation in terms of turbulent cascade

# Fragmentation process under scaling symmetry

✓ ***The fragmentation process:***

the production of random fragments (or particles) by the continuous breakup of clusters.




$F \;\; r,t \;\; ?$

time (kinetic evolution towards smaller sizes)

✓ ***Fragmentation under scaling symmetry***

- a parent particle splits into daughter particles $\boxed{r \Rightarrow \alpha r,\ 0 \le \alpha \le 1}$ $\int_0^1 Q(\alpha)\, d\alpha = 1$

  with a partition probability $Q(\alpha)$ which is independent of the parent particle size*

- the breakup rate of a fragment of size $r$ is the power function $\boxed{v(r) = v_0\, r^{\mu}}$

*Kolmogorov,41

# Population balance equation of fragmentation under scaling symmetry*

rate of change
of number of particles

number of parent particles
breaking per unit time

$$\frac{\partial}{\partial t}F(r,t)\Delta r=\underbrace{\int_0^1 q_0\,Q(\alpha)\,d\alpha\;\nu\left(\frac{r}{\alpha}\right)F\left(\frac{r}{\alpha}\right)\Delta\left(\frac{r}{\alpha}\right)}_{\text{gain of new particles due to the breakup of parent particles}}-\underbrace{\nu(r)F(r)\,\Delta r}_{\text{loss of particles of size r}};$$

✓ well-known equation of fragmentation:

$$\boxed{\frac{\partial}{\partial t}F(r,t)=\int_0^1 d\alpha\, q_0\, Q(\alpha)\frac{1}{\alpha}\,\nu\left(\frac{r}{\alpha}\right)F\left(\frac{r}{\alpha}\right)-\nu(r)F(r)}$$

$$\int_0^\infty F(r)\,dr\neq const$$ - number of particles increases with time

*Lubimov, 66

## Interpretation in terms of the energy density

- ✓ a small range $\left[r, r+d\,r\right]$ corresponds to definite set of disturbances with a certain characteristic length scale $r$

- ✓ the kinetic energy of larger length scale is handed down at random to smaller scales, and in such a way the population of smaller scales is evolving in time

*energy density, m/s²*

$$\underbrace{f(r)}_{\text{energy density, m/s}^2}dr \sim r^{\xi} F(r)dr$$ - energy in $\left[r \div r+d\,r\right]$

- ✓ in inviscid flows, the mechanical energy of liquid is conserved $\Rightarrow$ $\boxed{\int_0^{\infty} f(r)\,d\,r = const}$

$\Rightarrow$ mean number of smaller scales per one event of the energy transfer:

$$\boxed{\frac{\partial}{\partial t}\int_0^{\infty} r^{\xi} F(r)\,d\,r = 0 \;\Rightarrow\; q_0 = \frac{1}{\left\langle \alpha^{\xi} \right\rangle_Q}}$$

# Fragmentation integral equation for the energy per unit length

✓ from population balance equation

$$\int_0^r r^{\xi}\,dr\ :\quad \frac{\partial F(r)}{\partial t} = q_0 \int_0^1 \frac{d\alpha}{\alpha} Q(\alpha)\, v\left(\frac{r}{\alpha}\right) F\left(\frac{r}{\alpha}\right) - v(r)F(r)$$

to fragmentation equation* for the energy per unit length *f(r) = r* $^{\xi}$ *F(r)* in *r*-space:

$$\boxed{\frac{\partial f(r)}{\partial t} = \int_0^1 d\alpha\, q(\alpha)\, \frac{1}{\alpha} v\left(\frac{r}{\alpha}\right) f\left(\frac{r}{\alpha}\right) - v(r)f(r)}$$

$$\boxed{\int_0^{\infty} f(r)\,dr = const}$$

$$\boxed{v(r) = v_0\, r^{\mu}}$$

$v_0$ - the constant depending on how the turbulent cascade is started

$$q(\alpha) = \frac{\alpha^{\xi} Q(\alpha)}{\left\langle \alpha^{\xi} \right\rangle_Q}$$ - new partition probability function; $$\int_0^1 q(\alpha)\,d\alpha = 1$$


* Gorokhovski & Saveliev, Physica D, 2008
Gorokhovski & Herrmann, Annual Rev. in Fluid Mech. (2008)
Zamansky, Vinkovic & Gorokhovski, JFM (2013), JoP (2011)

## Motivation for the continuity form of the fragmentation equation

✓ Fragmentation equation in the integral form:

$$\frac{\partial f(r;t)}{\partial t} = \int_0^1 d\alpha\, q(\alpha) \frac{1}{\alpha} v\left(\frac{r}{\alpha}\right) f\left(\frac{r}{\alpha}\right) - v(r) f(r) \quad \Leftrightarrow$$

evolution of infinite number of breaking particles in *r*-space

✓ each particle jumps; it has no velocity in r-space

✓ lack of explicit expression for the energy-flux

✓ difference between gain and loss terms may be big

✓ equivalent statistics from both

✓ Expected renormalized equation (i.e. continuity form) :

energy -flux, $m^2/s^3$

$$\frac{\partial f(r;t)}{\partial t} = -\frac{\partial}{\partial r} j(r,t) \quad \Leftrightarrow$$

evolution of finite number of quasi-particles breaking in *r*-space

✓ explicit expression for the energy-flux (self-similar solution, integral of motion)

✓ each quasi-particle moves smoothly at the "effective" velocity $V = j(r)/f(r)$ ,

which is linked to the state of all other particles (non-locality)

✓ $j(r=0,t)$ "condensation" of zero-size quasi-particles (?mimics viscous dissipation?)

✓ renormalization of the fragmentation equation for the energy density

# Operator form of the fragmentation integral equation

- ✓ operator of scaling symmetry transformation:

$$\frac{1}{\alpha} f\left(\frac{r}{\alpha}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\ln^n \alpha}{n!} \left(\frac{\partial}{\partial r} r\right)^n f(r) = e^{-\ln\alpha \left(\frac{\partial}{\partial r} r\right)} f(r)$$

$\Uparrow$

Taylor's expansion of $e^{-\ln\alpha \left(\frac{\partial}{\partial r} r\right)}$

- ✓ the operator form of the production term:

$$\frac{\partial f(r;t)}{\partial t} = \int_0^1 d\alpha\, q(\alpha) \frac{1}{\alpha} \nu\left(\frac{r}{\alpha}\right) f\left(\frac{r}{\alpha}\right) - \nu(r) f(r)$$

$\Uparrow$

$$\boxed{\int_0^1 d\alpha\, q(\alpha)\, e^{-\ln\alpha \left(\frac{\partial}{\partial r} r\right)} \nu(r) f(r)}$$

## Exact renormalized form of the scaling transformation operator*

$$e^{-\ln\alpha\left(\frac{\partial}{\partial r}r\right)}=1+\frac{(-\ln\alpha)}{1!}\frac{\partial}{\partial r}r+\ldots+\frac{(-\ln\alpha)^{n-1}}{(n-1)!}\left(\frac{\partial}{\partial r}r\right)^{(n-1)}+$$

$$+\frac{(-\ln\alpha)^{n}}{n!}\left(\frac{\partial}{\partial r}r\right)^{n}\int_0^1 d\beta\, p_n(\beta)\, e^{-\beta(\ln\alpha)\left(\frac{\partial}{\partial r}r\right)}$$

$$n=1,2,3\cdots \qquad p_n(\beta)=n(1-\beta)^{n-1}; \qquad \int_0^1 d\beta\, p_n(\beta)=1$$

✓ the choice of $n$ leads to equivalent expressions. We use $n=1$ :

$$e^{-\ln\alpha\left(\frac{\partial}{\partial r}r\right)}=1+(-\ln\alpha)\frac{\partial}{\partial r}r\int_0^1 d\beta\, e^{-\beta(\ln\alpha)\left(\frac{\partial}{\partial r}r\right)}$$

*Saveliev & Nanbu, Phys. Rev. E, 2002
Saveliev & Gorokhovski, Phys. Rev. E, 2005

## … and introduction of presumed repartition function

- for n = 1, the renormalized form of the production term is* :

$$\int_0^1 \frac{d\alpha}{\alpha} q(\alpha) \nu\left(\frac{r}{\alpha}\right) f\left(\frac{r}{\alpha}\right) = [1+ \langle -\ln\alpha \rangle \frac{\partial}{\partial r} r \int_0^1 d\,\alpha\, q_1(\alpha) e^{-\ln\alpha\left(\frac{\partial}{\partial r} r\right)}]\, \nu(r)\, f(r)$$

$$q_1(\alpha) = \frac{1}{\langle -\ln\alpha \rangle} \int_0^1 q(\alpha\beta)\, d\beta; \qquad \int_0^1 q_1(\alpha)\, d\alpha = 1; \qquad \langle -\ln\alpha \rangle = \int_0^1 -\ln\alpha\, q(\alpha)\, d\alpha;$$

- presumed repartition function:

$$q(\alpha) = (\gamma+1)\alpha^{\gamma},\ -1 < \gamma \quad \Rightarrow \quad \langle -\ln\alpha \rangle = \frac{1}{\gamma+1}; \qquad q_1(\alpha) = q(\alpha)$$

$$\alpha^{\gamma} = e^{\gamma \ln\alpha} \quad \Rightarrow \quad \boxed{\left(1+\gamma-\frac{\partial}{\partial r} r\right) \int_0^1 d\alpha\ e^{\ln\alpha\left(\gamma - \frac{\partial}{\partial r} r\right)} = 1}$$

*Saveliev & Gorokhovski, Phys. Rev. E, 2012

## The final form of the renormalized fragmentation equation

✓ with renormalized production form and presumed repartition function:

$$\boxed{\frac{\partial f(r;t)}{\partial t} = -\frac{\partial}{\partial r}\, j}$$

$$\boxed{j = -r\left[\int_0^1 d\alpha\ e^{\ln\alpha\left(\gamma - \frac{\partial}{\partial r} r\right)}\right]\nu(r)\, f(r)}$$

$$\boxed{\nu(r) = \nu_0\, r^{\mu}}$$

✓ The next step is to construct the exact self-similar solution

$\Rightarrow$ intermediate long-time asymptotics

✓ auto-similar solutions (intermediate asymptotics)

## Self-similar transformations

- ✓ Non-dimensional time: $\bar{t} = t / \tau_0$ ; $\tau_0$ - some positive constant

- ✓ since $\nu(r) \sim r^{\mu}$ $\Rightarrow$ $\left|\bar{t}\right|^{1/\mu}$ is a characteristic scale of scaling transformations for $f(r,t)$

- ✓ then the direct and inverse self-similar transformations are :

$$f(r,t) = \left|\bar{t}\right|^{\mu^{-1}} \phi\left( \left|\bar{t}\right|^{\mu^{-1}} r, t \right)$$

$$\phi(r,t) = \frac{1}{\left|\bar{t}\right|^{\mu^{-1}}} f\left( \frac{r}{\left|\bar{t}\right|^{\mu^{-1}}}, t \right)$$

- ✓ applying it to $\dfrac{\partial f(r;t)}{\partial t} = -\dfrac{\partial}{\partial r} j$ we have the equation for self-similar solution

# Equation for the self-similar solution

$$\frac{\partial \phi(r;t)}{\partial t} = \frac{1}{|\bar{t}|}\frac{\partial}{\partial r}\, r\left[-\frac{|\bar{t}|}{\bar{t}}\frac{1}{\mu\tau_0} + \int_0^1 d\alpha \; e^{\ln\alpha\left(\gamma - \frac{\partial}{\partial r} r\right)} \nu\right]\phi$$

✓ the equation for the fixed point distribution:

$$\frac{\partial}{\partial t} = 0 \;\Rightarrow\; r\left[\mp\frac{1}{\mu\tau_0} + \int_0^1 d\alpha \; e^{\ln\alpha\left(\gamma - \frac{\partial}{\partial r} r\right)} \nu\right]\phi_0(r) = const = j_0$$

positive/negative times

fixed point distribution

bear flux - integral of motion (hereafter: $|j_0| \equiv \varepsilon_0$) appeared during evolution; it is the energy input per unit time

✓ using $\left(1+\gamma - \frac{\partial}{\partial r} r\right)\int_0^1 d\alpha \; e^{\ln\alpha\left(\gamma - \frac{\partial}{\partial r} r\right)} = 1$ we have an ordinary differential equation for $\phi_0(r)$

## Fixed point solution

- ✓ the linear first order differential equation for the fixed point:

$$\left[ r\frac{\partial}{\partial r} - \gamma + \mu\tau\nu \right]\phi_0(r) = -\frac{(1+\gamma)\mu\tau_0}{r}\, j_0$$

- ✓ here the negative and positive times are combined by the step function:

$$\tau(t) = \begin{cases} \tau_0, & t>0 \\ -\tau_0, & t<0 \end{cases}$$

- ✓ the fixed point solution :

$$\phi_0(r) = -(1+\gamma)\,\mu\tau\, j_0\, e^{-\nu\tau}\, r^{\gamma}\int_{r_0}^{r} e^{\nu\tau} r^{-\gamma}\,\frac{d\,r}{r^2}$$

- ✓ it may be rewritten to the form with standard integral

# Fixed point solution (continued)

- fixed point solution with standard integral :

$$\phi_0(r) = -\frac{(1+\gamma)\, j_0\, \tau}{r} v^{-a} e^{-\tau v} \int_{v(r_0)}^{v(r)} x^{a-1} e^{\tau x}\, dx; \quad a = -\frac{1+\gamma}{\mu}; \quad v(r) = v_0\, r^{\mu};$$

standard integral; may be expressed by confluent hypergeometric functions* $\Phi(1,1+a;x)$ and $\Psi\ 1,1+a;x$

- using inverse self-similar transformations we get intermediate self-similar solutions

- for integer $a = n,\ n = 1, 2, 3, \ldots$ these special functions can be expressed by elementary functions

- how to find $\mu$ in order to get integer $a$ ? With the help of physical assumption.

*Bateman, Higher Transcendental Functions, Mcgraw-Hill Book Company Inc., 1953

✓ the case of freely decaying isotropic turbulence

# Considerations on dimensional grounds

- dimensions of the distribution function and the flux :

$$[\ j\ ] = \frac{\mathrm{v}^2}{r} \cdot \mathrm{v} = \frac{r^2}{t^3}$$

- $j_0$ is the main parameter $\Rightarrow$

$$\nu_0 = |j_0|^y \quad \Rightarrow \quad \left[\ \frac{r^{-\mu}}{t}\ \right] = \left[\ \frac{r^2}{t^3}\ \right]^y$$

- whence follows :

$$\boxed{\begin{array}{c} y = \frac{1}{3}; \quad \mu = -\frac{2}{3}; \quad a = -\frac{1+\gamma}{\mu} = \frac{3}{2}(1+\gamma); \\ \\ \text{for } a \text{ integer:} \quad \gamma = 1 \Rightarrow a = 3 \end{array}}$$

# Decaying Turbulence: solutions expressed by elementary functions

- using usual notions : $\varepsilon_0 = |j_0|$ , $\varepsilon(t) = -j(r=0, t>0) \quad \Rightarrow \quad \nu(r) = \nu_0 r^{\mu} = \varepsilon_0^{\frac{1}{3}} r^{-\frac{2}{3}}$

$$\boxed{\begin{aligned} f(r,t) &= \frac{2\varepsilon(t)}{r\nu}\left(1-\frac{2}{\nu t}+\frac{2}{(\nu t)^2}-\frac{2e^{-\nu t}}{(\nu t)^2}\right) \\ j(r,t) &= -\,\varepsilon(t)\left[1-\frac{3}{\nu t}\left(1-\frac{2}{\nu t}+\frac{2}{(\nu t)^2}-\frac{2e^{-\nu t}}{(\nu t)^2}\right)\right], \quad j<0 \\ \varepsilon(t) &= \frac{\varepsilon_0\,\tau_0}{t}, \qquad -\infty<t<\infty \end{aligned}}$$

- Under the time translation $t \to t+\tau_0$ and setting $\tau_0 = \infty$ :

exact stationary solution $\Rightarrow$ $\boxed{f(r) = 2\varepsilon_0^{2/3} r^{-1/3}, \;\; j = -\varepsilon_0 < 0}$

## Comparison with K-O, 41

✓ the definition of the of second-order velocity structure function and K-O 41: ,

$$D_{\parallel}(r)=\left\langle \left[\mathbf{v}_{\parallel}(\mathbf{x}+\mathbf{r})-\mathbf{v}_{\parallel}(\mathbf{x})\right]^{2} \right\rangle_{\mathbf{x}}=C_2\,\varepsilon_0^{\frac{2}{3}}r^{\frac{2}{3}}$$

$$E(r)=\frac{3}{2}\frac{\partial}{\partial r}\left\langle \left[\mathbf{v}_{\parallel}(\mathbf{x}+\mathbf{r})-\mathbf{v}_{\parallel}(\mathbf{x})\right]^{2} \right\rangle_{\mathbf{x}}=\frac{3}{2}\frac{\partial}{\partial r}C_2\,\varepsilon_0^{\frac{2}{3}}r^{\frac{2}{3}}=C_2\,\varepsilon_0^{\frac{2}{3}}r^{-\frac{1}{3}}$$

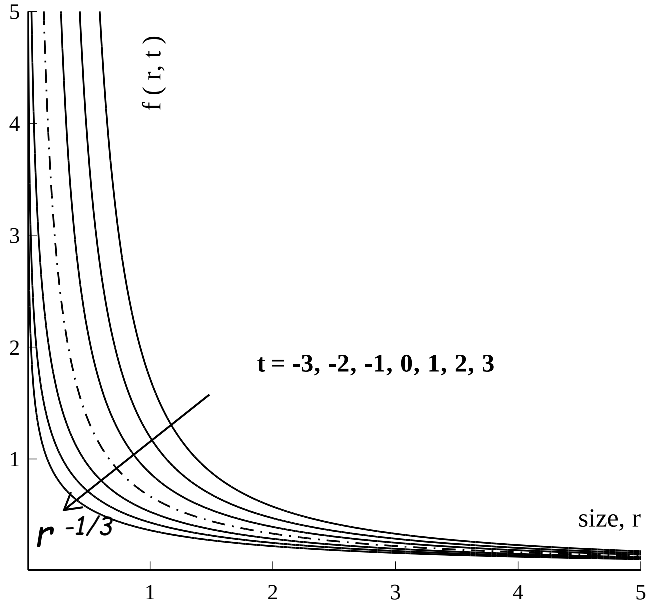
$$\int_0^{\infty}E(r)\,dr=3\left\langle \mathbf{v}_{\parallel}^{2}\right\rangle=\left\langle \mathbf{v}^{2}\right\rangle,\quad \mathbf{v}_{\parallel}=\frac{\mathbf{r}}{r}\mathbf{r}\cdot\mathbf{v}$$

✓ our long-time limit exact solution: $f(r)=2\,\varepsilon_0^{2/3}\,r^{-1/3}$

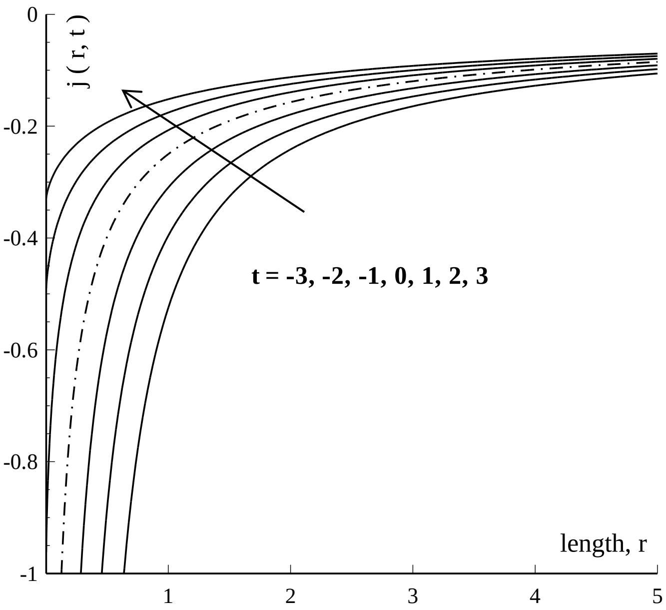
$C_2=2$ ! prediction includes also the empirical coefficient !

# Energy density distribution and its flux: linear-linear plots of evolution

energy distribution with time



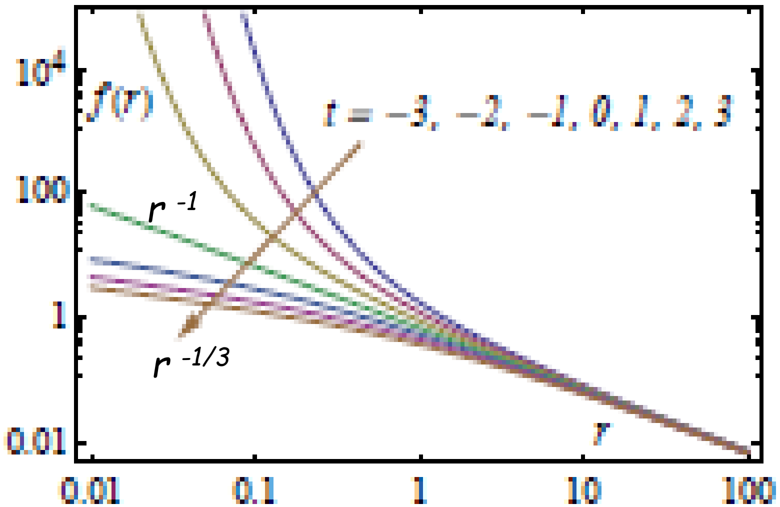

energy-flux distribution with time



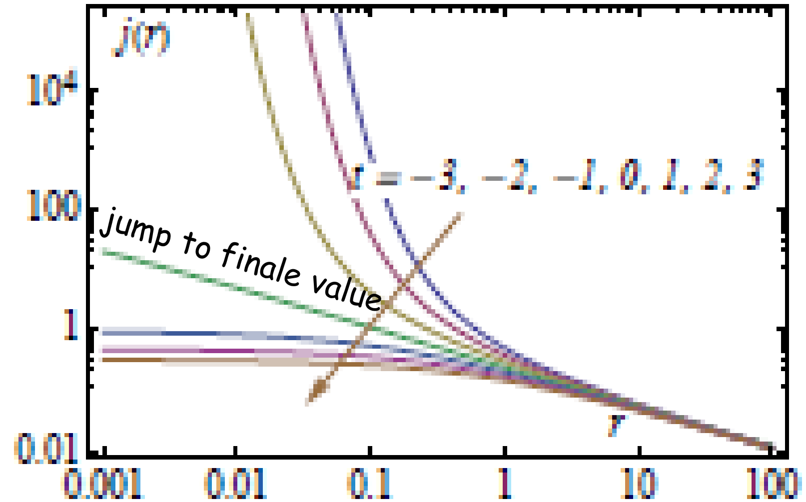

- ✓ *f(r, t)* evolves continuously with time on $-\infty < t < \infty$ towards K-O, 41
- ✓ *j(r, t)* evolves to equilibrium but will attain it only on very large times

## Energy density distribution and its flux: log-log plots of evolution

energy distribution with time

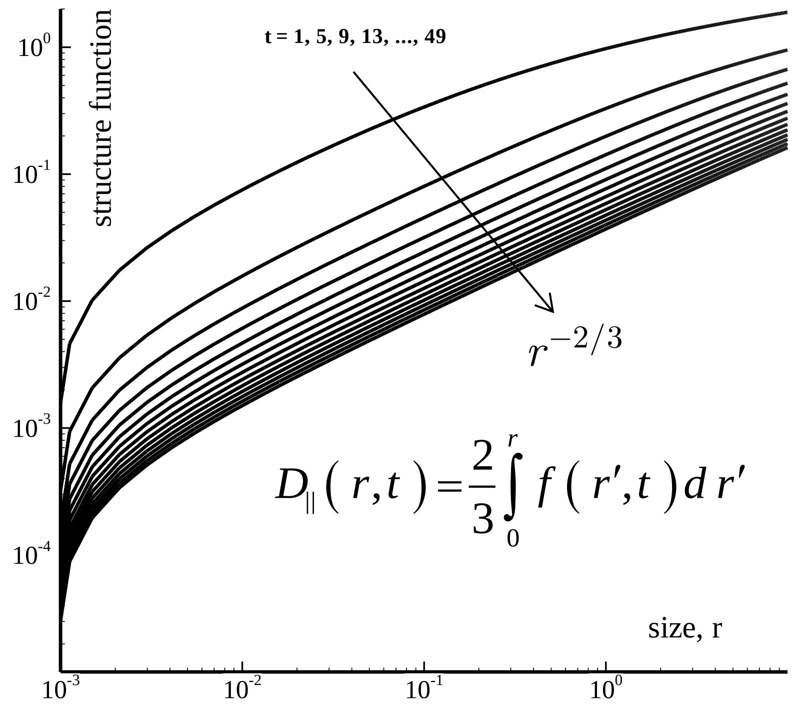



energy-flux distribution with time



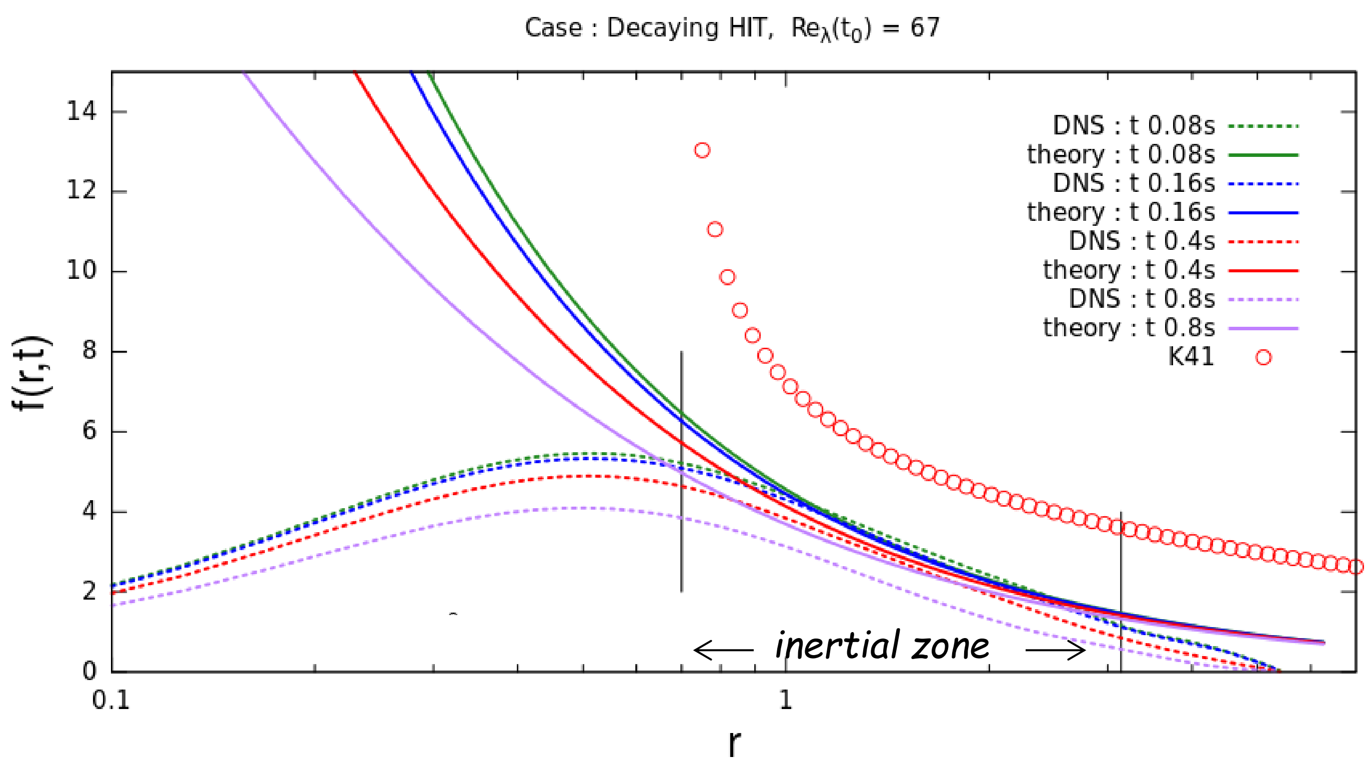

- ✓ **inset of turbulent cascade on $t < 0$ :** the most energy comes dominantly from small scales:
  with $r \downarrow$, $f(r) \uparrow$ exponentially; $r = 0$ : the energy flux is $\infty$
- ✓ **on $t = 0$:** "the second order phase transition", $f(r, t=0) \sim r^{-1}$,
  the energy-flux "jumps" discontinuously from infinity to the finite value
- ✓ **on $t > 0$,** the distributions approaches to the long-time limit distribution (K-O,41)

# The velocity structure function: log-log plot of evolution




✓ the velocity structure function evolves continuously towards “minus two-third” law

# Comparison with DNS* of decaying turbulence: energy density

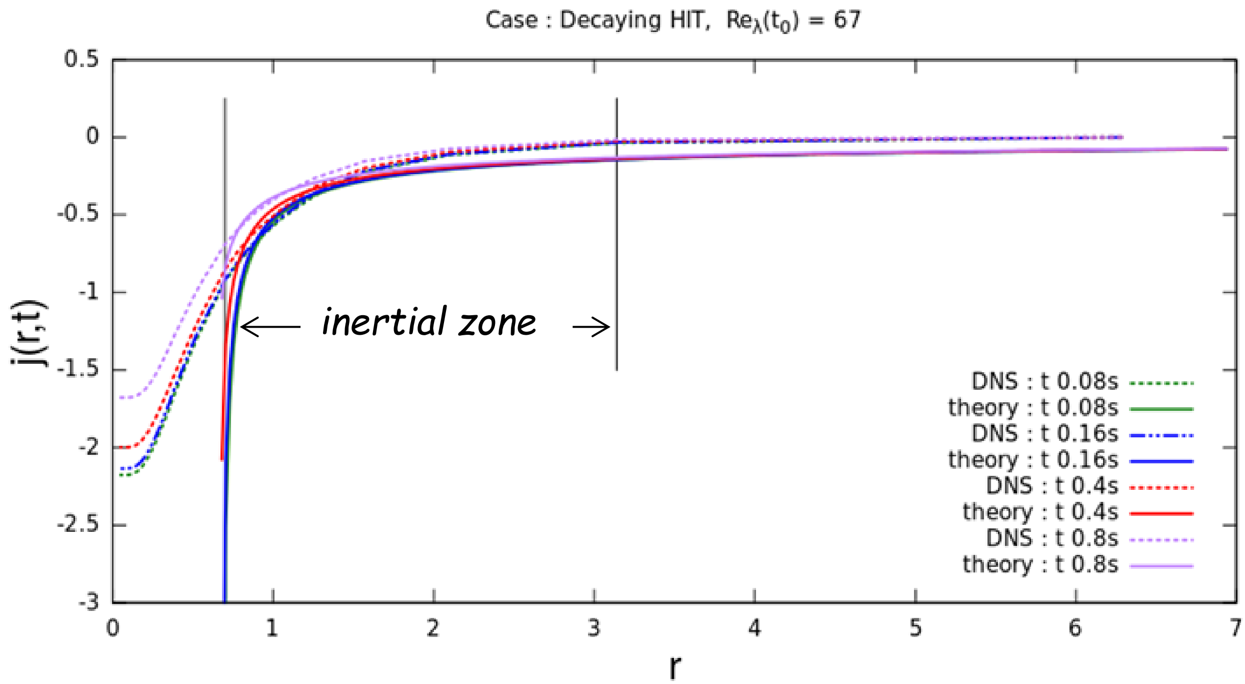



*solution:* $$f\left(r,t\right)=\frac{2\varepsilon\left(t\right)}{r\nu}\left(1-\frac{2}{\nu t}+\frac{2}{\left(\nu t\right)^{2}}-\frac{2e^{-\nu t}}{\left(\nu t\right)^{2}}\right),\quad \nu=\varepsilon_0^{\frac{1}{3}}r^{-\frac{2}{3}}$$

*DNS:* $$f\left(r,t\right)=\frac{3}{2}\frac{\partial}{\partial r}\left\langle\left[\mathbf{v}_{\parallel}\left(\mathbf{x}+\mathbf{r}\right)-\mathbf{v}_{\parallel}\left(\mathbf{x}\right)\right]^{2}\right\rangle_{\mathbf{x}}$$

*Alexis Barge, LMFA

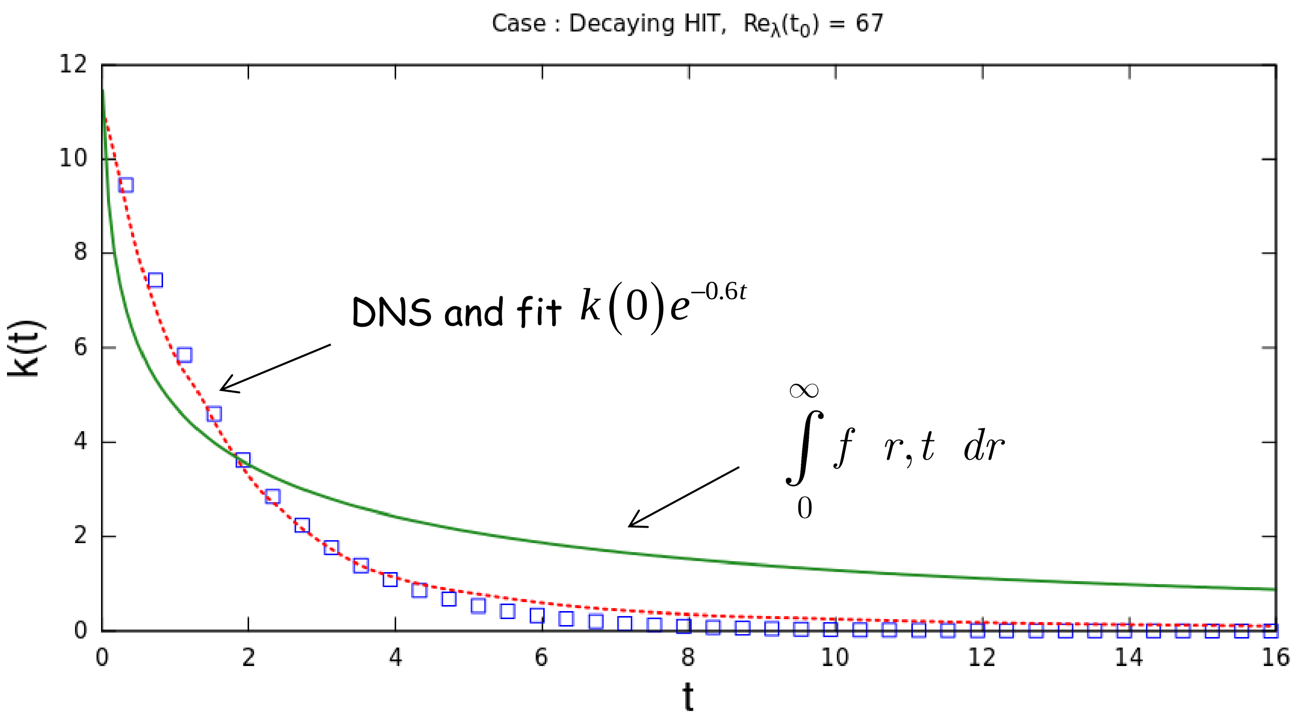
## Comparison with DNS of decaying turbulence: energy flux




*solution:* $$j(r,t) = -\varepsilon(t)\left[1-\frac{3}{\nu t}\left(1-\frac{2}{\nu t}+\frac{2}{(\nu t)^2}-\frac{2e^{-\nu t}}{(\nu t)^2}\right)\right],\quad j<0$$

*DNS: the energy flux from the energy balance in the Fourier space*

## Comparison with DNS* of decaying turbulence: decay rate



# Conclusion

**Main steps:**

- The ideas of renormalization $\rightarrow$ into phenomenological model of turbulent cascade
- The continuity equation for the specific energy in space of turbulent scales
- The exact auto-similar solutions in the case of decaying turbulence

  for the energy distribution and the energy-flux across turbulent length-scales
- Appearance of the integral of motion during the evolution towards fixed point solution

**Surprising scenario:**

✓ **on $t < 0$ :** the most energy comes dominantly from small scales: with $r \downarrow$ , $f(r) \uparrow$ exponentially;

$r = 0$ : the energy flux is $\infty$

✓ **on $t = 0$:** "the second order phase transition", $f(r, t{=}0) \sim r^{-1}$,

the energy-flux "jumps" discontinuously from infinity to the finite value

✓ **on $t > 0$,** the distributions approaches to the long-time limit distribution

✓ long-time limit distribution of the energy density predicts K-O 41 with precision up to the constant introduced usually by measurements