Eulerian and Lagrangian statistics in fully developed turbulence

Relations and Models

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Outline

Introduction

- ▷ Some basics about the statistical description of turbulence

Relations (something old)

- ▷ The Relation between Eulerian and Lagrangian Observables
- ▷ Case study: Relating increment statistics

Models (something new)

- ▷ Stochastic processes, smoothness and turbulence
- ▷ Case study: Eulerian intermittency

Introduction

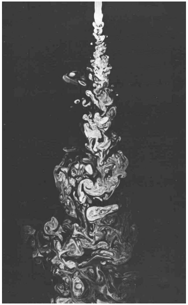
Introduction

Turbulence – Mathematical description

$$\dot{\boldsymbol{v}} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\nabla p + \frac{1}{\mathsf{Re}} \Delta \boldsymbol{v}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

Lagrangian description $\dot{\boldsymbol{X}}(t, \boldsymbol{y}) = \boldsymbol{v}(\boldsymbol{x}, t)$



[van Dyke]

During the talk incompressible turbulence is considered

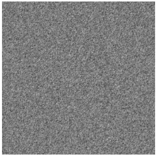
Introduction

Turbulence and the others ...

ordered

turbulence

random



Need for a statistical description, but . . .

Introduction

Sample from an experiment

$$v_1(x_1), v_2(x_2), \ldots, v_N(x_N)$$

Joint PDF

$$p(v_1(x_1), v_2(x_2), \ldots, v_N(x_N))$$

- ▷ Statistical independence $p(x, y) = p(x)p(y)$
- ▷ Conditional PDF $p(x|y) = \frac{p(x,y)}{p(y)}$
- ▷ Marginalization $p(x) = \int dx p(x, y)$
- ▷ Definition as δ-function $p(x) = \langle \delta(x - \tilde{x}) \rangle$
- ▷ PDFs can be estimated from data

Introduction

In principle one has to deal with

$$p(v_1(x_1), v_2(x_2), \ldots, v_N(x_N))$$

Idea to characterize complexity $\rightarrow u(r) = u(x+r,t) - u(x,t)$

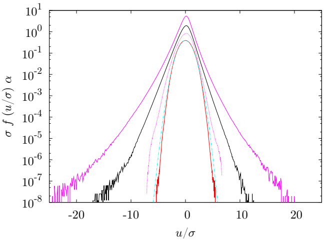
In HIT

$$f(v(r))$$

Moments of these PDFs are the structure functions

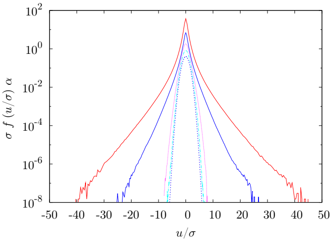
Introduction

Eulerian intermittency



$$u(\boldsymbol{r}, \boldsymbol{x}, t) = [\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}, t) - \boldsymbol{v}(\boldsymbol{x}, t)] \cdot \boldsymbol{e}_r$$

Lagrangian intermittency



$$u(\tau, t) = v(t+\tau, t) - v(t)$$

Introduction

Questions

- What is the relation between increments in Euler and Lagrange?
- How can we model the increment statistics from the viewpoint of stochastic processes?

Relations

Relations

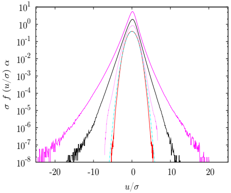
What is the relation between statistical quantities in the Eulerian and the Lagrangian frame?

Together with R. Friedrich, H.Homann, R. Grauer

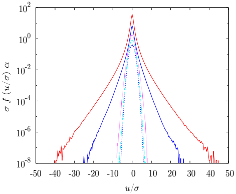
Relations

The Puzzle

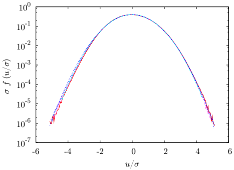
3D Euler



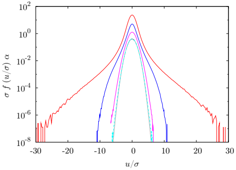
3D Lagrange



2D Euler



2D Lagrange



Realtions

Known approaches to translate from Euler to Lagrange are not compatible with the puzzle

Rely on $v_r \sim v_l$ and $r \sim v_e$

- M. S. Borgas *Phil. Trans.*, 342 (1993)
- L. Chevillard,S. G.Roux, E. Leveque, N. Mordant, J.-F. Pinton and A. Arneodo *Phys. Rev. Lett.*, 91, 214502 (2003)
- L. Biferale, G. Boffetta, A. Celani, B. J. Devenish, A. Lanotte and F. Toschi *Phys. Rev. Lett.*, 93, 064502 (2004)

Relations

The general view

N grid points/tracers, M time steps, $\tilde{\boldsymbol{x}}$ Lagrangian positions (depend on /labeled by their initial positions)

Euler

$f_E^{N\times M}(\boldsymbol{v}_1^1, \ldots, \boldsymbol{v}_N^M | \boldsymbol{x}_1^1, \ldots, \boldsymbol{x}_N^M, t^1, \ldots, t^M)$

field $f_E^{N\times 1}$

increment $f_E^{2\times 1}$

Lagrange

$f_L^{N\times M}(\boldsymbol{v}_1^1, \ldots, \boldsymbol{v}_N^M, \boldsymbol{x}_1^1, \ldots, \boldsymbol{x}_N^M | t^1, \ldots, t^M)$

trajectory $f_L^{1\times M}$

increment $f_L^{1\times 2}$

What kind of information has to be added to go from $f_E^{2\times 1}$ to $f_L^{1\times 2}$?

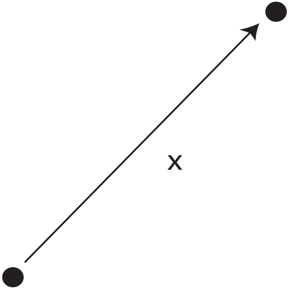
Remark: for HIT $f_E^{1\times 1}$ and $f_L^{1\times 1}$ are equal!

Example: Acceleration PDFs $f_E^{1\times 1}(a)$ and $f_L^{1\times 1}(a)$ are equal!

Relations

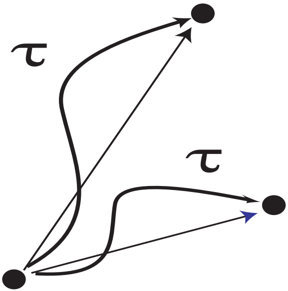
Case study: Relating increment statistics

Relations



$$u_e = v(\boldsymbol{y} + \boldsymbol{x}, t) - v(\boldsymbol{y}, t)$$

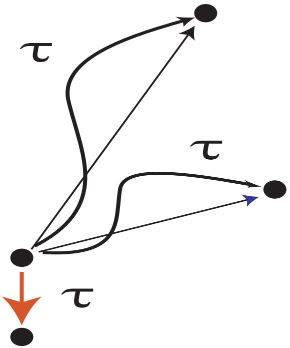
Relations



$$u_e = v(\boldsymbol{y} + \boldsymbol{x}, t) - v(\boldsymbol{y}, t)$$

$$u_{el} = v(\boldsymbol{y} + \tilde{\boldsymbol{x}}(\boldsymbol{y}, \tau, t), t) - v(\boldsymbol{y}, t)$$

Relations

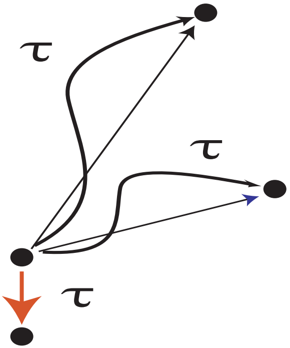


$$u_e = v(\boldsymbol{y} + \boldsymbol{x}, t) - v(\boldsymbol{y}, t)$$

$$u_{el} = v(\boldsymbol{y} + \tilde{\boldsymbol{x}}(\boldsymbol{y}, \tau, t), t) - v(\boldsymbol{y}, t)$$

$$u_p = v(\boldsymbol{y}, t) - v(\boldsymbol{y}, t - \tau)$$

Relations



$$u_e = v(\boldsymbol{y} + \boldsymbol{x}, t) - v(\boldsymbol{y}, t)$$

$$u_{el} = v(\boldsymbol{y} + \tilde{\boldsymbol{x}}(\boldsymbol{y}, \tau, t), t) - v(\boldsymbol{y}, t)$$

$$u_p = v(\boldsymbol{y}, t) - v(\boldsymbol{y}, t - \tau)$$

$$\begin{aligned} u_l &= v(\boldsymbol{y} + \tilde{\boldsymbol{x}}(\boldsymbol{y}, \tau, t), t) - v(\boldsymbol{y}, t - \tau) \\ &= u_p + u_{el} \end{aligned}$$

Relations

Eulerian increment:

$$u_e = v(\boldsymbol{y} + \boldsymbol{x}, t) - v(\boldsymbol{y}, t),$$

Fine grained Probability density function:

$$\hat{f}_e(v_e|\boldsymbol{x}, \boldsymbol{y}, t) = \delta(u_e - v_e).$$

Probability density function:

$$f_e(v_e|\boldsymbol{x}, \boldsymbol{y}, t) = \langle \hat{f}_e(v_e|\boldsymbol{x}, \boldsymbol{y}, t) \rangle,$$

Relations

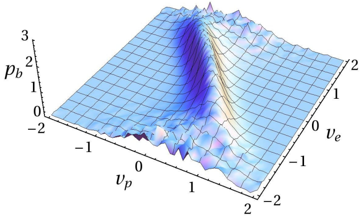
$$f_l(v_l|\tau) = \int dv_e p_b(\underbrace{v_l - v_e}_{v_p}|v_e,\tau) \underbrace{\int_0^\infty dr\ p_a(r|v_e,\tau)\ f_e(v_e|r)}_{f_{el}(v_e|\tau)}$$

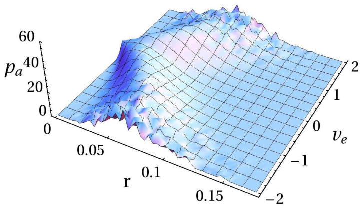
- assumptions: isotropy, stationarity, homogeneity
- for details see: Phys. Rev. E **79**, 066301 (2009)

- $p_a \rightarrow$ transport of tracers $\Rightarrow$ mixing of length scales
- $p_b \rightarrow$ time correlation of velocity field

Relations

Transition pdfs in 2d ($\tau = 0.1 T_I$)





- ▷ Particle path depends on velocity difference
- ▷ Negative correlation between increment v_p and v_e

Relations

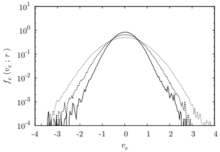
Transformation of pdfs (2D)

$$f_l(v_l|\tau) = \int dv_e p_b(\underbrace{v_l - v_e}_{v_p}|v_e, \tau) \underbrace{\int_0^\infty dr\ p_a(r|v_e, \tau)\ f_e(v_e|r)}_{f_{el}(v_e|\tau)}$$

Relations

Transformation of pdfs (2D)

$$f_l(v_l|\tau) = \int dv_e p_b(\underbrace{v_l - v_e}_{v_p}|v_e,\tau) \underbrace{\int_0^\infty dr\ p_a(r|v_e,\tau)\ f_e(v_e|r)}_{f_{el}(v_e|\tau)}$$



Relations

Transformation of pdfs (2D)

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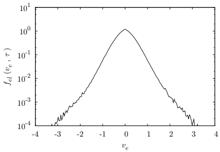

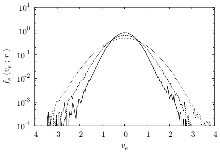

Relations

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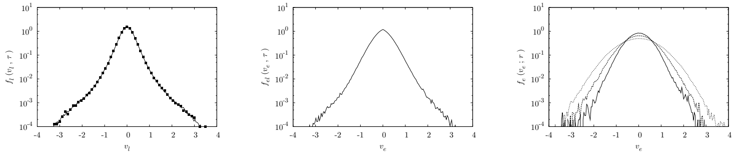
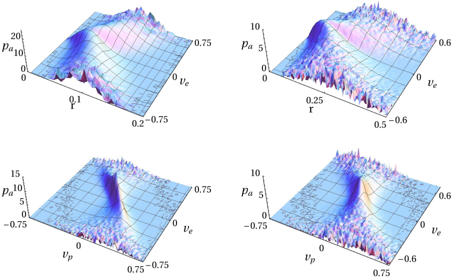


Relations

Transition PDFs in 3D NS and MHD ($\tau = 0.1T_I$)



- ▷ Similar structure as in 2D
- ▷ Differences in the details [New J. Phys. **11** 073020]

Relations

Making contact to other approaches

Approximation $r \sim v_e \tau$ and $v_l \sim v_e$ corresponds to
$p_a \sim \delta(r - v_e \tau)$ and $p_b \sim \delta(v_l - v_e)$

$$f_l(v_l; \tau) = \int dv_e \, \delta(v_l - v_e) \int_0^\infty dr \, \delta(r - v_e \tau) f_e(v_e; r) = f_e(v_l; v_l \tau)$$

Models

Models

Stochastic processes, smoothness and turbulence

Models

Time series from experiment or numerical simulation

- Time series sampled at $q(t_N), \dots, q(t_1)$ with $\tau = t_i - t_{i-1}$
- Full information is encoded in $f(q_N, \dots, q_1)$

Random process

$$f(q_N, \dots, q_1) = f(q_N) f(q_{N-1}) \times \cdots \times f(q_2) f(q_1)$$

Markov process:

$$f(q_N, \dots, q_1) = p(q_N | q_{N-1}) \times \cdots \times p(q_2 | q_1) f(q_1)$$

Important: In natural systems Markov property is fulfilled only for $\tau > t_{mar}$

Practical test: Measure distance between $p(q_N | q_{N-1}, q_{N-2})$ and $p(q_N | q_{N-1})$ for different τ

Models

Time evolution of $f(q,t)$ is given by **Kramers-Moyal** expansion

$$\frac{\partial}{\partial t} f(q,t) = \left[\sum_{n=0}^{\infty} \left(-\nabla^n\right) D^{(n)}(q,t) \right] f(q,t)$$

with **Kramers-Moyal coefficients**

$$D^{(n)}(q,t) = \frac{1}{n!} \lim_{\tau \to 0} \frac{1}{\tau} \langle [q(t+\tau) - q(t)]^n | q(t) \rangle$$

Pawula Theorem: if D^4 vanishes, all D^i with $i > 2$ vanish
Corresponds to Gaussian transition PDFs

Models

Fokker-Planck equation

$$\frac{\partial}{\partial t} f(q,t) = \left(-\frac{\partial}{\partial q} D^{(1)}(q) + \frac{\partial^2}{\partial q^2} D^{(2)}(q) \right) f(q,t)$$

Langevin equation

$$\dot{q} = D^{(1)}(q) + \sqrt{D^2(q)}\Gamma$$

Γ is Gaussian white noise

$$D^{(n)}(x,t) \approx \frac{1}{n!} \frac{1}{\tau_{min}} \langle [q(t+\tau) - q(t)]^n | q(t) \rangle$$

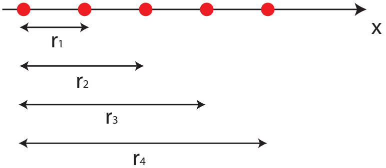
Coefficients can be estimated from experimental data!

Models

Application of Fokker-Planck approach to turbulence

[R. Friedrich and J. Peinke, PRL **78**, 863-866 (1997)]

Process in scale with increment $u(r_i) := u_i$ and $r_{i+1} - r_i = \Delta r$



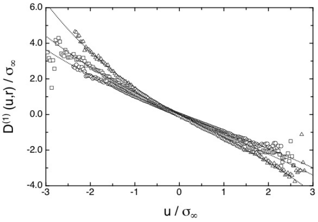
Complete statistical information: $f(u_4, u_3, u_2, u_1)$

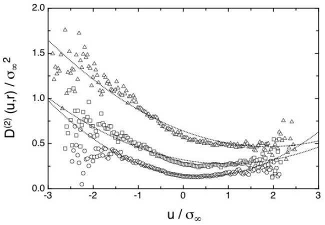
N-points: $f(u_N, u_{N-1}, \ldots, u_2, u_1)$

Markov process for $\Delta r > \lambda$

Models

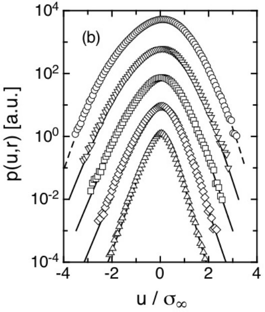
Estimates of $D^{(1)}(u, r)$ and $D^{(2)}(u, r)$

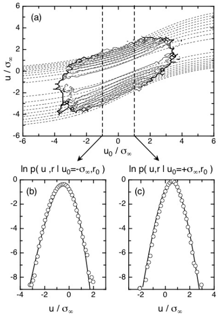




Models

Reconstruction via Fokker-Planck equation (right: $\Delta r = L/2$)





Models

Stochastic processes and smoothness

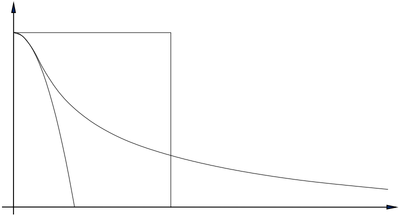
For continuously differentiable signals (like turbulence) the derivative of the correlation function is zero at the origin $\rightarrow$ Taylor length scale $C(x) \approx 1 - 0.5(x^2/\lambda^2)$

Simple stochastic process

$$\dot{q} = -\gamma q + \Gamma$$

Embedded stochastic process

$$\dot{\epsilon} = -\gamma_\epsilon \epsilon + \Gamma$$
$$\dot{q} = -\gamma q + \epsilon$$



Models

Case study: Eulerian intermittency

Models

The goal of the modeling approach

$$\dot{\boldsymbol{u}} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{1}{\mathsf{Re}} \Delta \boldsymbol{u} \quad \longrightarrow \quad u(x_1), u(x_2), \ldots u(x_n)$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

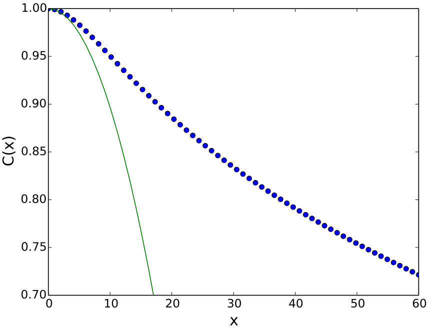
$$\partial_x u_m(x) = \text{model} \quad \longrightarrow \quad u_m(x_1), u_m(x_2), \ldots u_m(x_n)$$

$u(x_i)$ and $u_m(x_i)$ should share similar statistical features / η is base length

Models

A look into the data – Data set provided by J. Peinke

- ▷ Free jet experiment
- ▷ Re $= 2.7 \cdot 10^4$
- ▷ $\eta =$0.3mm
- ▷ λ=6.6mm
- ▷ L=67mm



Models

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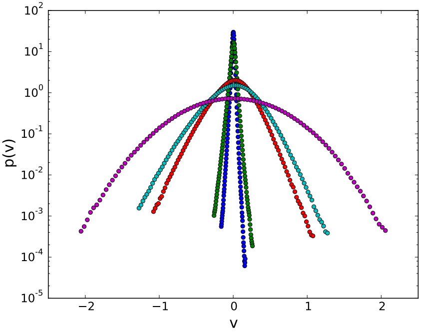


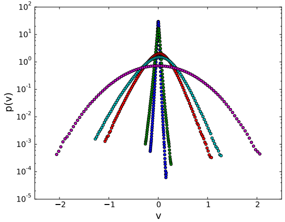
Models

The general model structure

$$\partial_x d(x) = D^{(1)}[d(x)] + \sqrt{D^{(2)}[d(x)]}\gamma$$

$$\partial_x u(x) = -\frac{1}{L}u(x) + d(x)$$

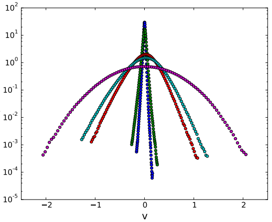
γ is Gaussian white noise



Modeling

Parametrization of $D^{(1)}$ and $D^{(2)}$

$$D^{(1)}[u(x)] = a_1^1 d(x) + a_2^1 d(x)^2 + a_3^1 d(x)^3$$
$$D^{(2)}[u(x)] = a_1^2 + a_2^2 d(x)^2$$

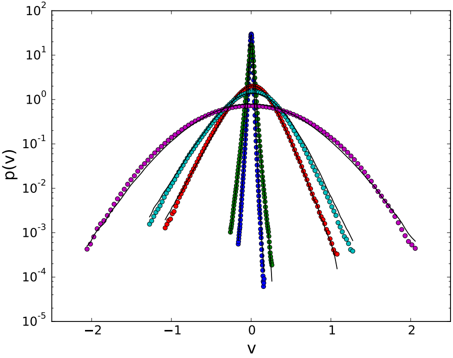


- a_1^1, a_3^1 needed for stability
- a_2^1 asymmetry (energy flux)
- a_1^2 Gaussian tip
- a_2^2 non Gaussian tails

Modeling

Parameters from minimizing distance between increments PDFs

- Lagrangian frame
- Further tests



Work in progess [1]

[1] O. Kamps, A Stochastic Model for intermittency in fully developed turbulence, in preparation

Conclusion

- If you want to translate Observables from the Eulerian to the Lagrangian frame you have to provide information encoded in transition probabilities
- If you want to make dynamical stochastic models for multipoint statistics you have to care for the smoothness of the underlying process