Inertial Wave Turbulence in Rotating Flows

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Simple “boundary conditions”, yet the flow field is “crumpled” and consists of many scales

Outline

•A “cartoon” of turbulence

•Turbulence in rotating systems

•Inertial waves

•The experimental system

•Energy spectrum evolution – long times

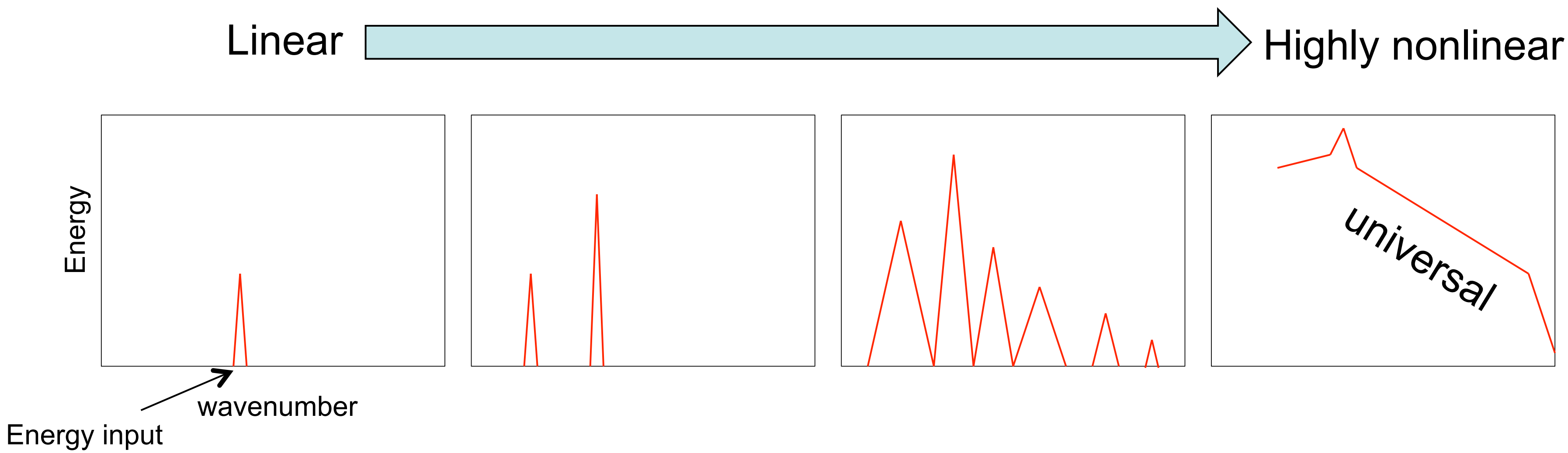
•Energy spectrum evolution – short times

•Inertial wave turbulence

Turbulence – the highly nonlinear state of a system - an illustration

Explicit solutions
of the equations of motion

Characterization of
Steady state statistics



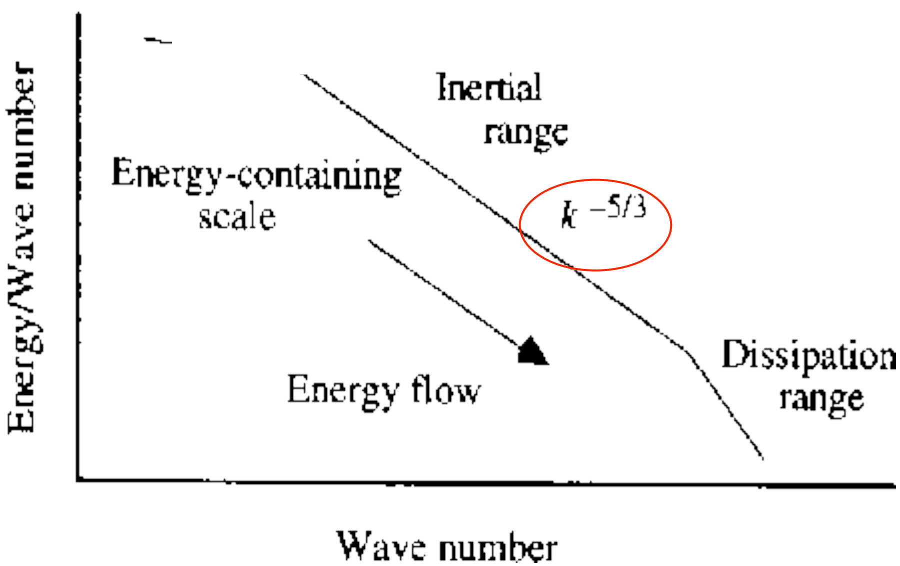
Not in a thermodynamic equilibrium

3D, 2D, Boundary layers, elastic …

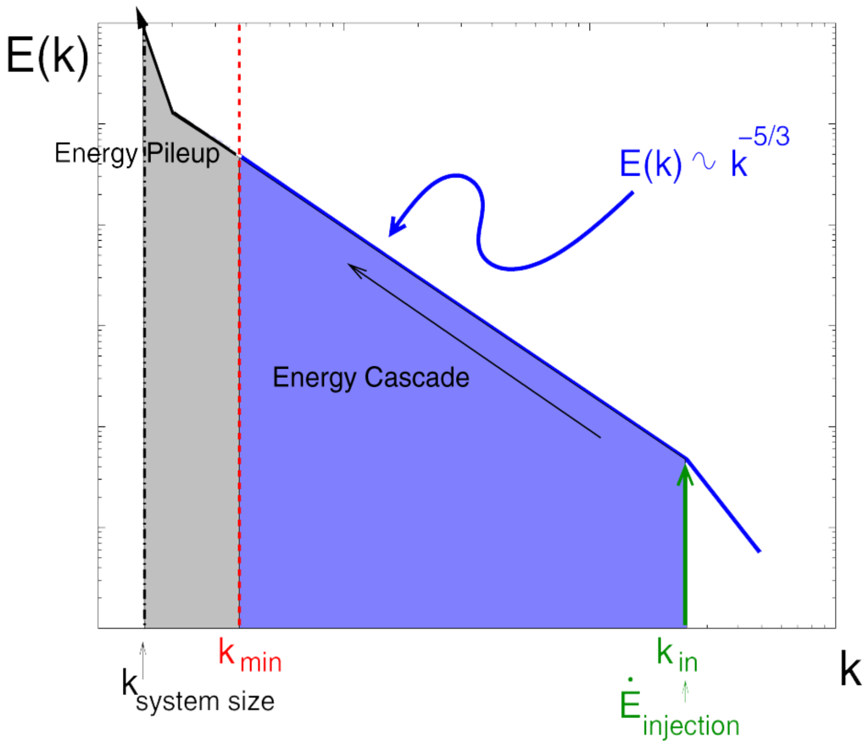
The picture of an Energy Cascade

Energy spectrum E(K)

Homogeneous 3D – forward cascade



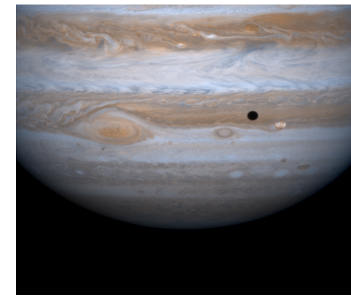
Homogeneous 2D inverse cascade



Turbulence under rotation

Atmosphere, Oceans, Flows within the Earth’s mantle

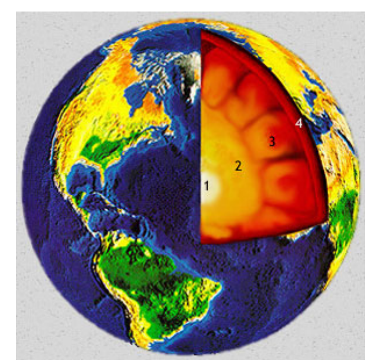
Highly turbulent flows ($Re \sim 10^9$)



Strong rotation

(Sometimes) driven by
a homogeneous small-scale
energy source

Long lived coherent structures



Equations and numbers

$$\cancel{\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}} = -\frac{\nabla p}{\rho} + \cancel{\nu \nabla^2 \vec{u}} - 2\vec{\Omega} \times \vec{u} \qquad \text{and} \qquad \nabla \cdot \vec{u} = 0$$

Inertial Viscous **Coriolis**

Taylor-Proudman Theorem
Dominance of rotation implies:

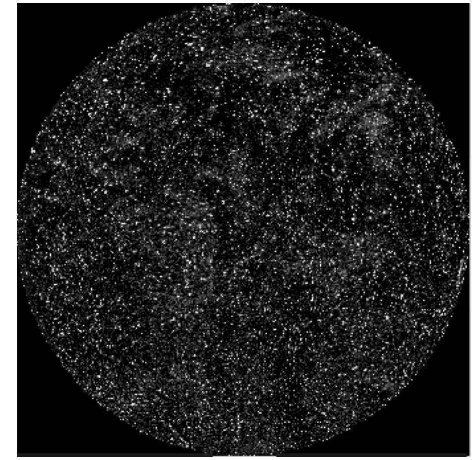
$$2\vec{\Omega} \times \vec{u} = -\frac{\nabla p}{\rho} \quad \text{(Taking the Curl)} \quad \Longrightarrow \vec{\Omega} \cdot \nabla \vec{u} = 0$$

"Quasi 2D"

In Atmospheric flows

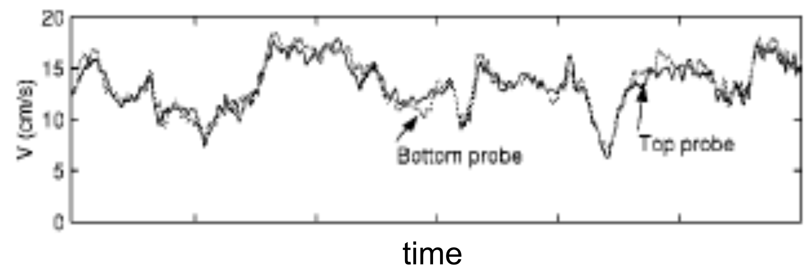
Reynolds number	$\mathrm{Re} = \frac{inertial}{viscous} \approx \frac{UL}{\nu} \sim 10^9$	turbulent
Rossby Number	$\mathrm{Ro} = \frac{inertial}{Coriolis} \approx \frac{\omega}{2\Omega} \sim 10^{-2}$	Rotation dominates
Ekman Number	$\mathrm{Ek} = \frac{viscous}{Coriolis} \approx \frac{\nu}{2\Omega L^2} \sim 10^{-6}$	invicid

What is the proper framework for the description of deep rotating turbulence?



Option 1: Using the formalism of 2D turbulence

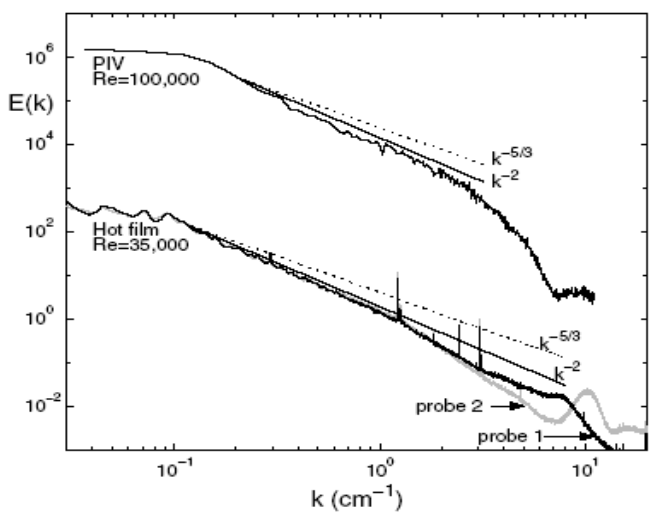
- Build up of large scales via energy cascade
(see McEwan 1970, de Verdiere 1980, Hopfinger 1982…)
- 2D in the large scales (Baroud, Plapp and Swinney, 2003)



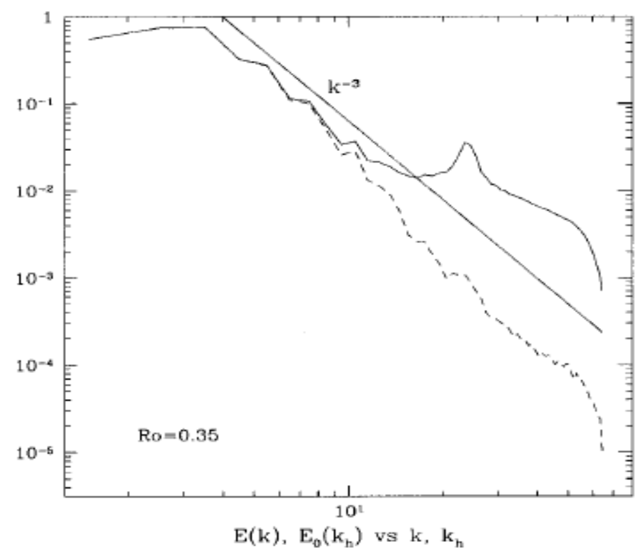
But also:

Different energy power spectra:

Baroud et. al. 2002

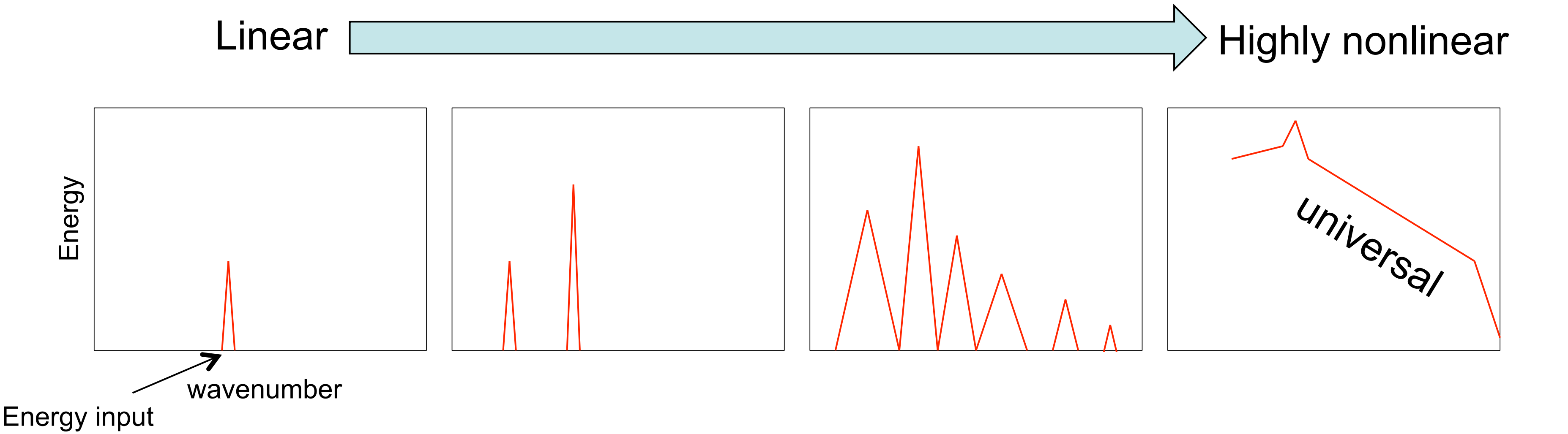


Smith and Waleffe 1999



An alternative direction: Wave Turbulence

(See: Zakharov, L'vov and Falkovich, wave Turbulence, Springer, 1992, Nazarenko 2011, Newell and Rumpf, *Ann. Rev. Flu. Mech* 2011)

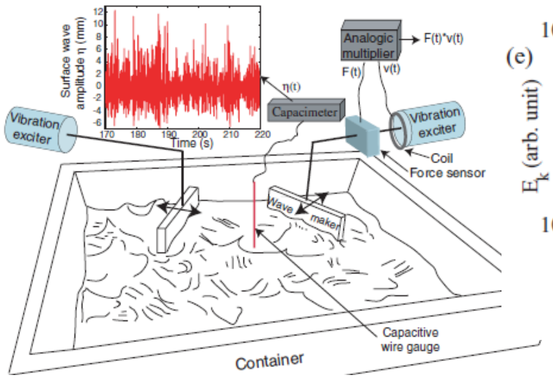


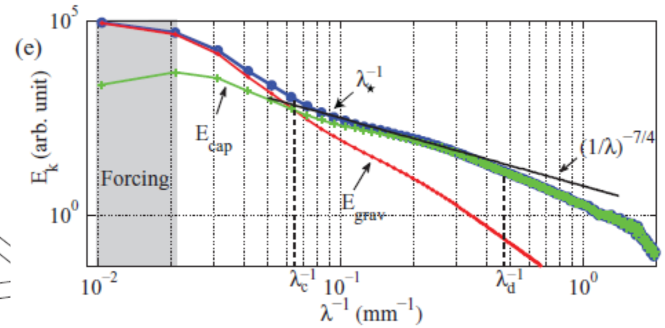
- A system that supports linear waves
- A unique dispersion relation $\omega(k)$
- Nonlinearity via resonant interactions of these waves (time dependant amplitudes)
- Possible evolution of an (out of equilibrium) ensemble of uncorrelated interacting waves with universal statistical properties (various possible closure assumptions).
- Expected to hold in “moderate” nonlinearity

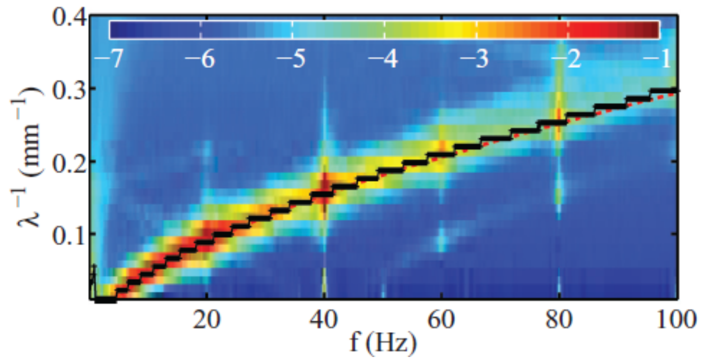
Wave turbulence in other systems – experimental observations

Capillary surface waves

E. Falcon, C. Laroche, and S. Fauve PRL **98, 094503 (2007),**
M. Berhanu and E. Falcon, PRE, **87, 033003 (2013)**

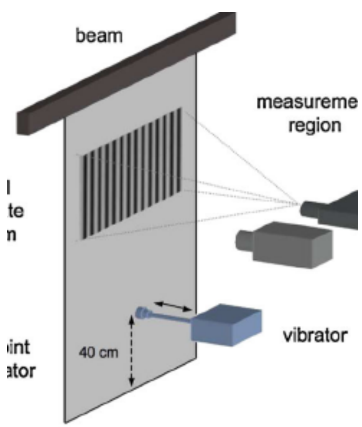


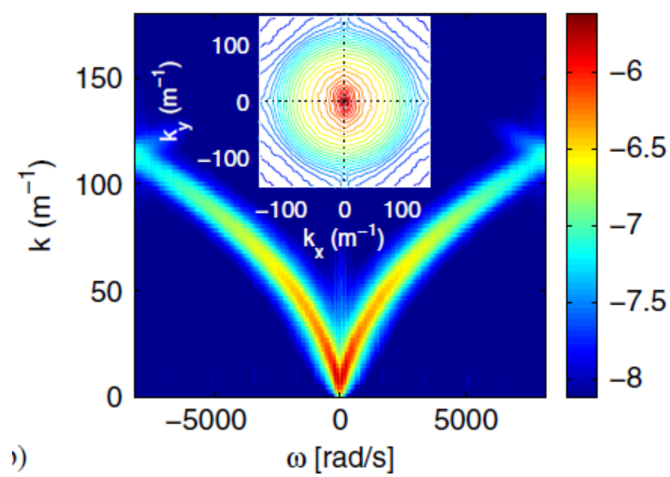




Elastic surface bending waves

P. Cobelli et al. PRL 103, 204301 (2009)





Broad spectrum, energy is concentrated along the dispersion relation

Inertial Waves in rotating flows

Navier-Stokes equation in a rotating frame:

$$\frac{\partial \vec{u}}{\partial t} + \cancel{(\vec{u}\cdot\vec{\nabla})\vec{u}} = -2\vec{\Omega}\times\vec{u} - \vec{\nabla}\left(\frac{\tilde{p}}{\rho}\right) + \cancel{\nu\nabla^2\vec{u}} \qquad , \qquad \vec{\nabla}\cdot\vec{u} = 0$$

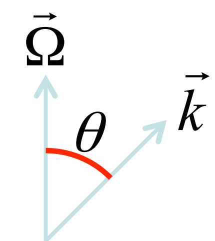
This equation supports the propagation of *inertial waves:*

$$u_{x,y,z} = a_{x,y,z}\, e^{i(\vec{k}\cdot\vec{x}-\omega t)} + c.c. \qquad\qquad \omega = \pm 2\frac{\vec{\Omega}\cdot\vec{k}}{\left|\vec{k}\right|}$$

Properties of inertial waves:

- $|\omega| < 2\Omega$
- $\omega = \omega(\theta) = \pm 2\Omega\cos(\theta)$

θ being the angle between $\vec{k}$ and $\vec{\Omega}$.



- $\vec{v}_g = \pm\frac{\vec{k}\times(2\vec{\Omega}\times\vec{k})}{k^3} \qquad \vec{v}_p = 2\frac{\vec{\Omega}\cdot\vec{k}}{\left|\vec{k}\right|^3}\vec{k} \Longrightarrow \qquad \vec{v}_p\cdot\vec{v}_g = 0$

- Right/Left helical modes

Can rotating turbulence be described as a wave turbulence of inertial waves?

If so

- A 3D description

- Based on a controlled approximation to N.S. Eq.

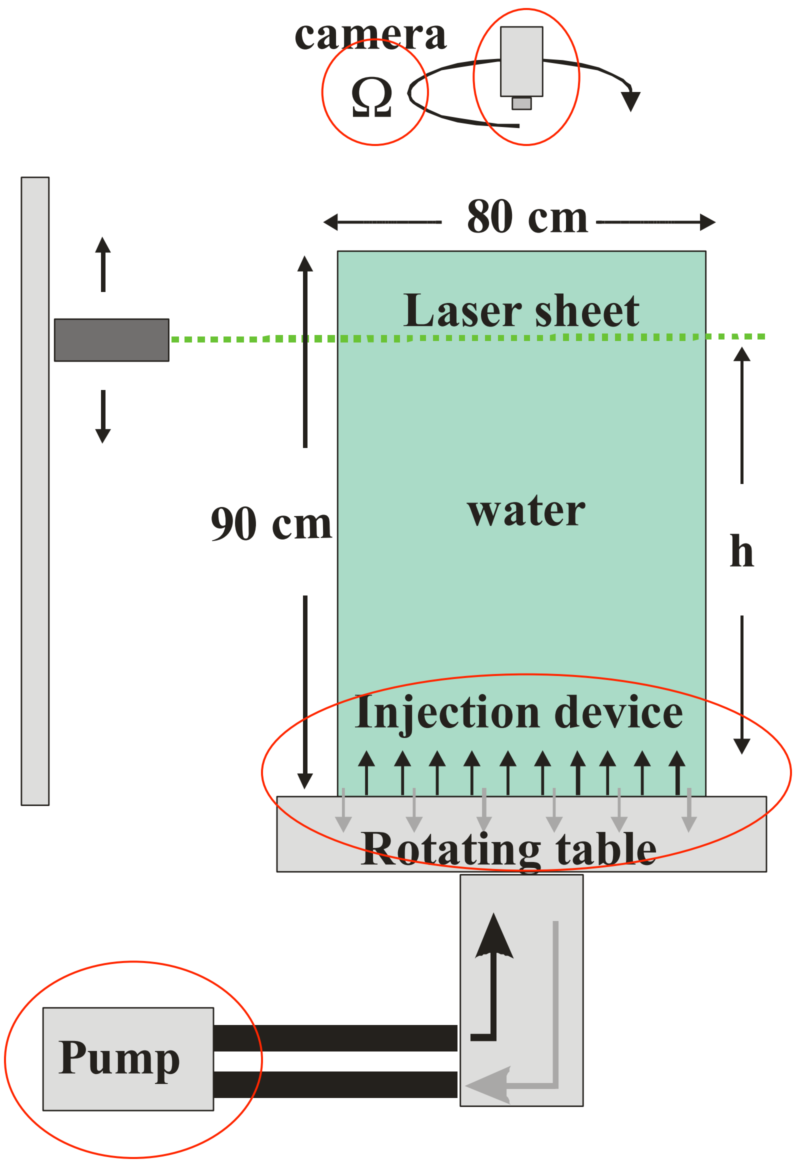
- "Solvable" – unique predictions

Various theoretical predictions and numerical results
See: : Zakharov, L'vov and Falkovich 1992, Nazarenko 2011, Newell and Rumpf 2011, Cambon and Godeferd 1996-2006, Smith and Waleffe 1999, Galtier 2003, 2014…

Observation of inertial waves in non-turbulent flows (Greenspan 1968, Moisy 2012) or during transients (Bewley 2007, Davidson 2006, Kolvin 2009)

No experimental evidence for the existence of steady inertial wave turbulence

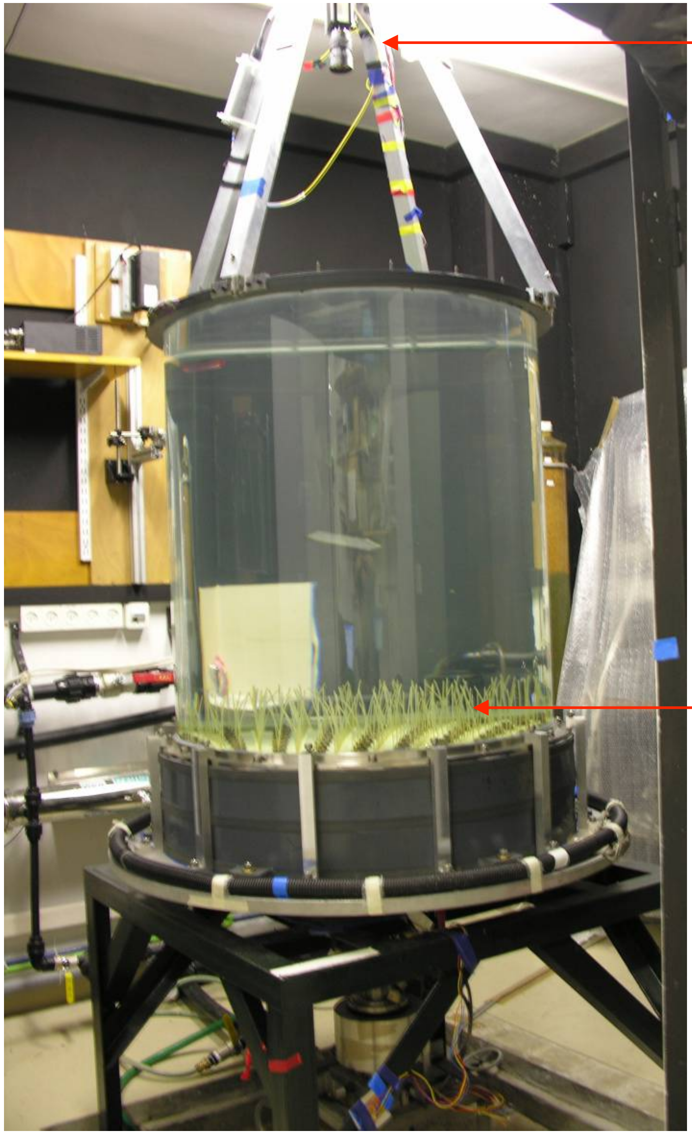
Experimental system



Ω up to 16 Rad/s

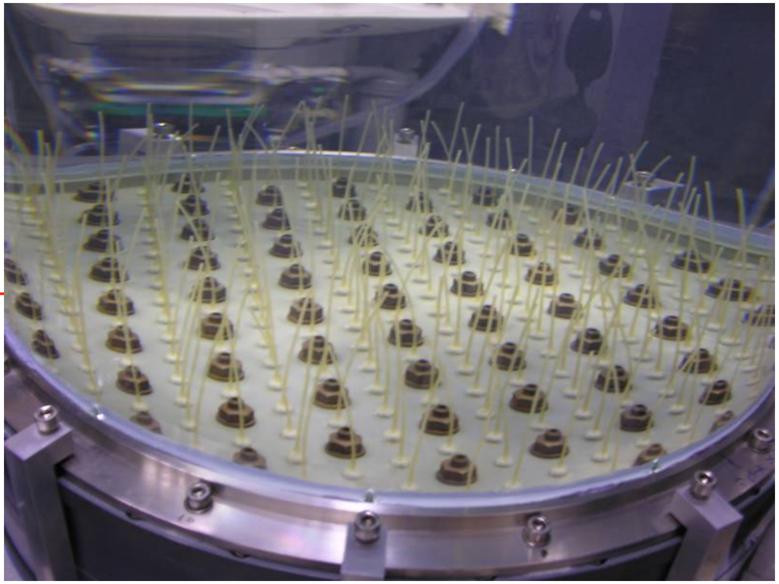
Max. flow rate 3 L/s => ~ 300 W

250 outlets and 70 inlets in hexagonal lattice

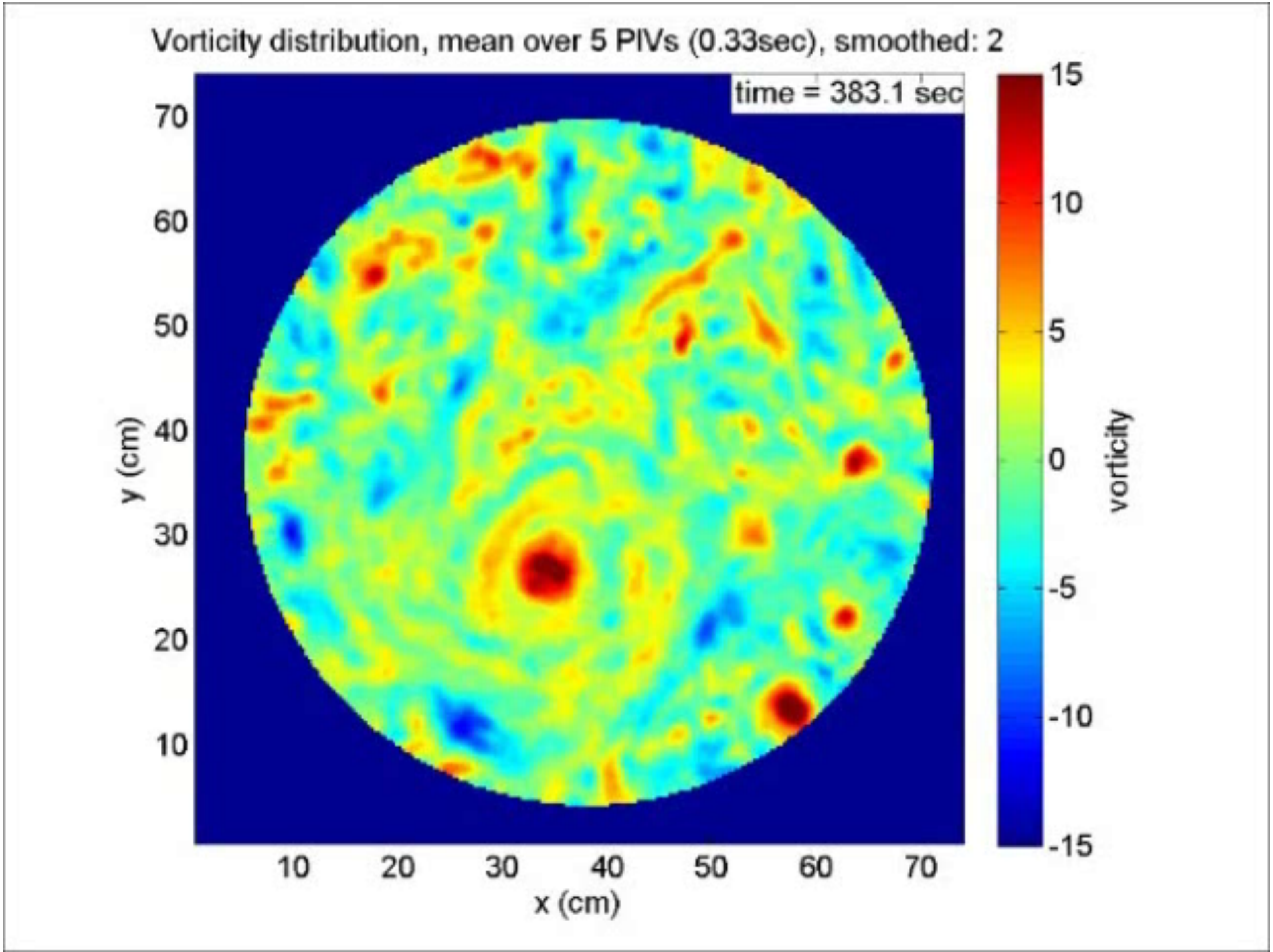


Camera

Injection Nozzles



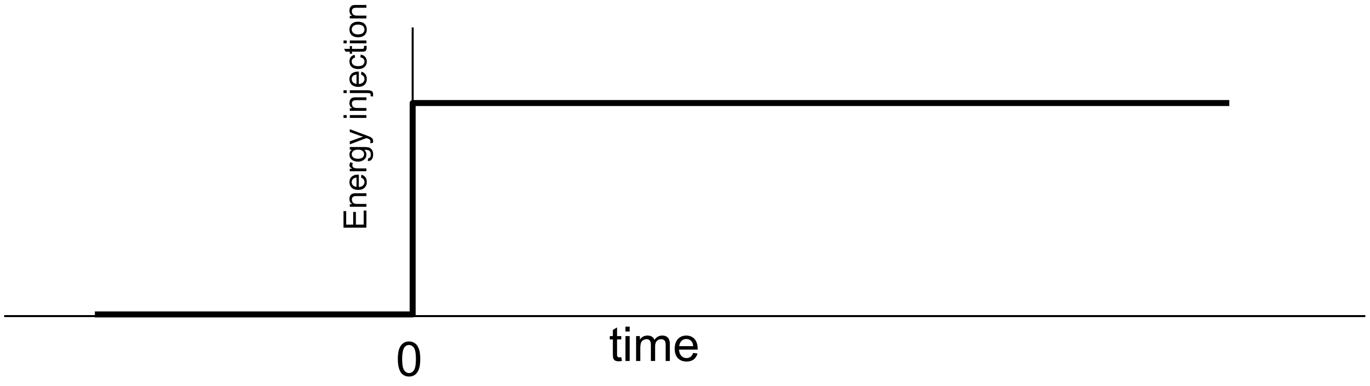
In a steady state (vorticity field)



Experiment 1: Turbulence Buildup – long times

The system is brought to a solid body rotation (u=0) at a given rotation rate Ω.

At t=0, we start injecting energy at a given flow rate (generating a step function in the injected power)

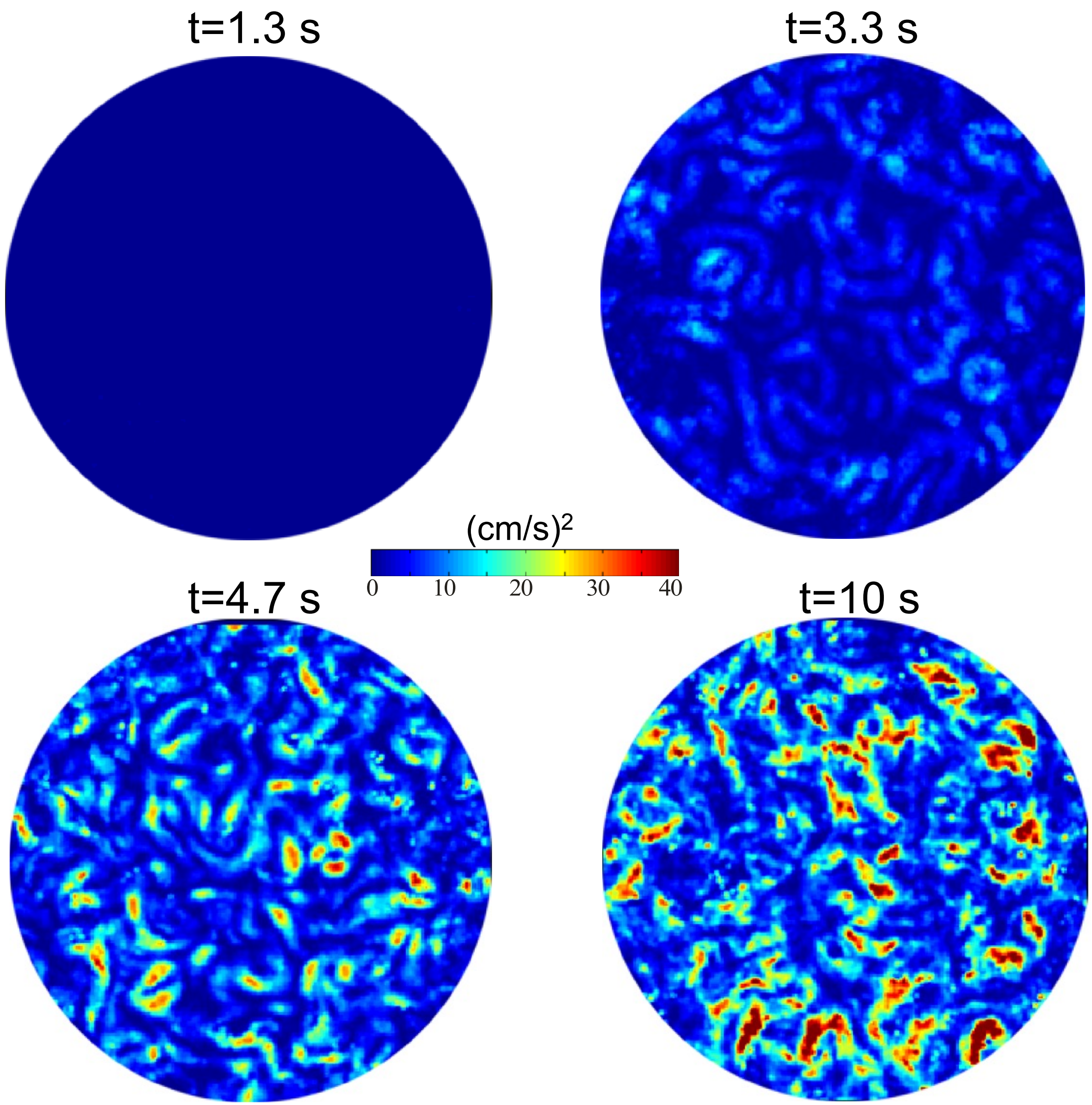


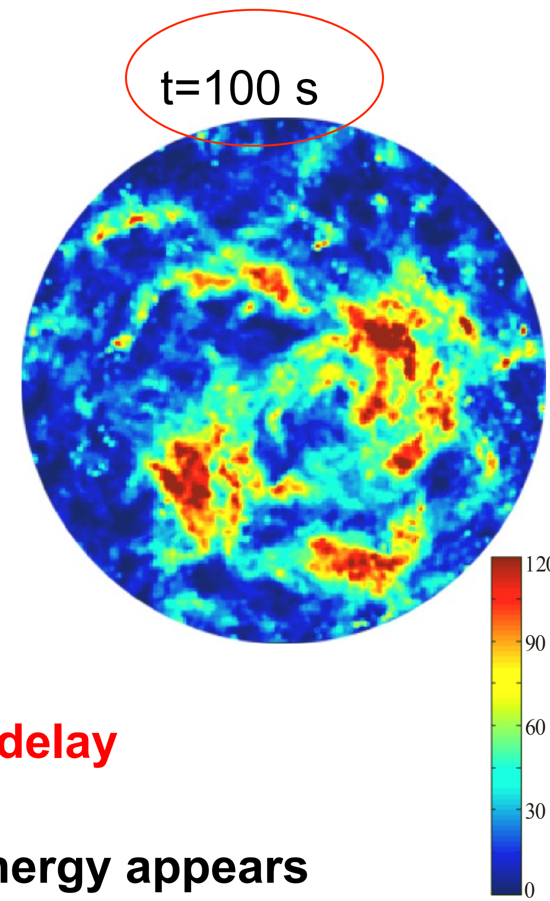
We measure the horizontal velocity (u,v) field at height H

Deriving energy power spectrum, E(k) and energy density "map", (u^2+v^2), as functions of time.

Energy Density evolution

Ω=10 Rad/S,
Re~10^3-10^4
R_o~1



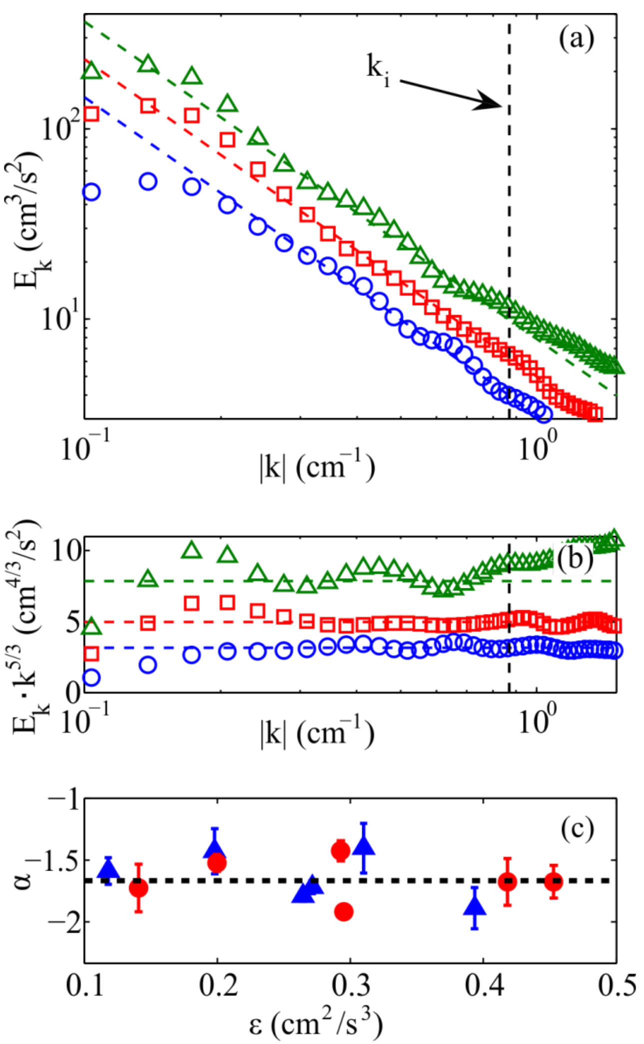


A delay

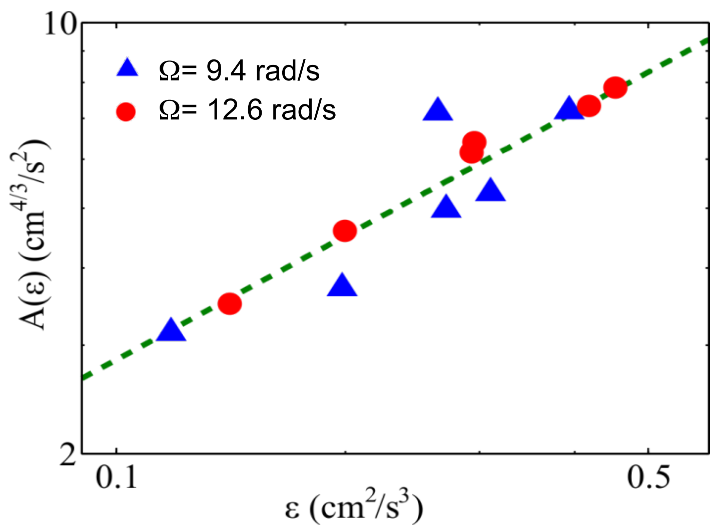
Energy appears simultaneously across the entire cross section

The steady state spectrum

$$E_{HOR}(k) = \tfrac{1}{2}\int_0^{2\pi} (|v_k|^2 + |u_k|^2) k\, d\theta$$



In 2D turbulence: $E_{2D}(k) = C\varepsilon^{2/3} k^{-5/3}$



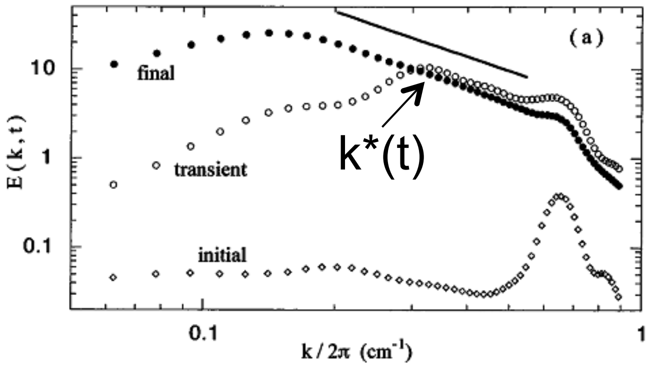
Scaling with k: $E_{HOR}(k) \sim k^{\alpha}$ with $\alpha = -1.65 \pm 0.08$

Scaling with ε: $E_{HOR}(k) = A(\varepsilon) k^{-5/3}$ with $A(\varepsilon) \sim \varepsilon^{0.7 \pm 0.2}$

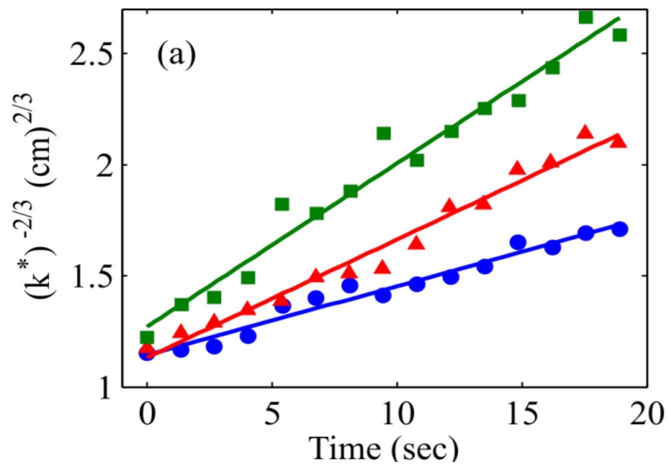
Is there a cascade of energy?

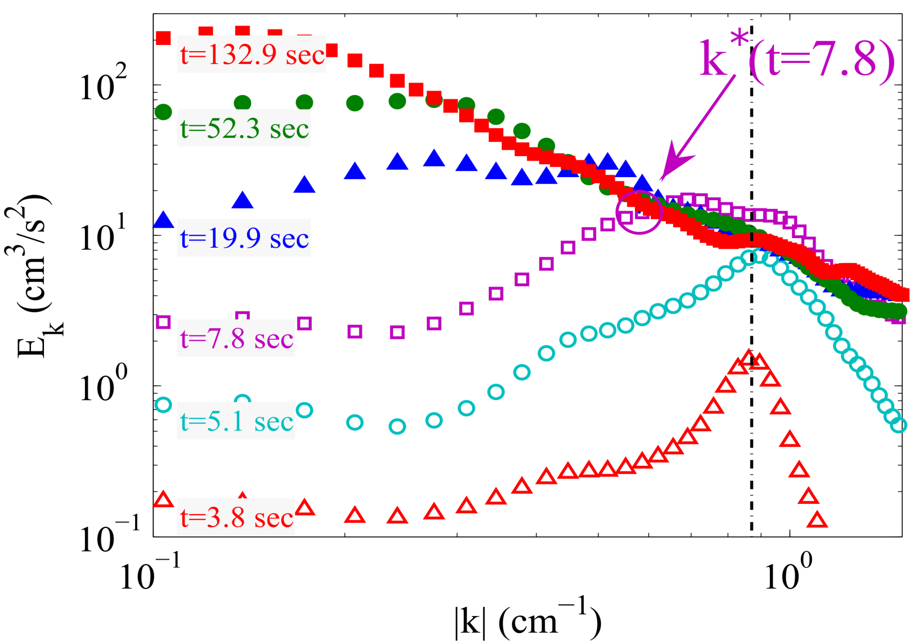
The evolving spectrum

In 2D turbulence: $k^*(t) \propto \varepsilon^{-1/2} \cdot t^{-3/2}$



PRL 79 (21), (1997) - J. Paret, P. Tabeling
See: Kraichnan 1967, Smith and Yakhot 1994



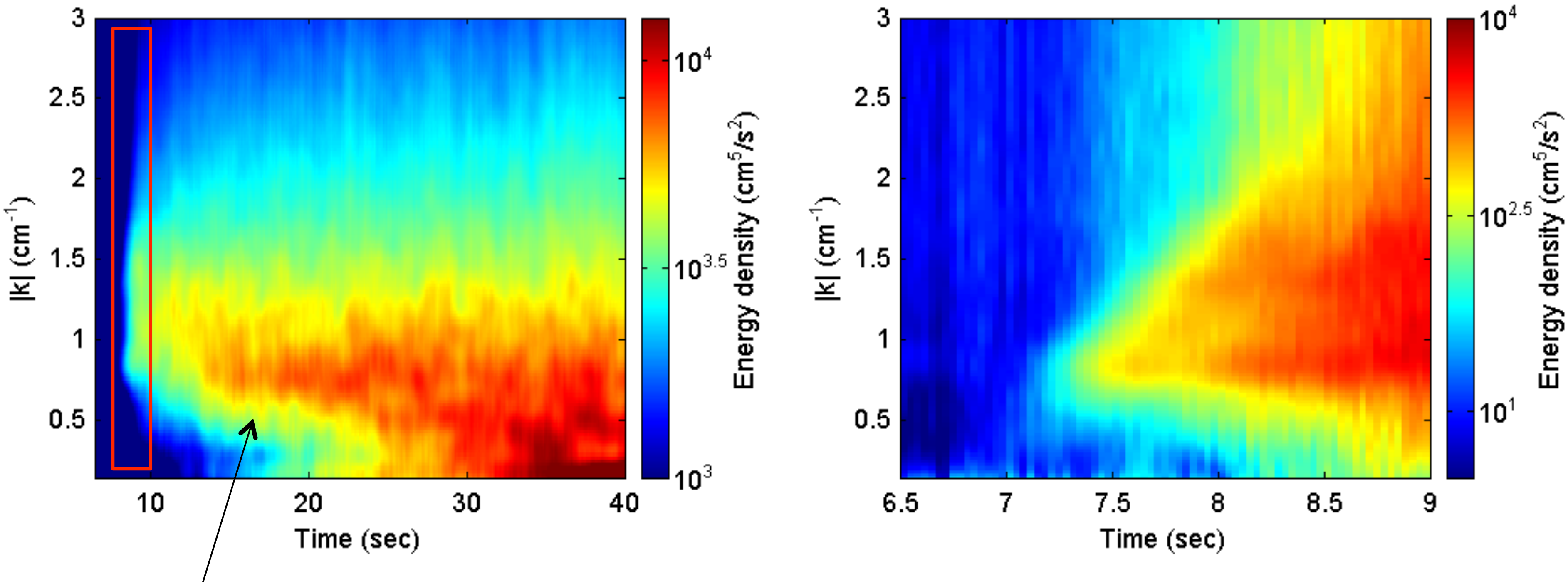


Slope: $\varepsilon^{0.3\pm0.1}$

E, Yarom , Y. Vardi and E. S (2013), *Phys. Fluids*, **25**, 085105

Short vs. long times

Spectrum evolution in time (a different representation)



cascade

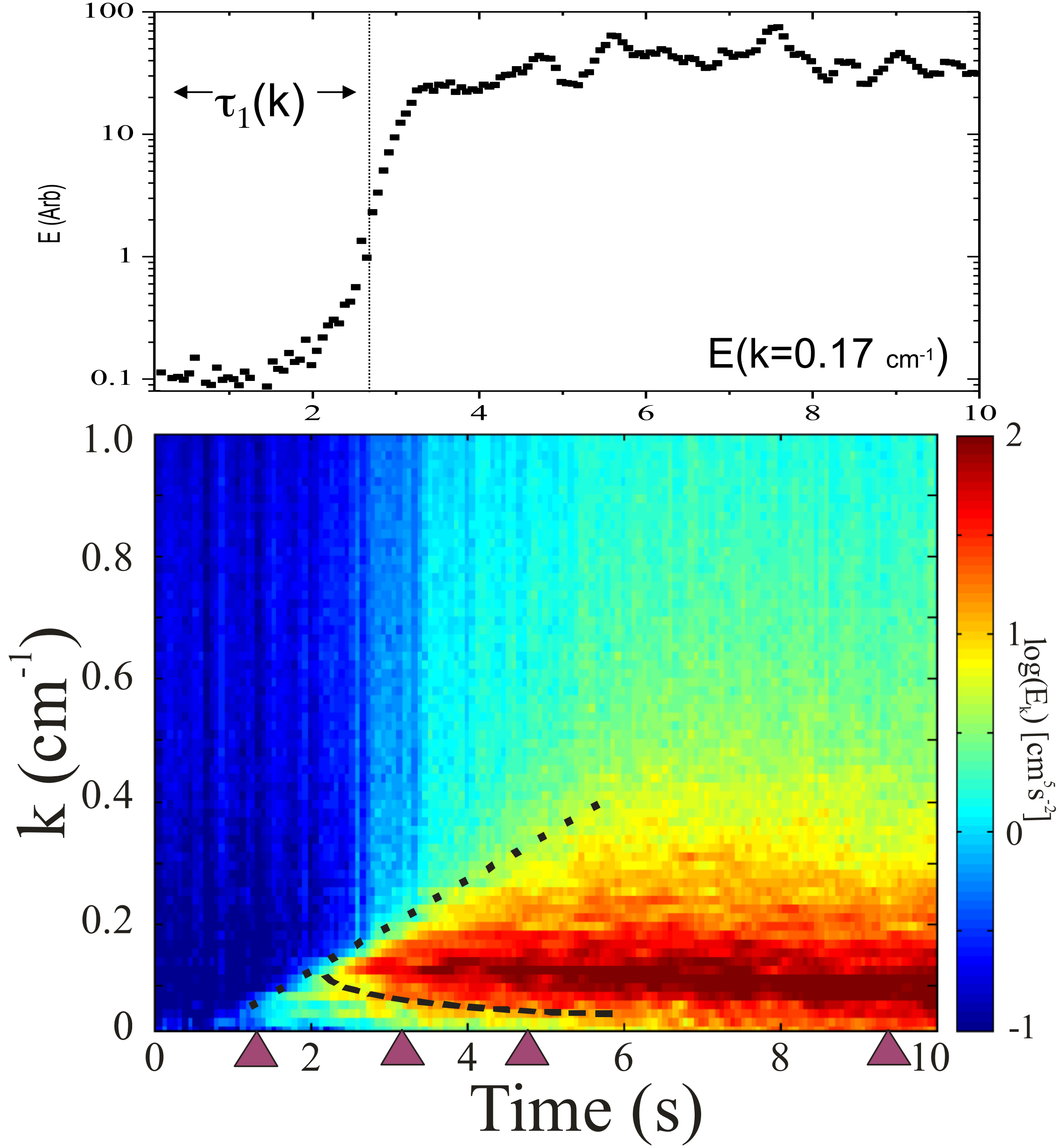
Evolution of the energy spectrum – Short times

Sharp "arrival" of energy

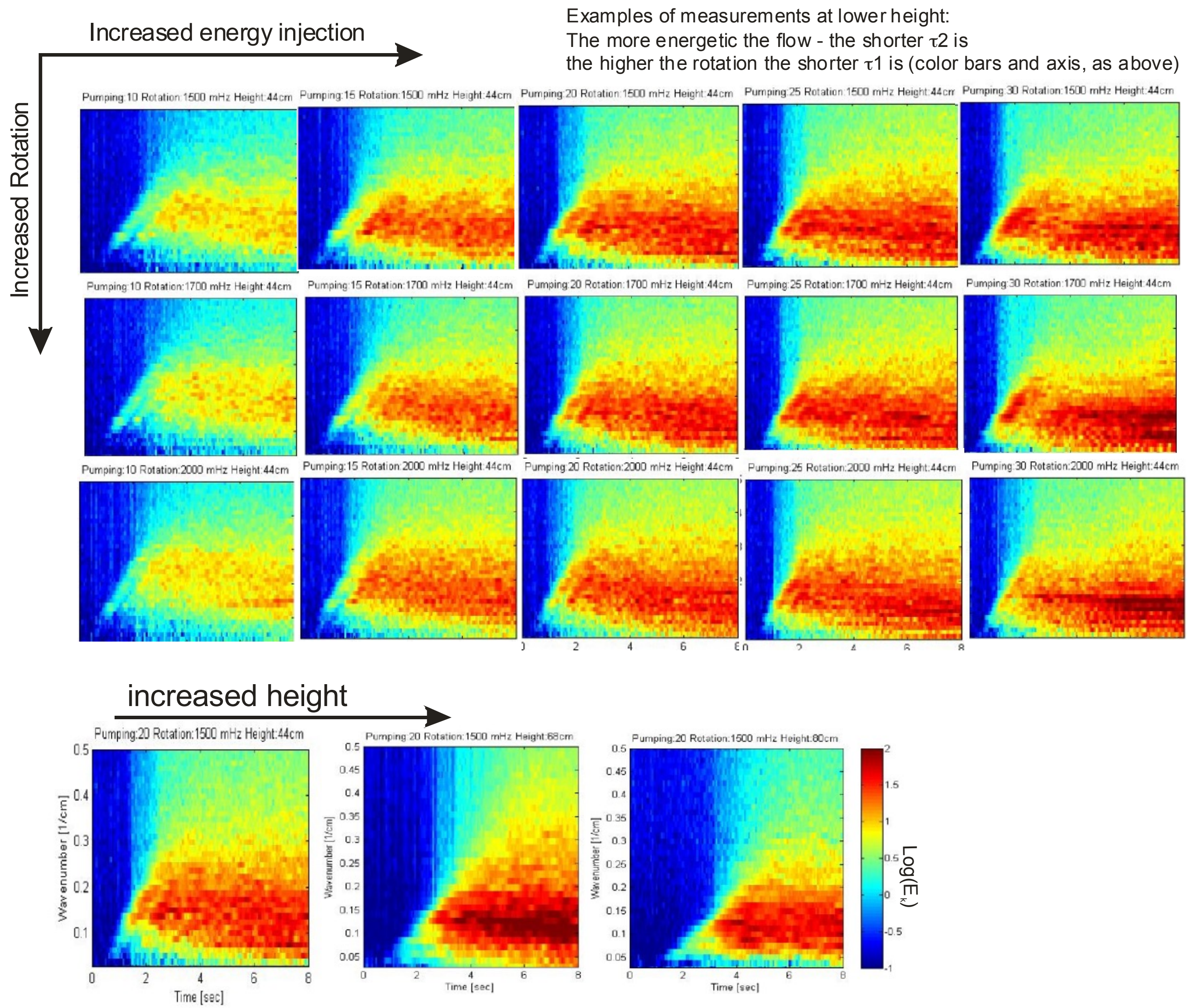
Two "Populating fronts" in the spectrum

One is linear in k, defining $\tau_1(k)$

The second defines $\tau_2(k)$, which decreases with k

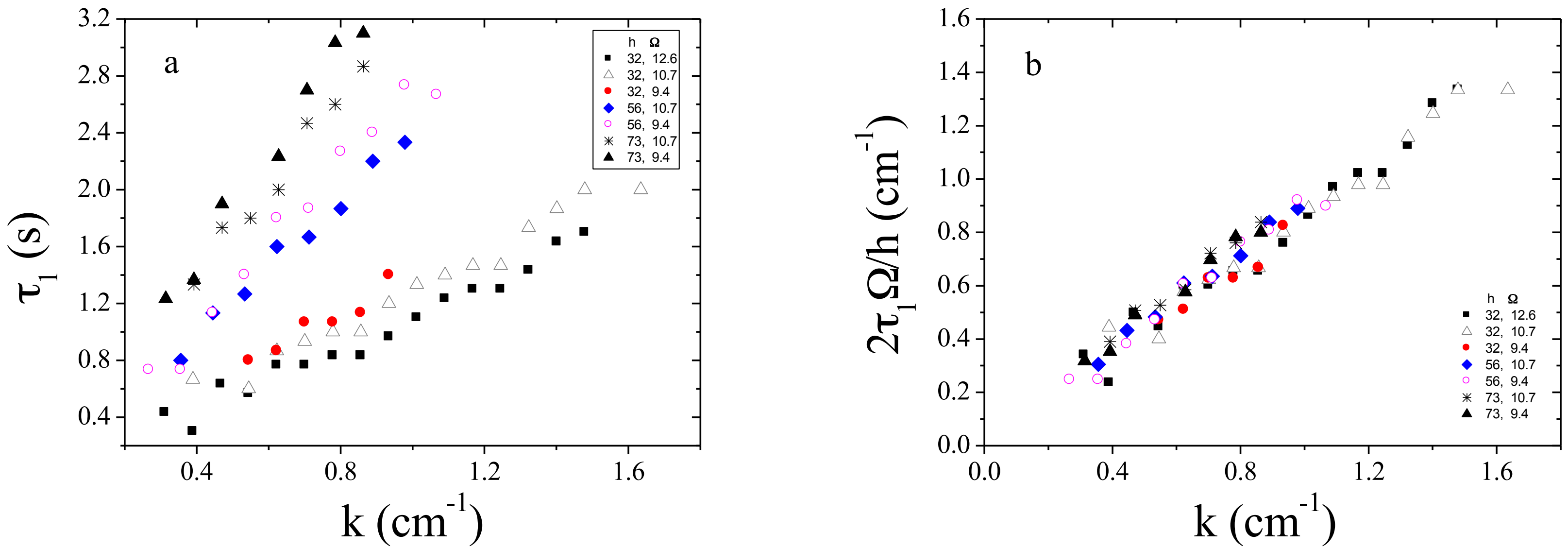


Variation of fronts properties with rotation, energy injection and height



Scaling of τ_1

The energy carried by an inertial wave of wave number k would arrive at h by a time: $t = \frac{h}{v_g} = \frac{hk}{2\Omega}$



τ_1 is the traveling time of inertial waves to the measuring plane
(typical velocities ~ 1 m/s)

All the energy transfer along z is done by inertial waves

I. Kolvin, K. Cohen, Y. Vardi and E. S., *Phys. Rev. Lett*. 102**,** 014503,(2009).

Searching for inertial waves in steady state

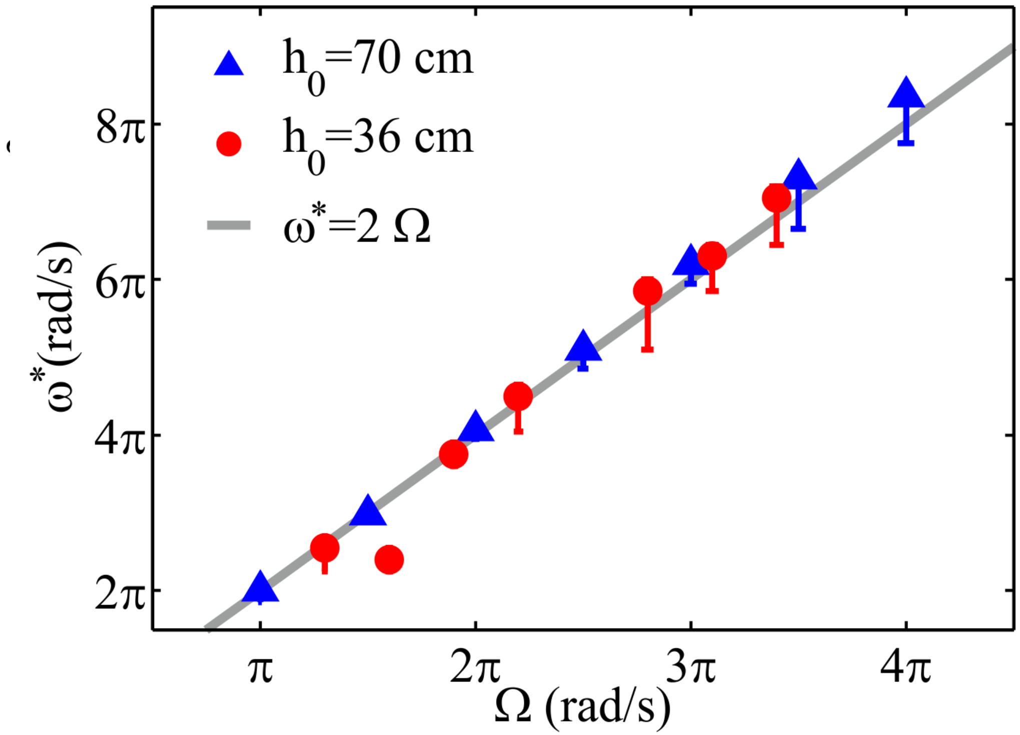
$|\omega| < 2\Omega$

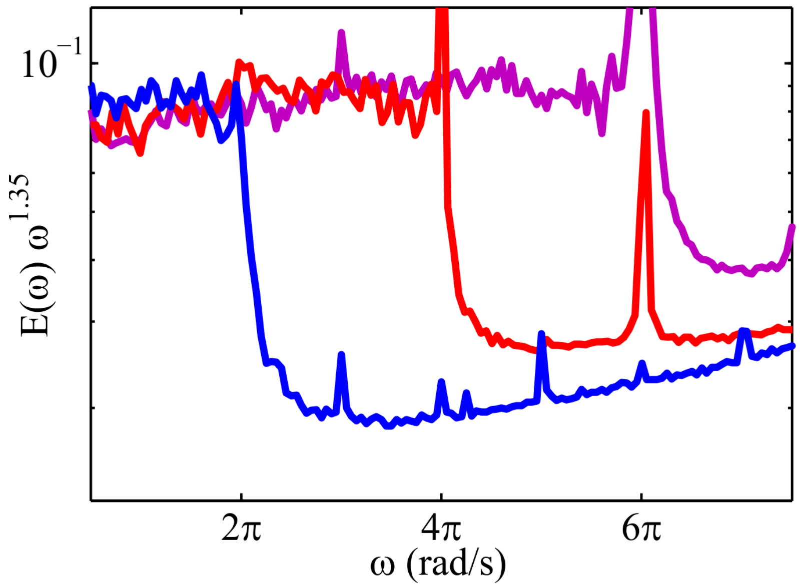
$\omega = \omega(\theta) = \pm 2\Omega\cos(\theta)$

Temporal energy spectrum

$$\tilde{\vec{u}}_\perp(x,y,\omega) = \frac{1}{\sqrt{2\pi}}\int \vec{u}_\perp(x,y,t)e^{-i\omega t}dt$$

$$E_\perp(\omega) = \frac{1}{2ST}\iint\left(\left|\tilde{u}_x\right|^2 + \left|\tilde{u}_y\right|^2\right)dxdy$$





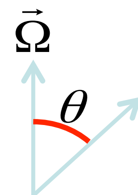
Ω (rad/s)	1π	1.5π					

A “signature “of inertial waves in steady state!

Is the energy concentrated along the dispersion curve? yes

Dispersion relation:

$$\omega = \pm 2\Omega\cos(\theta)$$



A need for 3D measurement

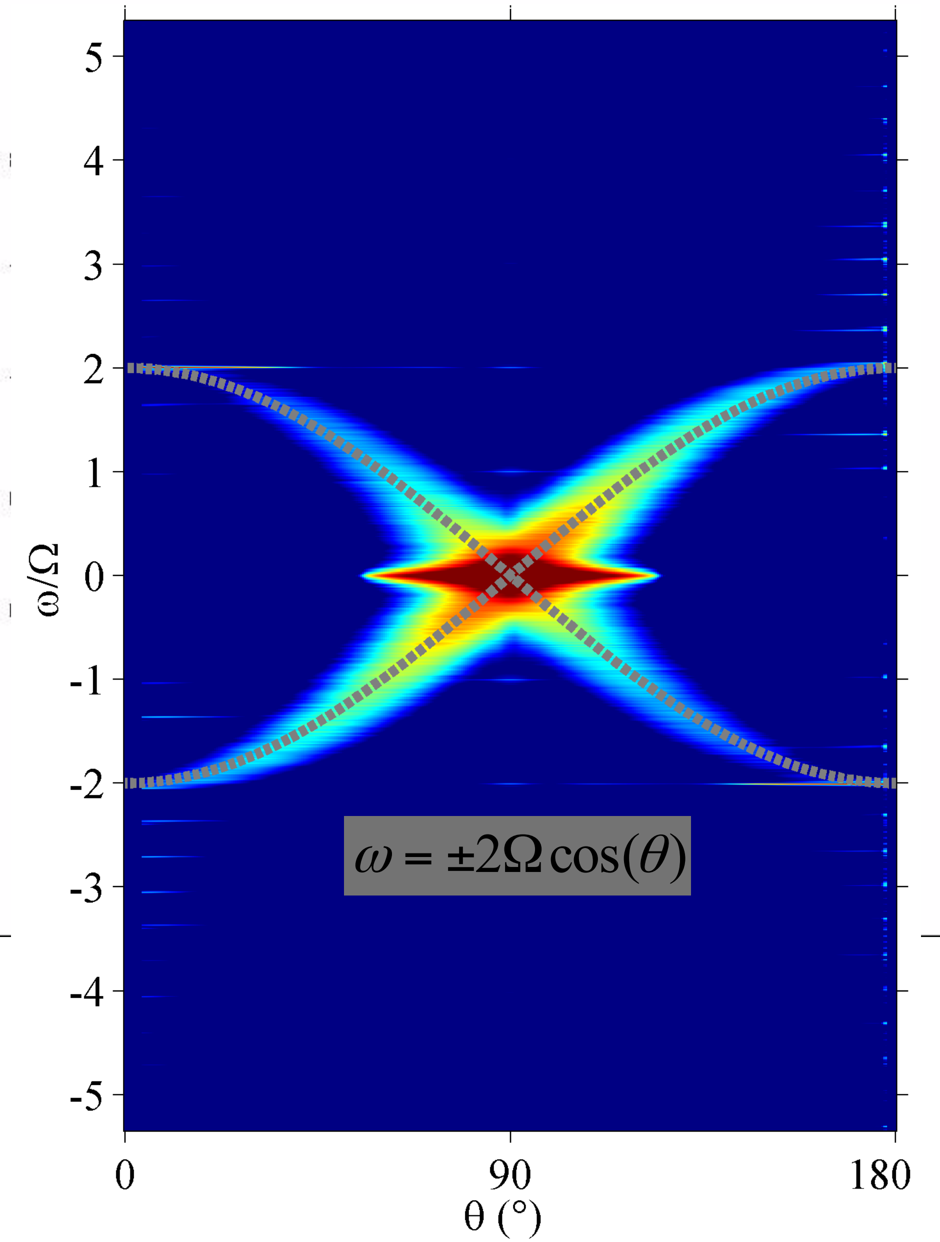
$$\tilde{u}_\perp(\vec{k},\omega) = \frac{1}{(2\pi)^2}\iint \vec{u}_\perp(x,y,z,t)e^{-i(\vec{k}\cdot\vec{r}+\omega t)}d\vec{r}dt$$

$$E_\perp(\vec{k},\omega) = \frac{1}{2}\left(\left|\tilde{u}_x\right|^2 + \left|\tilde{u}_y\right|^2\right)$$

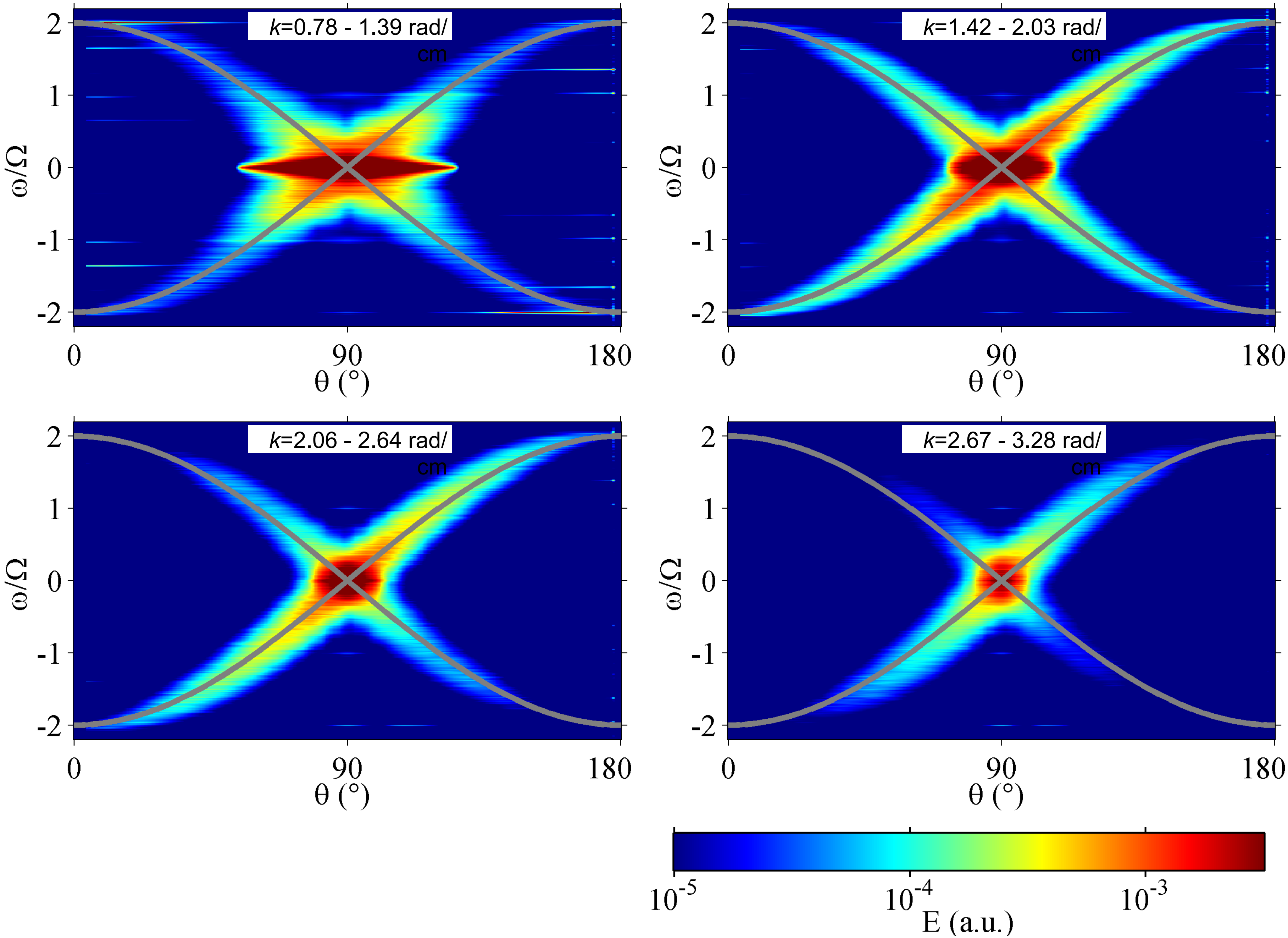
- ~ 1000 fps
- 30 measurement
- ~ 30 “blocks” /s

$$E_\perp(\omega,\theta) = \frac{1}{VT}\iint k^2\sin(\theta)E(\vec{k},\omega)dkd\varphi$$

3D energy spectrum for $\Omega=4\pi$/s

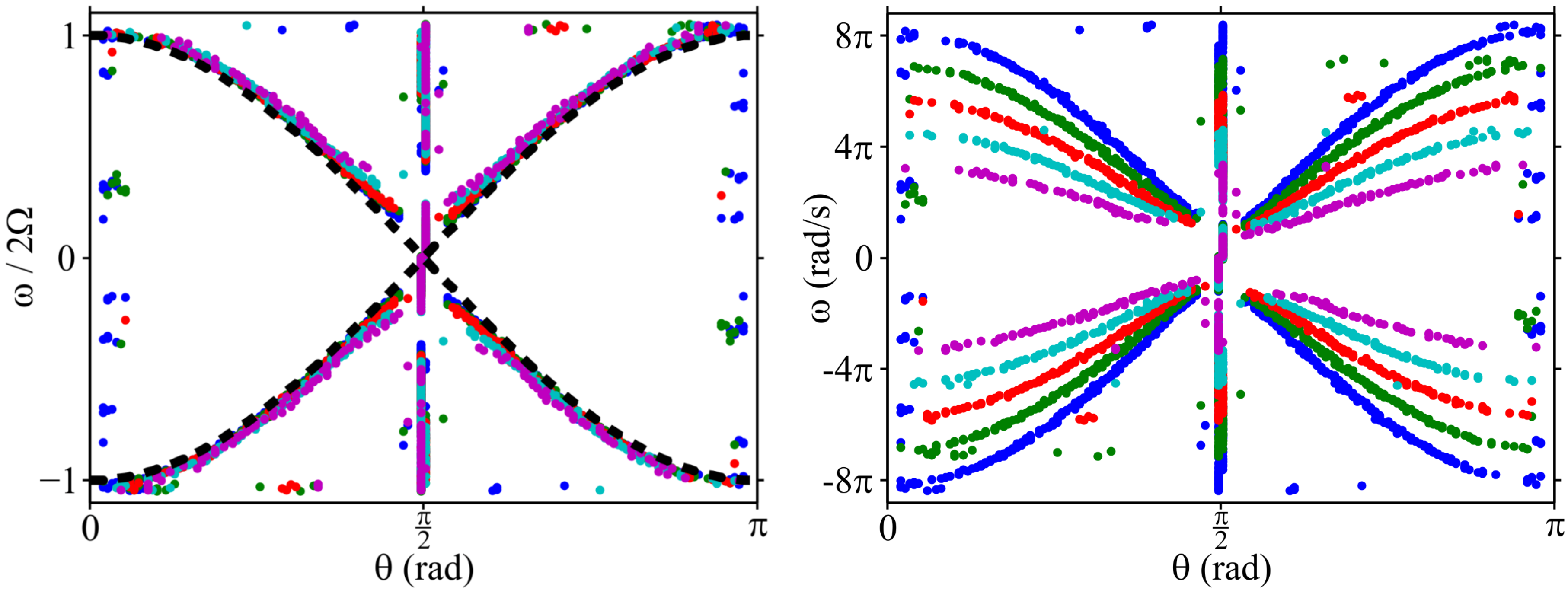


|k| independence of the dispersion curve



The variation with Ω

$$\omega = \pm 2\Omega \cos(\theta)$$



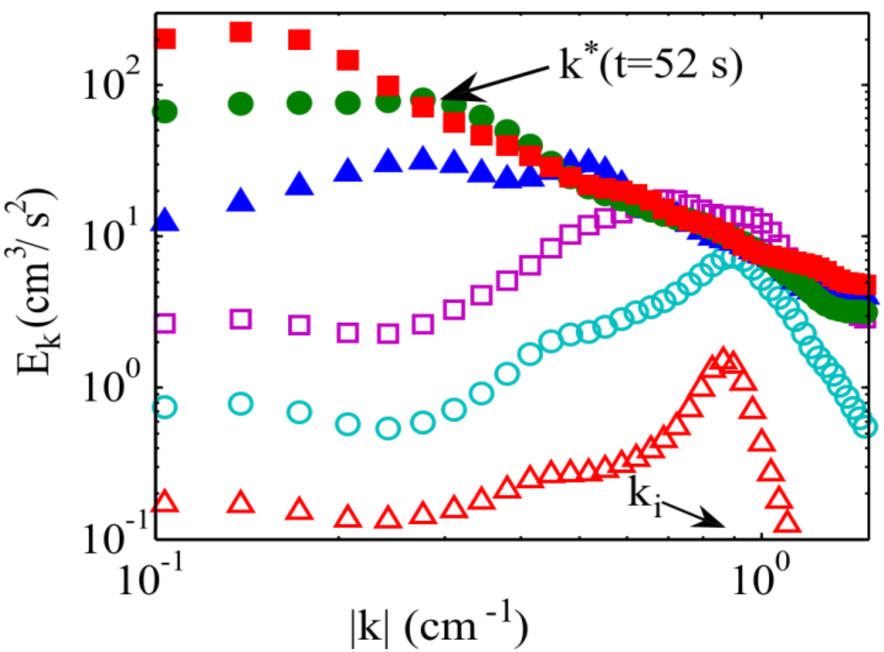
Ω (rad/s)	4π	3.4π	2.8π	2.2π	1.6π

E. Yarom and E.S. *Nature Physics* 2014

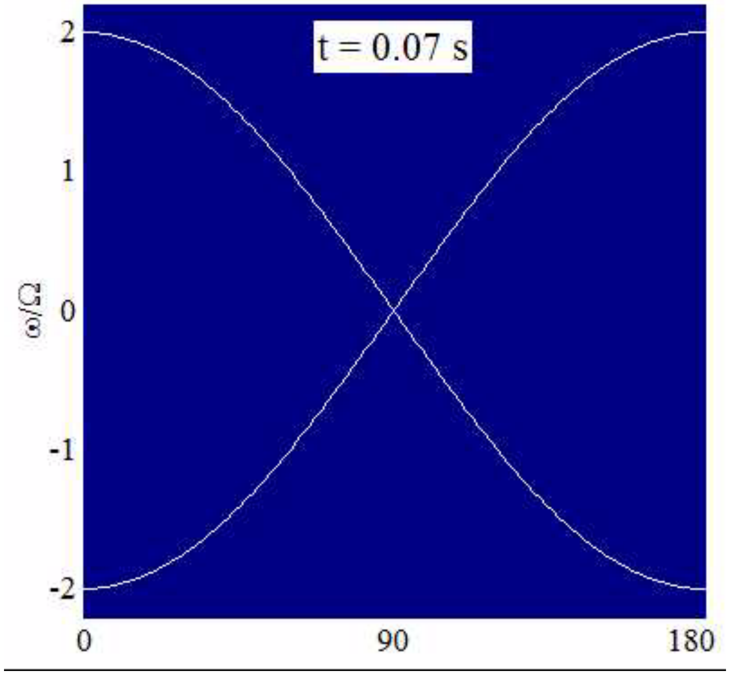
Is energy being transferred via wave interactions?

Measuring the spectrum evolution towards steady state

In 2D



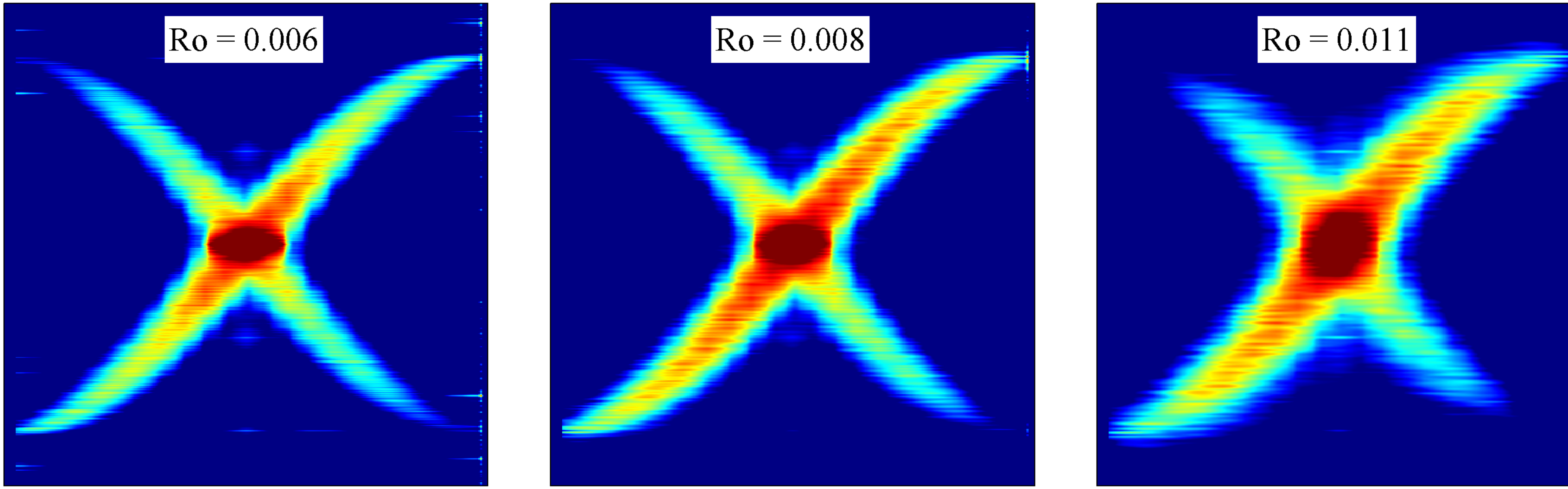
In 3D (ω,θ – plane)

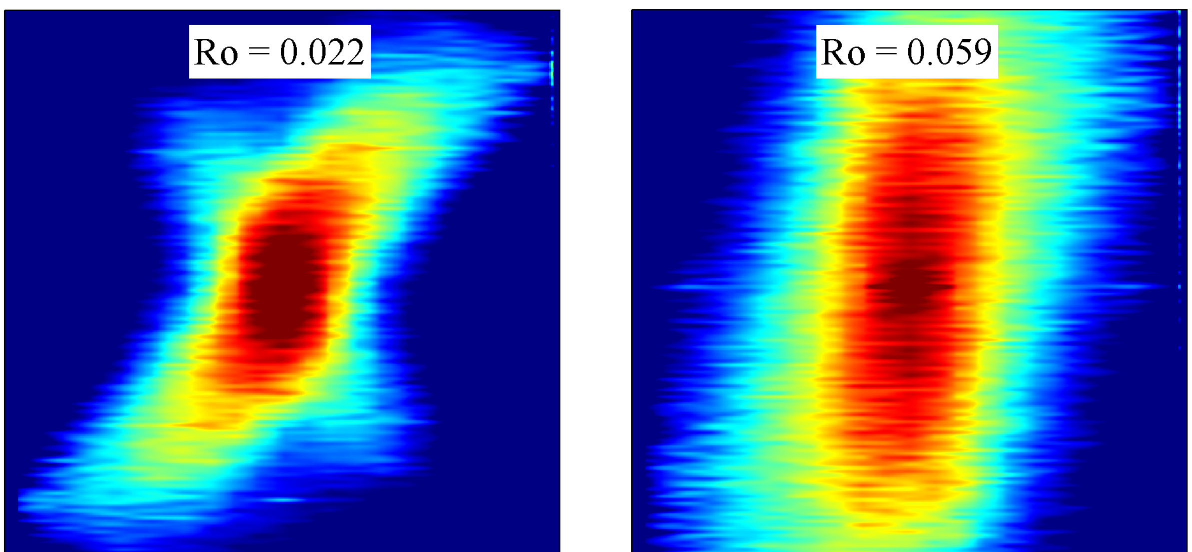


The entire spectrum evolution is confined to the dispersion relation.

Consistent with transfer via wave interactions.

Limits of the wave turbulence behavior - Spectrum at different Rossby





$\Omega = 0.2 – 2$ *hz*

$k = 1.73 – 1.85$ *rad/cm*

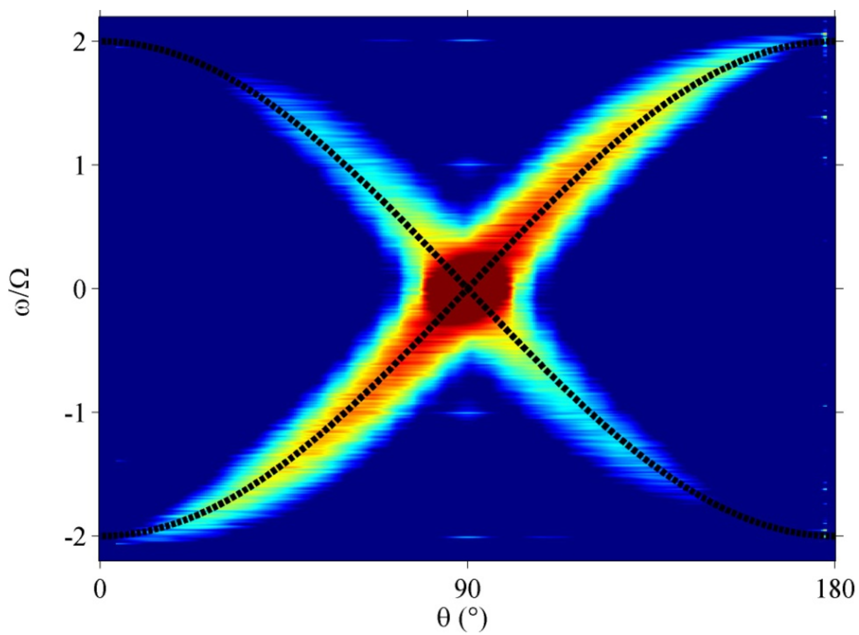
Conclusions

The energy spectrum of deep rotating turbulence and its evolution are quantitatively consistent with the idea of inverse energy cascade.

For moderate Re (~10^3) The energy is contained and transferred by 3D inertial waves

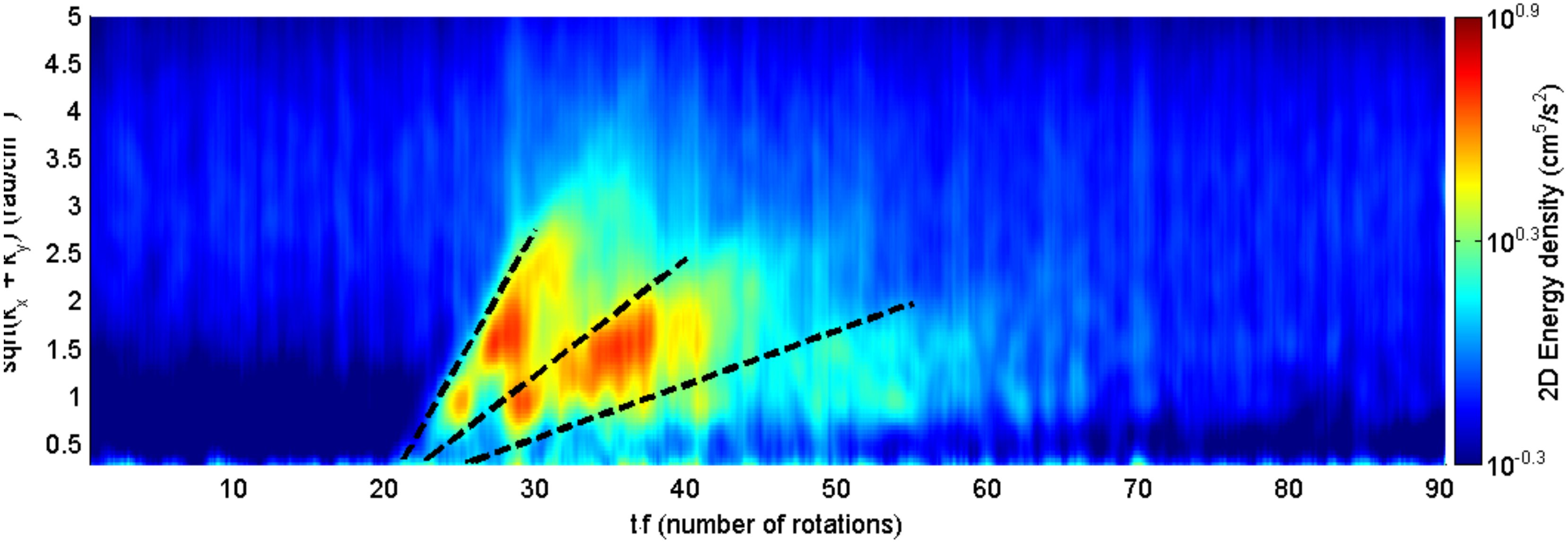
The entire 3D turbulent field is, therefore, a wave turbulence of inertial waves

Further quantitative study is needed: the $E(\omega) \sim \omega^{-4/3}$ scaling was not predicted theoretically



Thank you

A single pulse



Helical modes spectrum

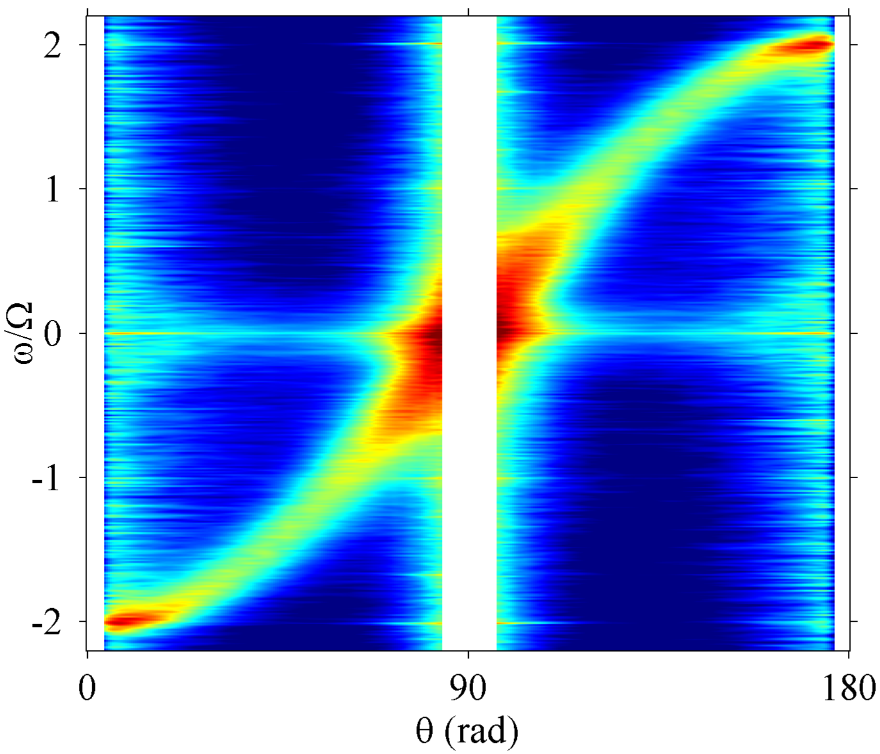
$$\vec{h}^{s}_{\vec{k}} = \left(\hat{k}\times\hat{\Omega}\right)\times\hat{k} + i\,s\left(\hat{k}\times\hat{\Omega}\right)$$

$$\vec{u} = a^{+}_{k} e^{i\omega_{+}t}\vec{h}^{+}_{\vec{k}} + a^{-}_{k} e^{i\omega_{-}t}\vec{h}^{-}_{\vec{k}}$$

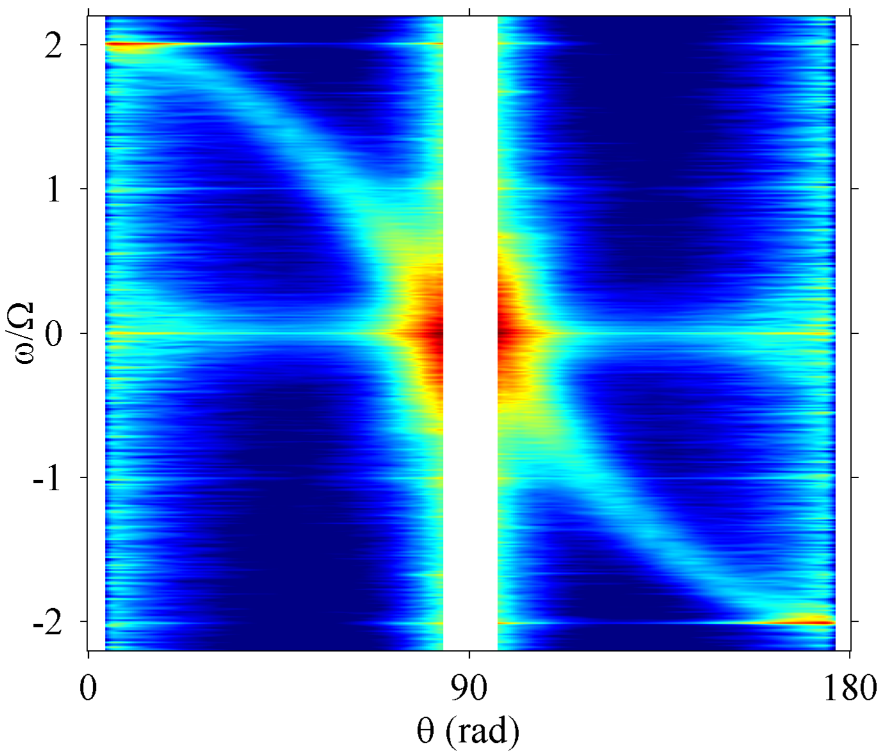
$$a^{s}_{\vec{k}} e^{i\omega_{s}t} = \frac{\vec{u}\cdot\vec{h}^{-s}_{\vec{k}}}{2(k_{\perp}/k)^{2}}$$

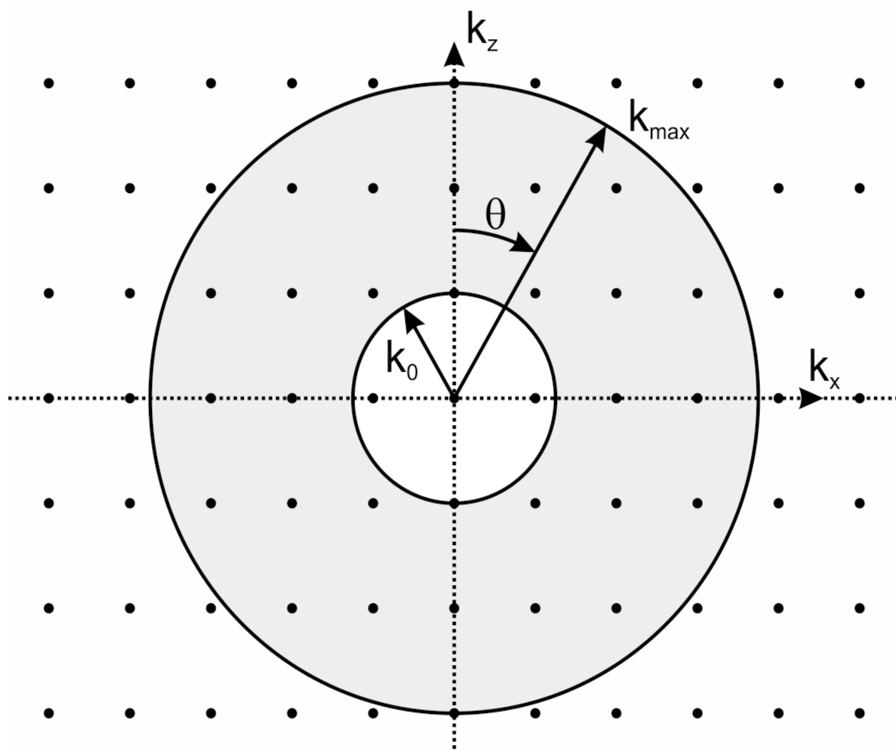
$$E^{s}(\vec{k},\omega) = \frac{1}{2}\left|a^{s}_{k}\right|^{2}$$

Positive Helicity: E^{+}



Negative Helicity: E^{-}





k_z
k_{max}
θ
k_0
k_x

Inertial Waves

$$\vec{u} \sim e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)} \qquad \omega = s2\vec{\Omega}\cdot\hat{k} = s2\Omega\cos(\theta) \qquad , \; s = \pm 1$$

$$\vec{c}_g = s\frac{2}{\left|\vec{k}\right|^3}\vec{k}\times\left(\vec{\Omega}\times\vec{k}\right) \qquad c_g = s\frac{2\Omega}{k}\sin(\theta)$$

$$c_p = s\frac{2\Omega}{k}\cos(\theta)$$

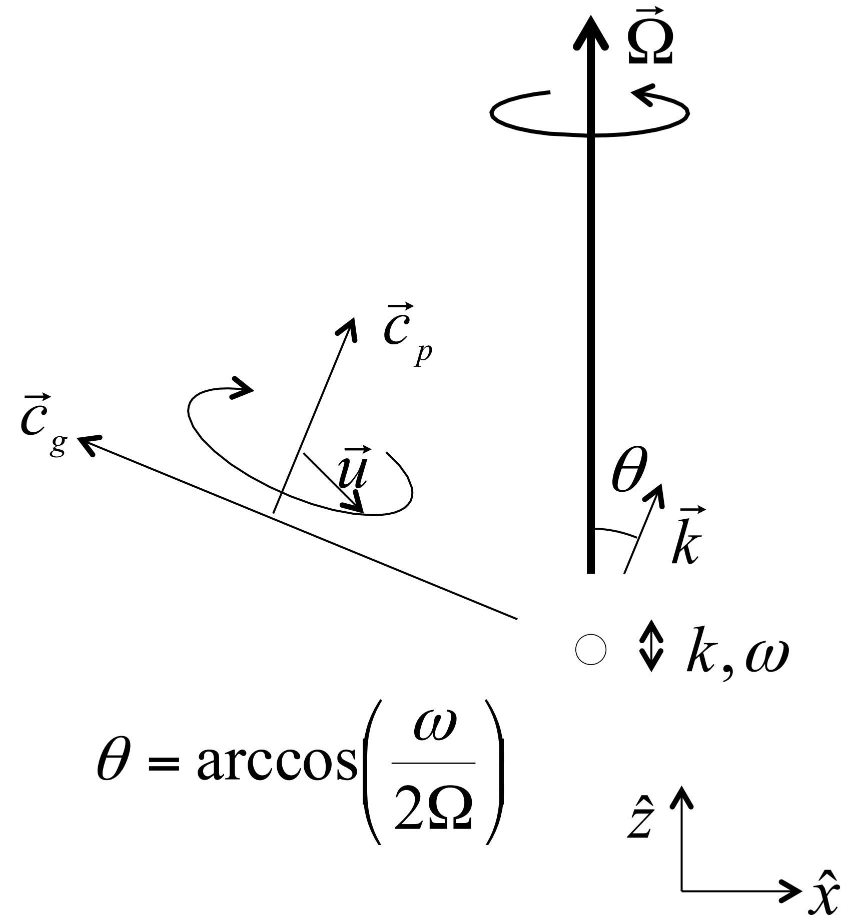
- Define: $\vec{\Omega} = \Omega\,\hat{z}$, $k_y = 0$

$$u_x = ik_z u_0 e^{i(k_x x + k_z z - \omega t)} + c.c$$

$$u_y = sku_0 e^{i(k_x x + k_z z - \omega t)} + c.c$$

$$u_z = -ik_x u_0 e^{i(k_x x + k_z z - \omega t)} + c.c$$

$$\vec{u}\cdot\vec{k} = 0$$

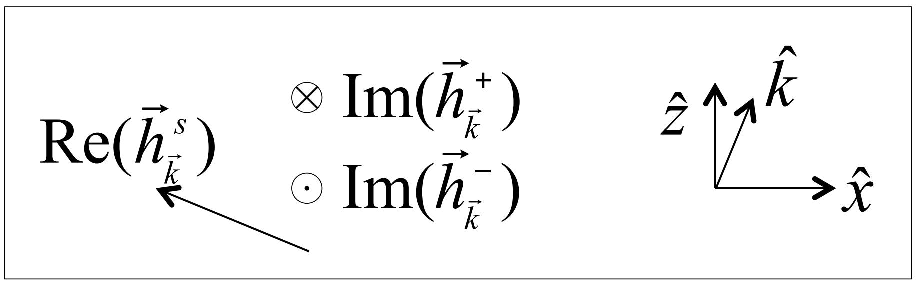


$$\theta = \arccos\left(\frac{\omega}{2\Omega}\right)$$

Inertial Waves – Helical Modes

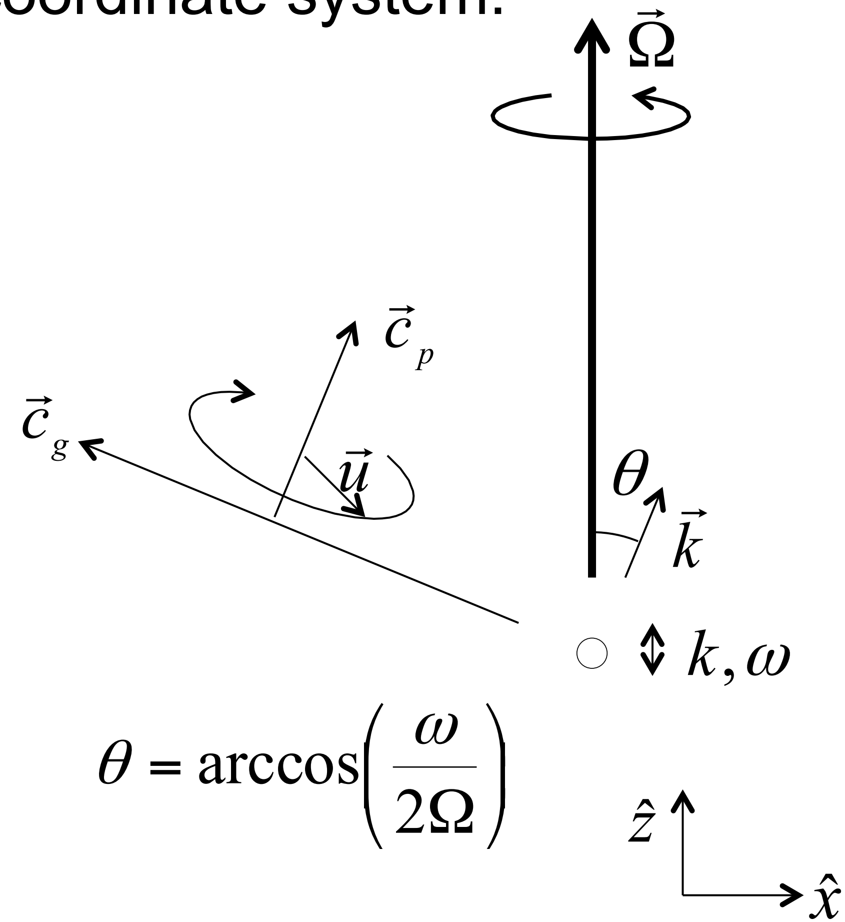
- Vorticity of mode $\vec{k}$ $\quad \vec{\varpi} \equiv \vec{\nabla} \times \vec{u} = -s\,k\,\vec{u}$
- Helicity: $h \equiv \vec{\varpi} \cdot \vec{u} = -s\,k\,u_0^2$
- Define helical modes as new coordinate system:

$$\vec{h}_{\vec{k}}^{s} = \left(\hat{k} \times \hat{\Omega}\right) \times \hat{k} + i\,s\left(\hat{k} \times \hat{\Omega}\right)$$



$$\vec{u} = a_k^{+} e^{i\omega_+ t} \vec{h}_{\vec{k}}^{+} + a_k^{-} e^{i\omega_- t} \vec{h}_{\vec{k}}^{-}$$

$$a_{\vec{k}}^{s} e^{i\omega_s t} = \frac{\vec{u} \cdot \vec{h}_{\vec{k}}^{-s}}{\vec{h}_{\vec{k}}^{s} \cdot \vec{h}_{\vec{k}}^{-s}} = \frac{\vec{u} \cdot \vec{h}_{\vec{k}}^{-s}}{2(k_\perp / k)^2}$$



$$\theta = \arccos\left(\frac{\omega}{2\Omega}\right)$$