Laws of unsteady turbulence

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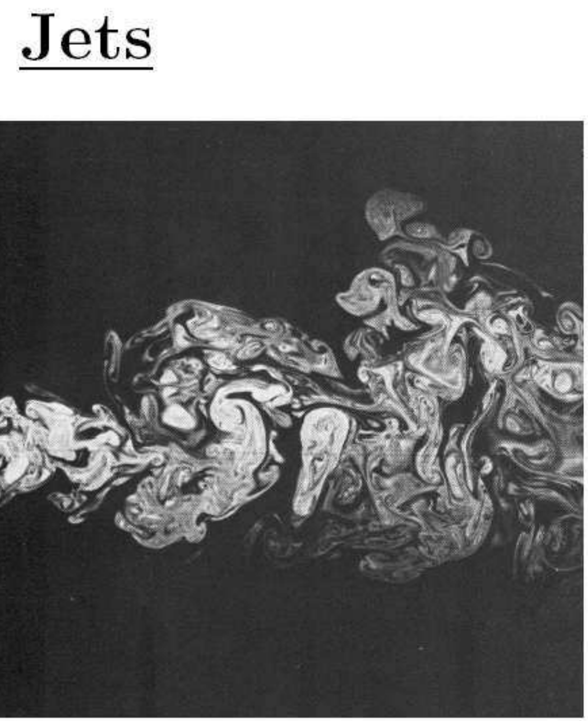
Free shear turbulent flows away from walls

Work (in chronological order over past ten years) with
R. Seoud, N. Mazellier, P. Valente, J. Nedic,
T. Dairay, M. Obligado & S. Goto

Turbulent jets

(Picture from album of fluid motion)



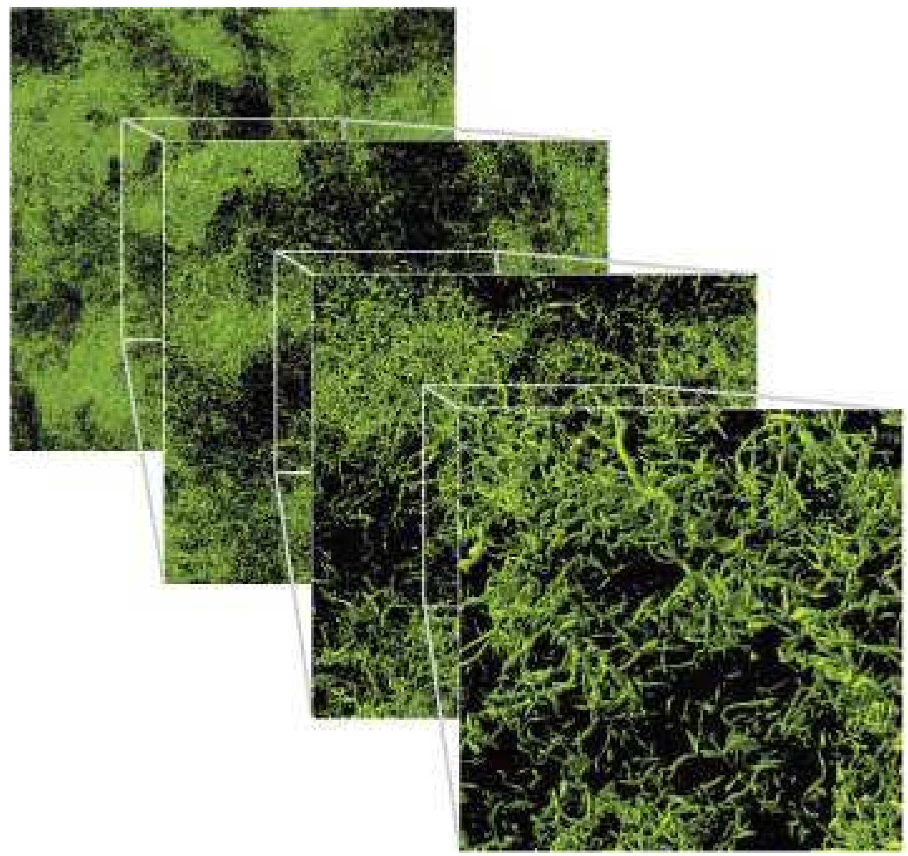
Turbulent wakes

(Picture downloaded from the web)

FIG. 1.

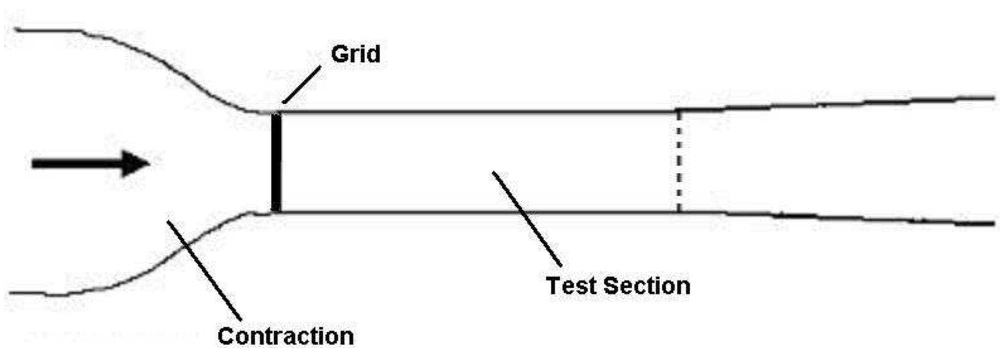
Homogeneous turbulence

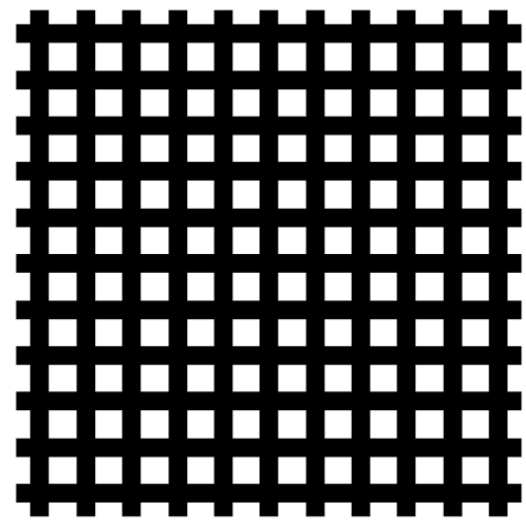
(from Ishihara et al, early/mid 2000s, Japan, Earth Simulator)



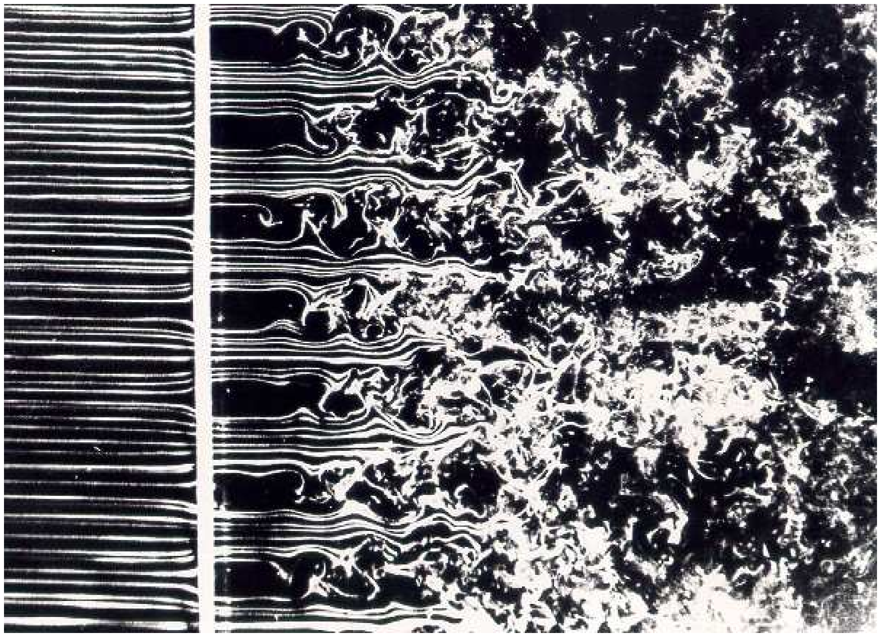
Grid-generated turbulence

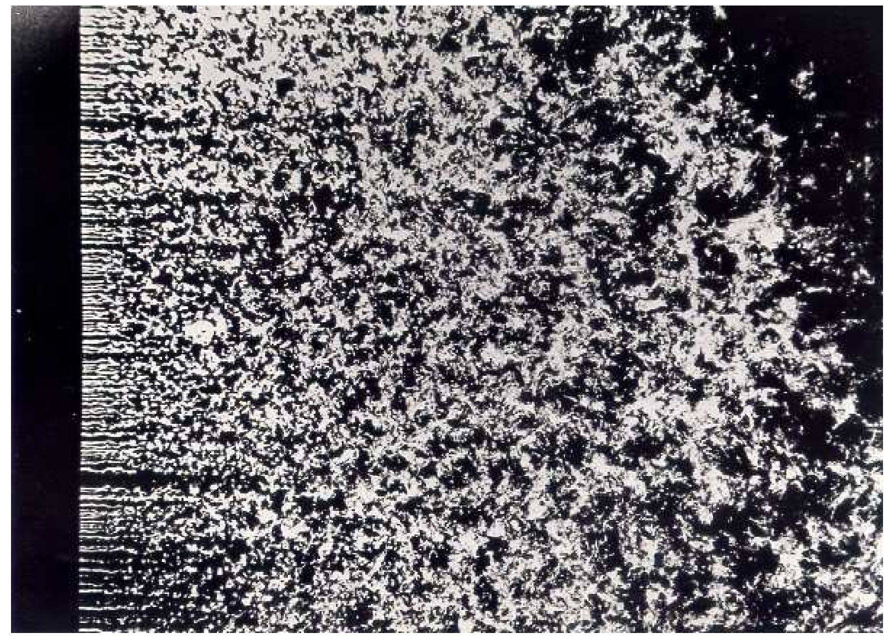
1. Attempt at reasonably homogeneous isotropic turbulence to check theories of turbulence: 1934 to this day.
2. Attempt at more stirring/mixing than with canonical free-shear flows (individual wakes, jets, mixing layers)





Grid-generated turbulence





Reynolds decomposition

Turbulent flows are fluctuating randomly around a mean

Decompostion as mean $\mathbf{U}$ + fluctuations $\mathbf{u}$

i.e. full velocity field $= \mathbf{U} + \mathbf{u}$

Wide range of scales of motion in fluctuating velocity field $\mathbf{u}$ when Reynolds number is high.

Richardson-Kolmogorov cascade

Mechanism of turbulence dissipation at high Reynolds number

Kinetic energy of velocity fluctuations cascades from large scales of motion to small scales of motion.

When it reaches a scale small enough for viscous dissipation to be effective, it dissipates.

This cascade is an equilibrium cascade where the rate with which kinetic energy crosses a length-scale l where the turbulent fluctuations have a characteristic velocity $u(l)$ is the same from the largest to the smallest length-scale.

Estimate this rate dimensionally as $u(l)^3/l$ (no viscosity) and equate it to the kinetic energy dissipation $\epsilon = \nu < \mathbf{s}^2 >$

Richardson-Kolmogorov cascade

Estimate this rate dimensionally as $u(l)^3/l$ (no viscosity) and equate it to the kinetic energy dissipation $\epsilon = \nu < \mathbf{s}^2 >$

The largest of these length-scales l must be of the order of the integral (correlation) length-scale L where
$u(l) = u(L) \sim u' \equiv \sqrt{< \mathbf{u^2} >}$

Hence, $\epsilon \sim u'^3/L$

We write $\epsilon = C_\epsilon u'^3/L$ where the dimensionless dissipation coefficient C_ϵ is a constant independent of Reynolds number.

R-K equilibrium cascade in equations

What is $u(l)$?

1. Define $\delta \mathbf{u} \equiv \mathbf{u}(\mathbf{x} + \frac{1}{2}\mathbf{l}, t) - \mathbf{u}(\mathbf{x} - \frac{1}{2}\mathbf{l}, t)$

2. $< |\delta \mathbf{u}|^2 >$ is a function of $\mathbf{x}$ and $\mathbf{l}$ and is our definition of $u^2(l)$. The vector $\mathbf{l}$ has a norm $l = |\mathbf{l}|$.

3. Write down the Navier-Stokes equations and incompressibility at both $\mathbf{x} + \frac{1}{2}\mathbf{l}$ and $\mathbf{x} - \frac{1}{2}\mathbf{l}$.

$$\frac{\partial}{\partial t}(\mathbf{U} + \mathbf{u}) + (\mathbf{U} + \mathbf{u}) \cdot \nabla_\xi (\mathbf{U} + \mathbf{u}) = -\nabla_\xi (P + p) + \nu \nabla_\xi^2 (\mathbf{U} + \mathbf{u})$$

$$\text{and } \nabla_\xi \cdot \mathbf{U} = 0,\ \nabla_\xi \cdot \mathbf{u} = 0$$

at both $\xi = \xi_+ \equiv \mathbf{x} + \frac{1}{2}\mathbf{l}$ and $\xi = \xi_- \equiv \mathbf{x} - \frac{1}{2}\mathbf{l}$.

R-K equilibrium cascade in equations

$$\frac{D^*}{Dt} < |\delta \mathbf{u}|^2 > +\nabla_{\mathbf{l}} \cdot < (\delta \mathbf{u} + \delta \mathbf{U}) |\delta \mathbf{u}|^2 >= P^* + T^*_{\mathbf{x}} + D^* - \epsilon^*$$

where

$$\frac{D^*}{Dt} \equiv \frac{\partial}{\partial t} + \frac{1}{2}[\mathbf{U}(\xi_+) + \mathbf{U}(\xi_-)] \cdot \nabla_{\mathbf{x}}$$

This is the fully generalised Karman-Howarth-Monin equation following Reginald Hill’s work from the mid/late 1990s to the early 2000s. We should perhaps call it the Karman-Howarth-Monin-Hill or KHMH equation.

R-K equilibrium cascade in equations

$$\frac{D^*}{Dt} < |\delta \mathbf{u}|^2 > + \nabla_{\mathbf{l}} \cdot < (\delta \mathbf{u} + \delta \mathbf{U}) |\delta \mathbf{u}|^2 > = P^* + T^*_{\mathbf{x}} + D^* - \epsilon^*$$

where

$$\frac{D^*}{Dt} \equiv \frac{\partial}{\partial t} + \frac{1}{2} [\mathbf{U}(\xi_+) + \mathbf{U}(\xi_-)] \cdot \nabla_{\mathbf{x}}$$

(i) Consider high enough Reynolds numbers so that the two-point viscous diffusion term D^* may be neglected; (ii) consider regions of turbulent flows where the integral scale L of the turbulent fluctuating velocity is small compared to length-scales characterising spatial variations in $\mathbf{x}$ of mean fow statistics. Then for $l << L$ and $l >> \eta_{viscous}$

$$\frac{\partial}{\partial t} < |\delta \mathbf{u}|^2 > + \mathbf{U}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} < |\delta \mathbf{u}|^2 > + \nabla_{\mathbf{l}} \cdot < \delta \mathbf{u} |\delta \mathbf{u}|^2 > = -4\epsilon$$

Kolmogorov's 1941 assumption

Characteristic times of small scale ($l << L$) motions very small compared to time scale of turbulence decay hence:

$$\frac{\partial}{\partial t} < |\delta\mathbf{u}|^2 > + \mathbf{U}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} < |\delta\mathbf{u}|^2 > \approx 0$$

The equilibrium cascade then follows in the form

$$\nabla_{\mathbf{l}} \cdot < \delta\mathbf{u}|\delta\mathbf{u}|^2 > \approx -4\epsilon$$

Integrate both sides over a sphere of radius $|\mathbf{l}| = l$ and get

$$\int \hat{\mathbf{l}} \cdot < \delta\mathbf{u}|\delta\mathbf{u}|^2 > d\Omega \approx -\frac{16\pi}{3}\epsilon l$$

If you assume isotropy this becomes $< (\delta\mathbf{u} \cdot \hat{\mathbf{l}})^3 > \approx -\frac{4}{5}\epsilon l$

R-K equilibrium cascade: dissipation

For $l \sim L$, expect $\int \hat{\mathbf{l}} \cdot < \delta\mathbf{u} |\delta\mathbf{u}|^2 > d\Omega \sim u'^3$

which, with equilibrium, then implies $u'^3 \sim \epsilon L$.

Equilibrium dissipation law:
$\epsilon = C_\epsilon u'^3 / L$ with $C_\epsilon = Const$

Notes:
(1) Equilibrium OVER THE ENTIRE INERTIAL RANGE, THAT IS INCLUDING $l \sim L$, is required to obtain $\epsilon = C_\epsilon u'^3 / L$ with $C_\epsilon = Const$

(2) The inverse is not necessarily true: $\epsilon = C_\epsilon u'^3 / L$ with $C_\epsilon = Const$ does not mean there must be equilibrium.

R-K equilibrium cascade: spectra

Define $S \equiv < (\delta \mathbf{u} \cdot \hat{\mathbf{l}})^3 > / < (\delta \mathbf{u} \cdot \hat{\mathbf{l}})^2 >^{3/2}$ so that

$$< (\delta \mathbf{u} \cdot \hat{\mathbf{l}})^2 > \approx C_2 (\epsilon l)^{2/3} \text{ where } C_2 = (\frac{-4}{5S})^{2/3}$$

Assuming $S = Const$, this last relation is often given in its equivalent energy spectral form (in the range $2\pi/L \ll k_1 \ll 2\pi/\eta_{viscous}$)

$$E_{11}(k_1) \approx C_1 \epsilon^{2/3} k_1^{-5/3} \text{ where } C_1 \approx C_2/4$$

Note consistency:

$u'^2 \sim \int_0^{L^{-1}} E(k)dk + \int_{L^{-1}}^{+\infty} E(k)dk$

which with $\int_0^{L^{-1}} E(k)dk \sim u'^2$ and $E(k) \sim \epsilon^{2/3} k^{-5/3}$

implies $\epsilon \sim u'^3/L$

Range of scales

1. A sufficient condition for the diffusion term $\nu\nabla_{\mathbf{l}}^2 < |\delta u|^2 >$ to be negligible compared to 4ϵ and therefore drop out so as to be left with

$$\frac{\partial}{\partial t} < |\delta \mathbf{u}|^2 > + \mathbf{U}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} < |\delta \mathbf{u}|^2 > + \nabla_{\mathbf{l}} \cdot < \delta \mathbf{u} |\delta \mathbf{u}|^2 > = -4\epsilon$$

is that $l \gg \lambda$ where $\lambda^2 \equiv \nu u'^2/\epsilon$.

(See Laizet, V & Cambon, FDR **45**(6), 061408, 2013).

2. In small-scale isotropic turbulence, $\lambda \sim \overline{d}$ where $\overline{d}$ is the average distance between stagnation points of the fluctuating velocity. The fluctuating velocity is “rougher” at scales larger than λ than it is at scales smaller than λ. (See Goto & V, PoF **21**, 035104, 2009).

Range of scales

Use $\lambda^2 \equiv \nu u'^2/\epsilon$ and $\epsilon = C_\epsilon u'^3/L$ to obtain

$$L/\lambda \sim C_\epsilon Re_\lambda$$

where $Re_\lambda \equiv \frac{u'\lambda}{\nu}$ is a local Reynolds number dependent on the position in the flow since u' and λ depend on $\mathbf{x}$.

Demonstrates that $L \gg \lambda$ if $Re_\lambda \gg 1$ and therefore that intermediate scales l where $\lambda \ll l \ll L$ exist.

For the Richardson-Kolmogorov cascade, the higher Re_λ the higher the range of scales required for the turbulent energy to be dissipated. This follows from the R-K equilibrium consequence that $C_\epsilon = const$.

The R-K equilibrium cascade matters

BECAUSE:

The turbulence problem is to reliably reduce the number of degrees of freedom (either universally or in different ways in different universality classes) and the turbulence dissipation scaling and the turbulence cascade seem to be essential stepping stones in this direction

This reduction of number of degrees of freedom may take the form of
(i) a moment closure (e.g. k-ϵ if 1-point, EDQNM if 2-point)
(ii) a filtering approach, e.g. Large Eddy Simulations (LES)
(iii) a dynamical systems approach (state-space attractors)

The R-K equilibrium cascade matters

BECAUSE:

(i) the turbulent eddy viscosity ν_t in one-point RANS models of turbulence is estimated using $\nu_t \sim u' L$ and $\epsilon = C_\epsilon u'^3/L$ where $C_\epsilon = Const$:

$$\nu_t \sim C_\epsilon u'^4/\epsilon;$$

(ii) two-point turbulence modelling such as Large Eddy Simulation relies on the R-K equilibrium cascade;

(iii) the number of degrees of freedom is usually estimated as $(L/\eta)^3$ where $\eta = (\nu^3/\epsilon)^{1/4}$ is the Kolmogorov microscale. The equilibrium relation $\epsilon = C_\epsilon u'^3/L$ where $C_\epsilon = Const$ is crucial in determining that $(L/\eta)^3 \sim C_\epsilon^{3/4} Re_L^{9/4}$; Re_L is another local Reynolds number based on u' and L.

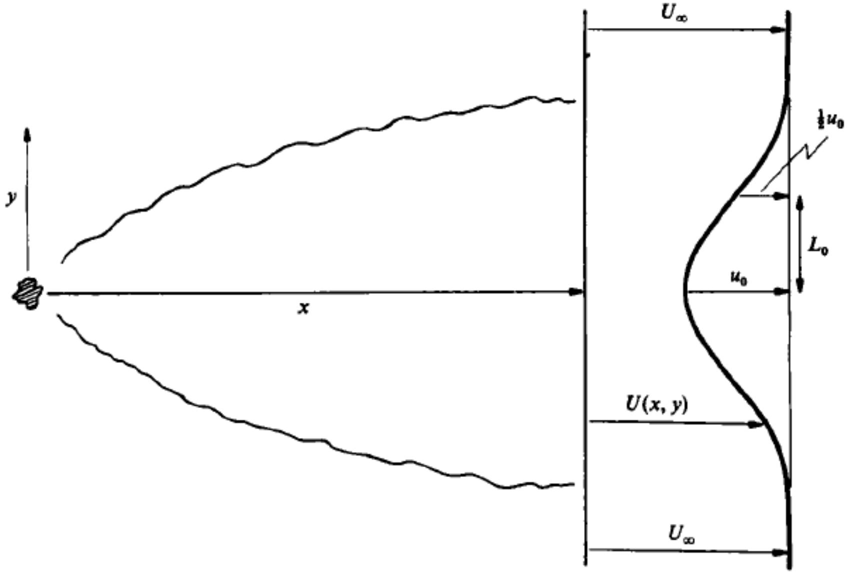
And because

(iv) the combination of the equilibrium dissipation relation $\epsilon = C_\epsilon u'^3/L$ and an invariant quantity (Saffman,Loitsianski or other) determines the turbulence decay of homogeneous turbulence.

(v) the equilibrium dissipation relation $\epsilon = C_\epsilon u'^3/L$ determines the streamwise development of mean profiles of self-preserving turbulent free shear flows.

Quote from Lumley (1992): “What part of modeling is in serious need of work? Foremost, I would say, it is the mechanism that sets the level of dissipation in a turbulent flow, particularly in changing circumstances.”

Self-preserving wake profiles



$$U_\infty - U(x, r) = u_0(x) f[r/L_0(x)]$$

We assume that the wake becomes axisymmetric at some point donwstream.

Townsend 1976, George 1989

The Reynolds-averaged streamwise momentum equation for an axisymmetric wake in a uniform and constant stream,

$$U_\infty \frac{\partial}{\partial x}(U_\infty - U) = -\frac{1}{r}\frac{\partial}{\partial r} r < u'_x u'_r >$$

has the general self-preserving solution

$$U_\infty - U = u_0(x) f[r/L_0(x)]$$

and

$$< u'_x u'_r >= R_0(x) g[r/L_0(x)]$$

under the conditions

$\frac{d}{dx} L_0(x) \sim \frac{R_0}{U_\infty u_0}$ and $u_0 L_0^2 \sim U_\infty \theta^2 = Const$

where θ is the conserved momentum thickness.

2 conditions for 3 unknowns (L_0, u_0, R_0), so make use of kinetic energy equation.

Townsend 1976, George 1989

The Reynolds-averaged kinetic energy equation for an axisymmetric wake in a uniform and constant stream,

$$U_\infty \frac{\partial}{\partial x} K = - < u'_x u'_r > \frac{\partial U}{\partial r} + Transport - \epsilon$$

has the general self-preserving solution

$$K(x, r) = K_0(x) k[r/L_0(x)],$$

$$Transport = T_0(x) t[r/L_0(x)]$$

and

$$\epsilon = D_0(x) e[r/L_0(x)]$$

under the additional conditions

$$\frac{d}{dx} L_0(x) \sim \frac{T_0}{K_0 U_\infty} \sim \frac{D_0 L_0}{K_0 U_\infty} \text{ and } K_0 \sim u_0^2$$

IN TOTAL: 5 conditions for 6 unknowns u_0, L_0, R_0, K_0, T_o and D_0, so need one more relation:

$$D_0 \sim K_0^{3/2} / L_0$$

Townsend 1976, George 1989

$D_0 \sim K_0^{3/2}/L_0$ is $\epsilon = C_\epsilon u'^3/L$ with $C_\epsilon = const$ adapted to this self-preserving flow

and implies

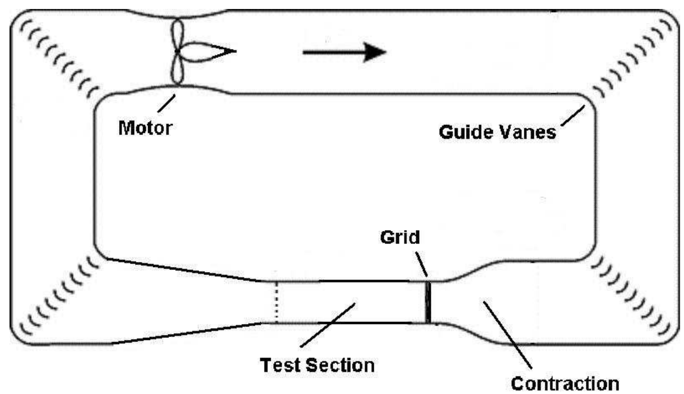
$$u_0/U_\infty \sim (\frac{x-x_0}{\theta})^{-2/3}$$

&

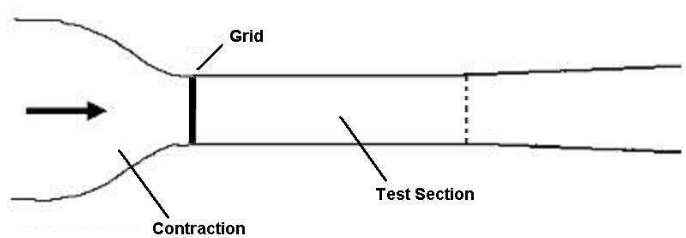
$$L_0/\theta \sim (\frac{x-x_0}{\theta})^{1/3}$$

Wind tunnels

$0.91^2 m^2$ width; test section $4.8m$; max speed $45m/s$; background turbulence ≈ 0.25%.



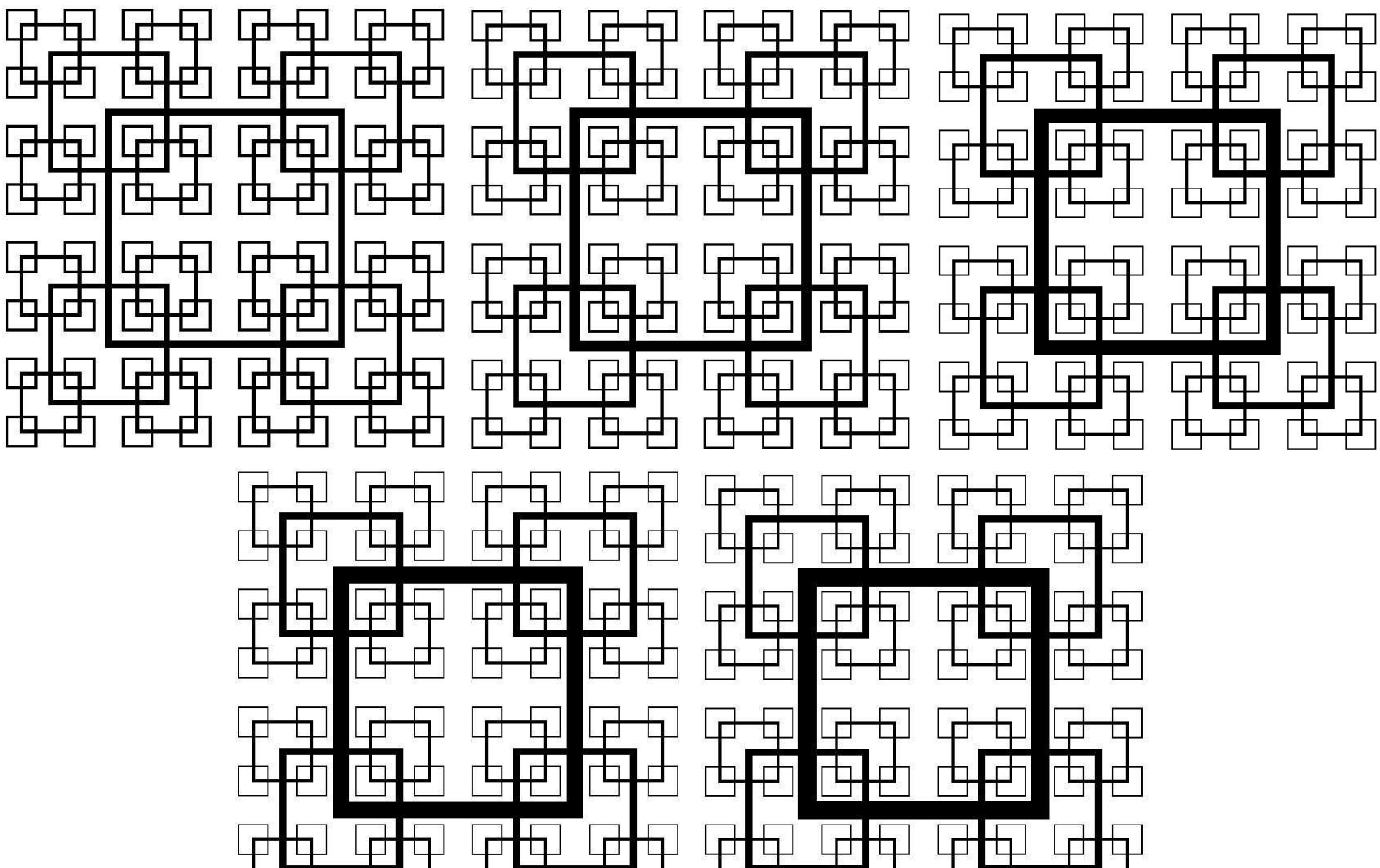
$0.46^2 m^2$ width; test section $\approx 4.0m$; max speed $33m/s$; background turbulence ≈ 0.4%.



$D_f = 2$, $\sigma = 25$% fractal square grids

and equal $M_{eff} \approx 2.6cm$, $L_{max} \approx 24cm$, $L_{min} \approx 3cm$, $N = 4$, $T = 0.46m$.

BUT $t_r = 2.5, 5.0, 8.5, 13.0, 17.0$



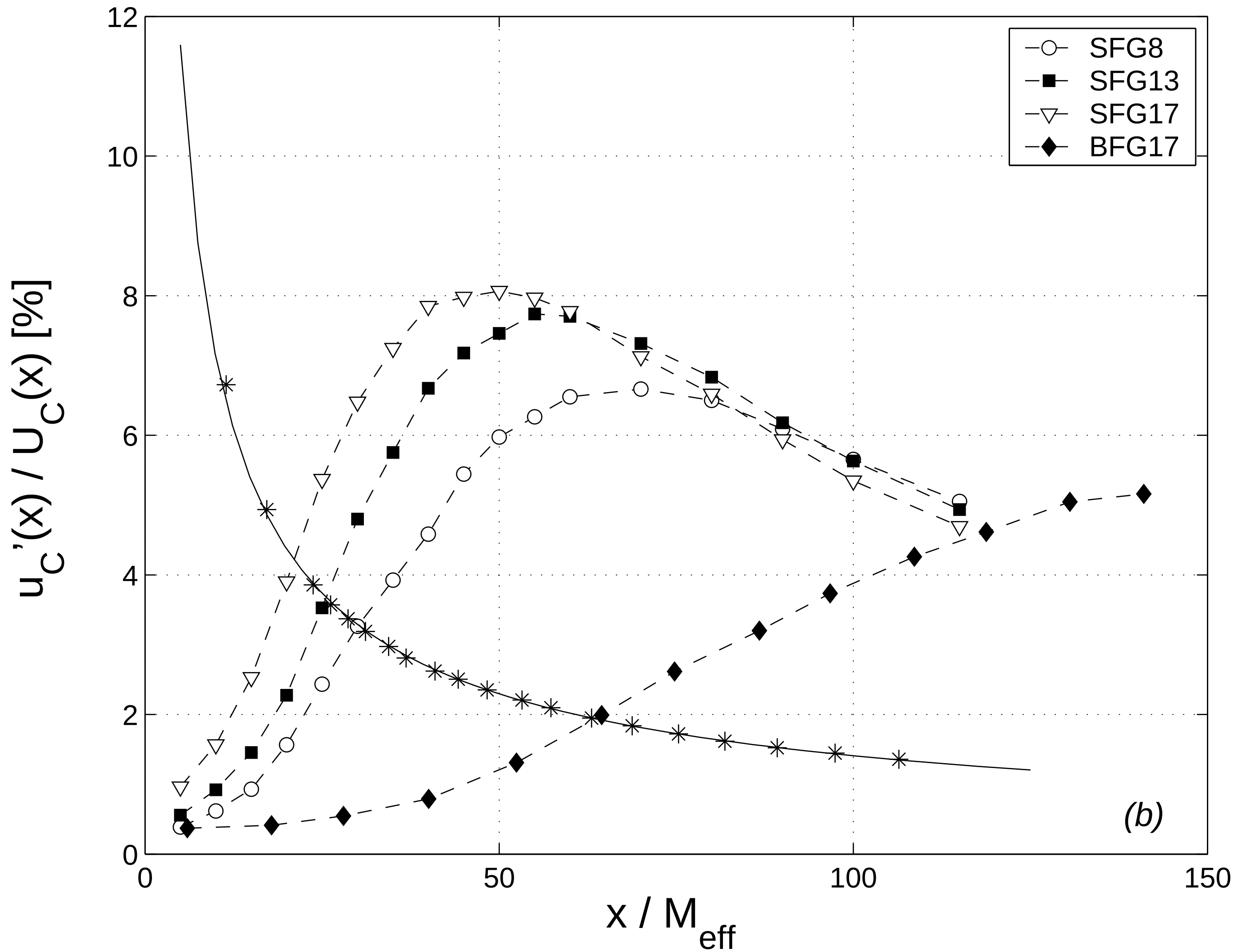
Recent grid & wake turbulence research

One main outcome

Grid and axisymmetric wake turbulence experiments have shown that, when $Re_I = \frac{U_\infty L_b}{\nu}$ is large enough, a significant turbulence decay region exists where $E_{xx}(k_x) \sim k_x^{-5/3}$ over wide range but $C_\epsilon \sim Re_I/Re_L \sim \sqrt{Re_I}/Re_\lambda$, i.e. $\epsilon \sim U_\infty L_b u'^2/L^2$, where $Re_L = \frac{u'L}{\nu}$ and $Re_\lambda = \frac{u'\lambda}{\nu}$.

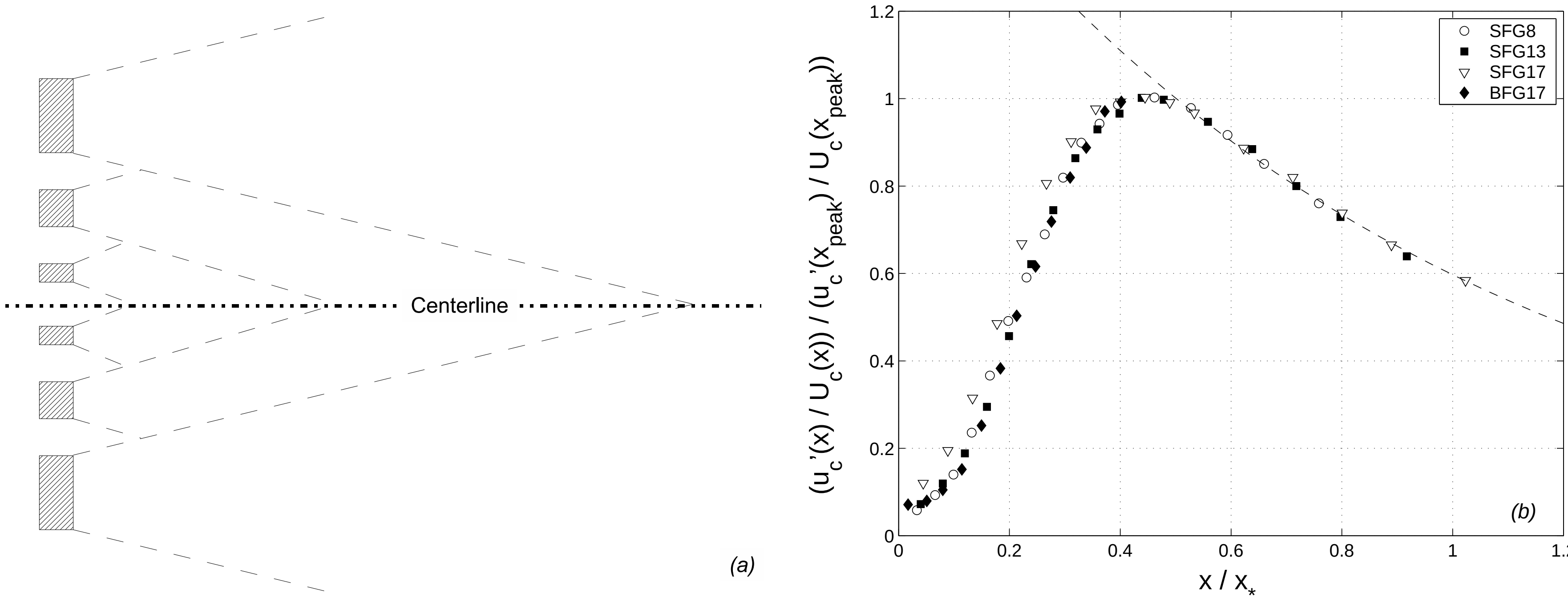
Seoud & V (PoF 2007), Mazellier & V (PoF 2010), Valente & V (JFM 2011, PRL 2012, JFM 2014), Gomes-Fernandes et al (JFM, 2012), Dairay, Obligado & V (JFM 2015) from our group; but also Nagata et al (PoF June 2013) from Nagoya, Japan; Discetti et al (FDR October 2013) from Arizona State, USA; Hearst & Lavoie (JFM 2014) from Toronto, Canada; Isaza et al (JFM 2014) from Cornell, USA.

Streamwise turbulence intensity



From Mazellier & V (PoF 2010)

Wake-interaction length-scale

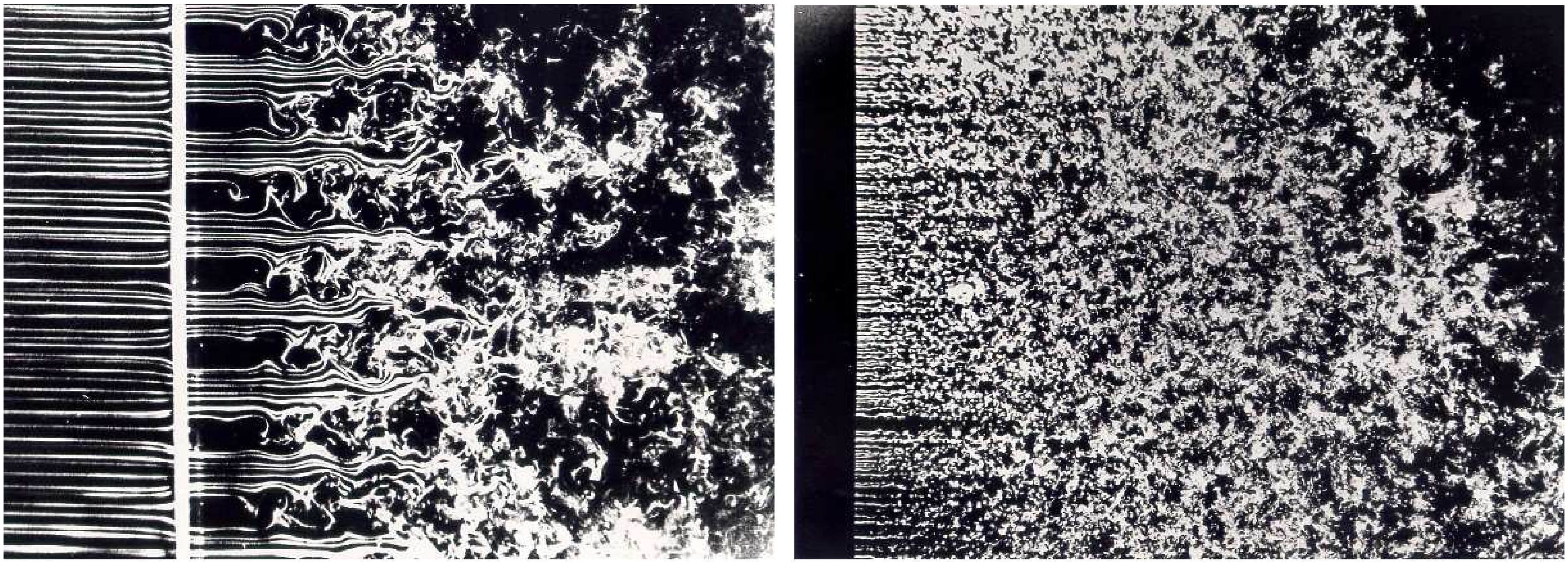


$(u'/U)/(u'/U)_{peak}$ versus x/x_* where $L_0 = \sqrt{t_0 x_*}$

$$x_{peak} \approx 0.45 x_*$$

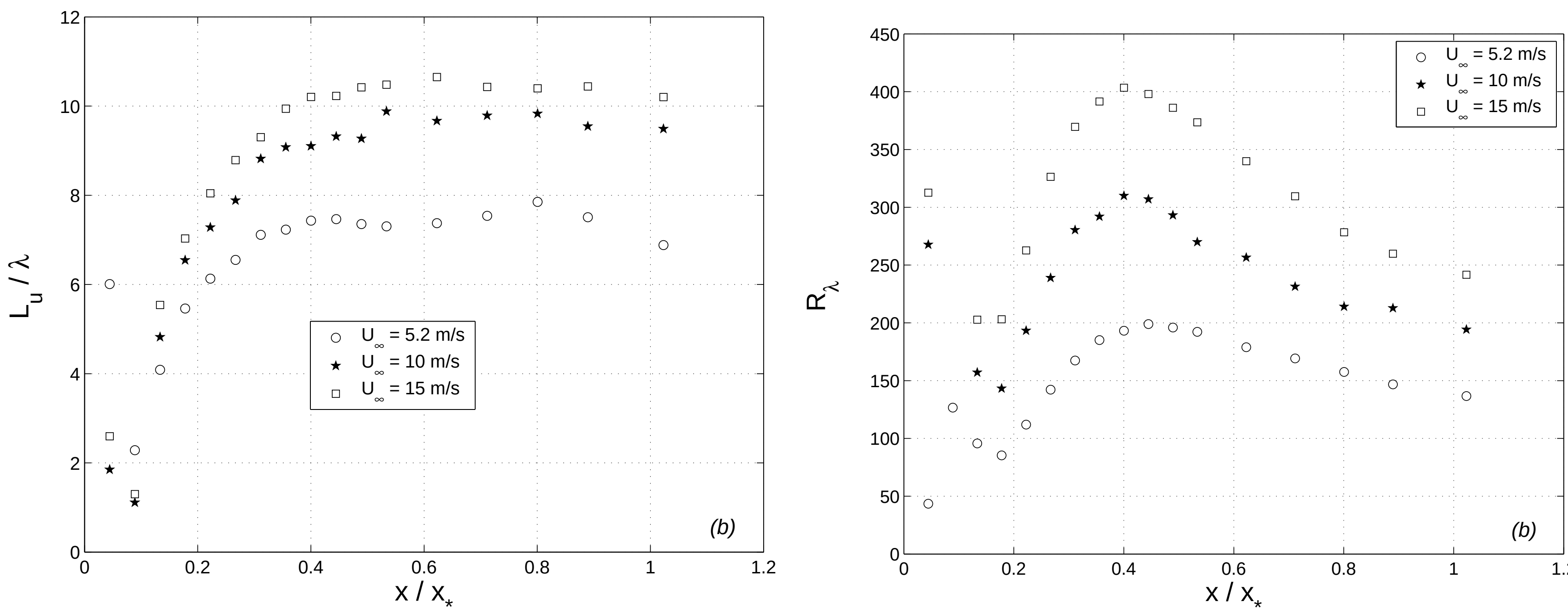
From Mazellier & V (PoF 2010)

Wake-interactions



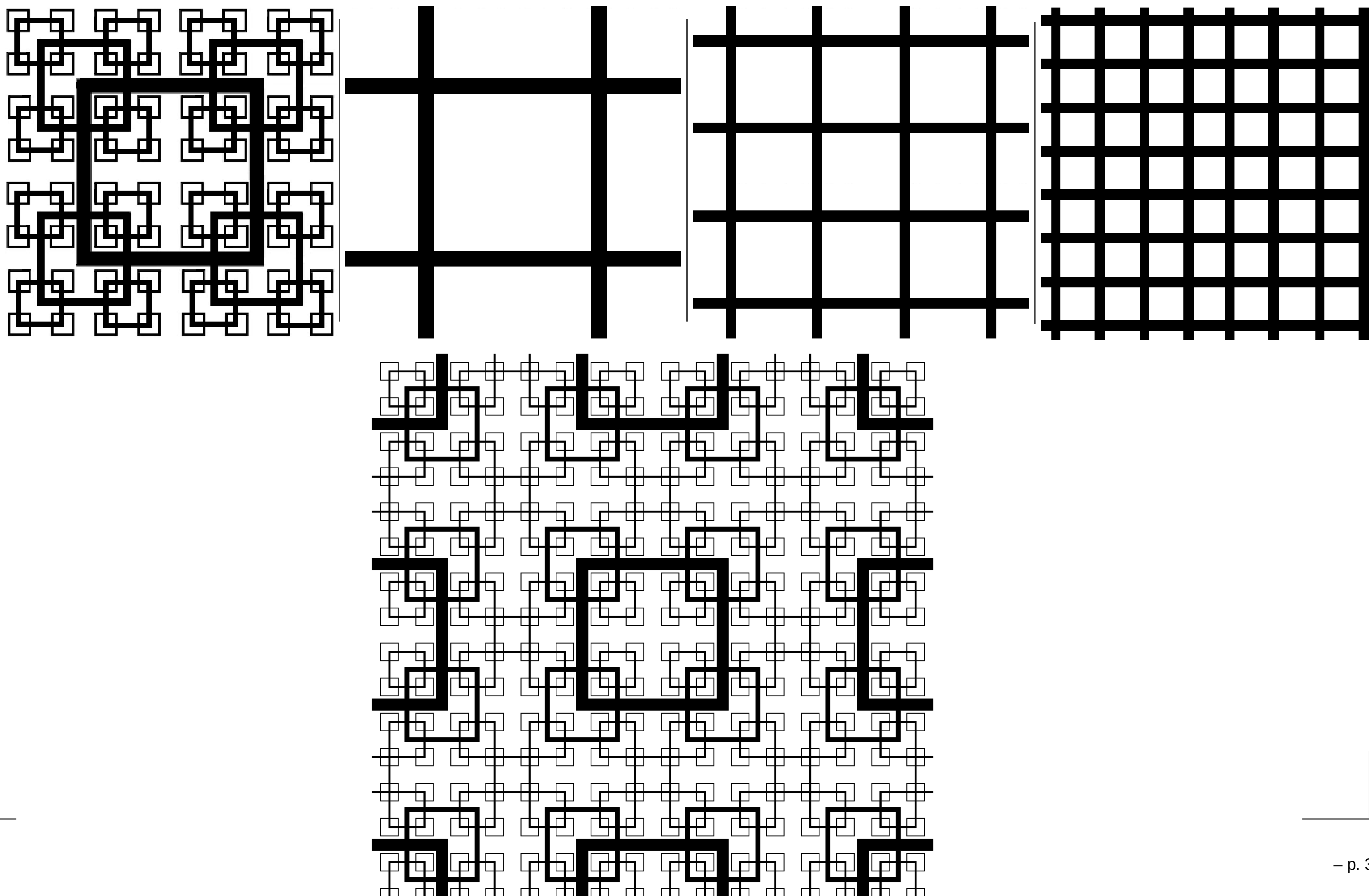
L_u/λ and Re_λ in FSG turbulence

L_u/λ is about constant where Re_λ decays

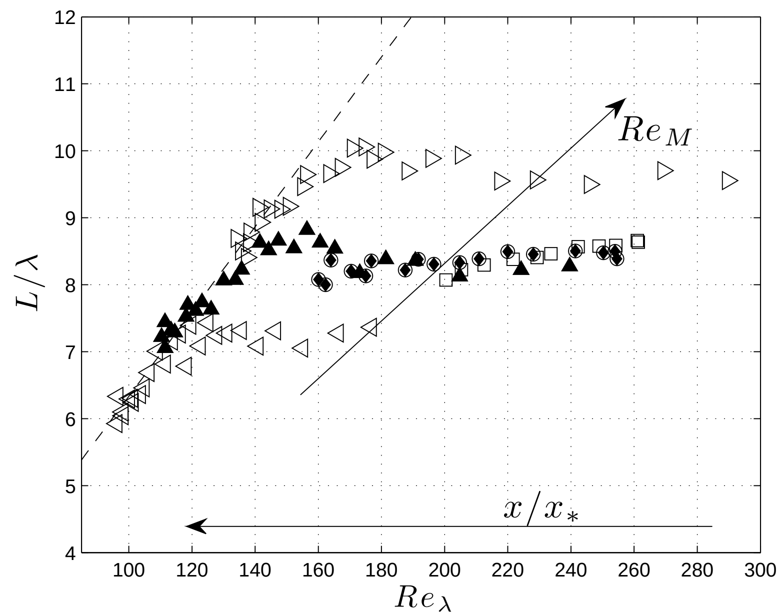


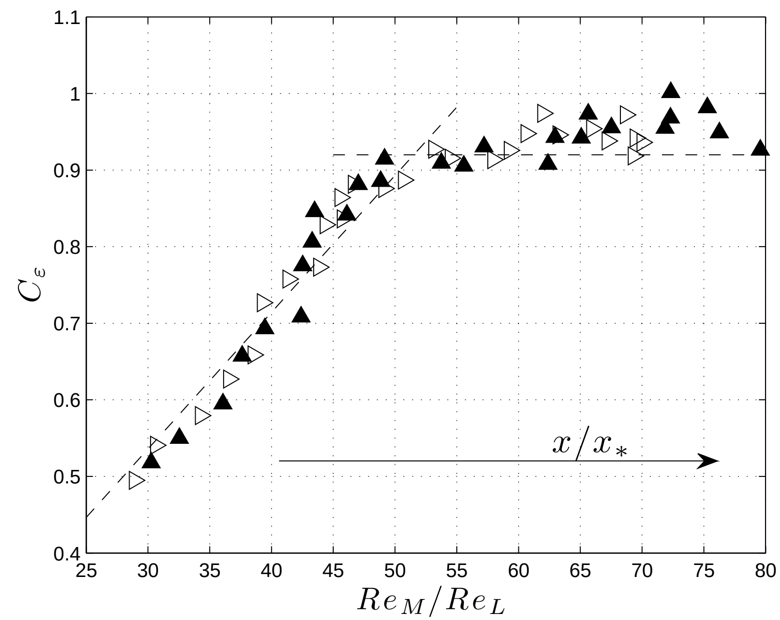
This approximately constant value is set by $Re_I = U_\infty L_b/\nu$.
The constant L/λ increases with increasing U_∞.
$L/\lambda = \frac{C_\epsilon}{15} Re_\lambda$, hence $L/\lambda = const$ iff $C_\epsilon \sim Const/Re_\lambda$.
From Mazellier & V (PoF 2010)

How universal is this?



$C_\epsilon \sim Re_I^m / Re_L^n$ **with** $m \approx 1 \approx n$



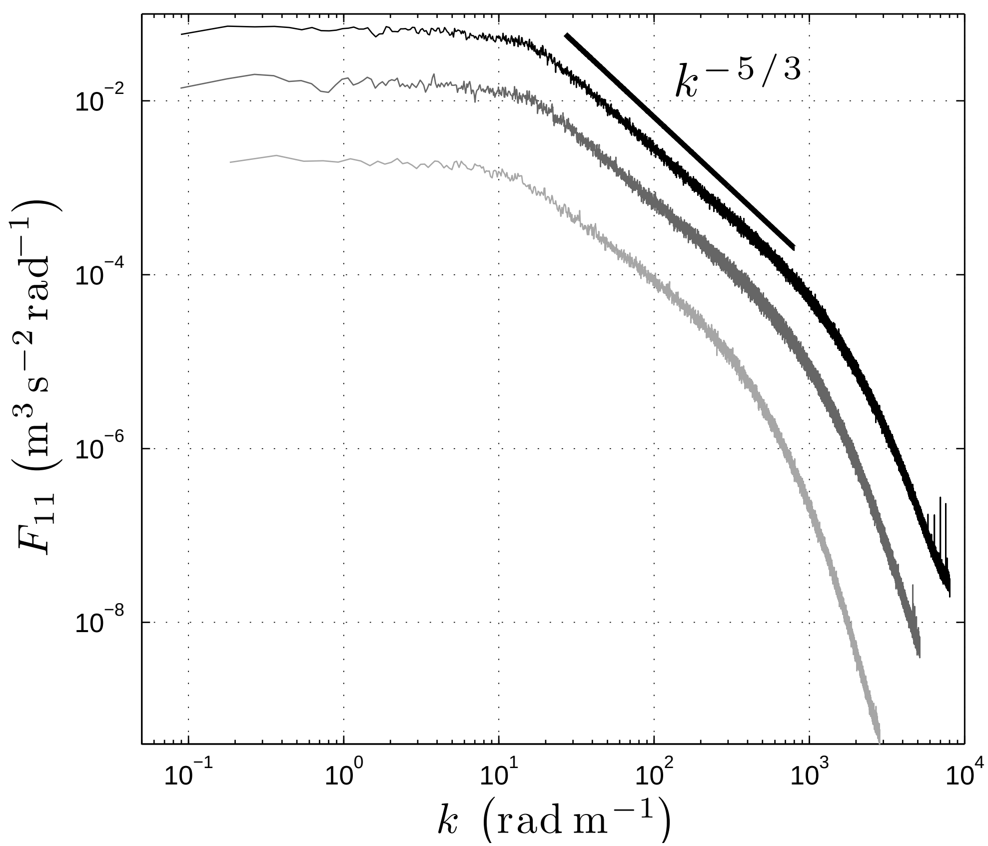


$C_\epsilon \sim Re_I^m / Re_L^n$ (plots above with $m = n = 1$) in the high-Re_L decay region followed by $C_\epsilon \sim const$ in the further downstream low-Re_L decay region.

Note low-Re_L far-downstream region $x > 5x_{peak} \approx 2x_*$ where $C_\epsilon \approx Const$ for RG60: 4 eddy turnovers from x_{peak} to $5x_{peak}$ and 3-4 eddy turnovers from $5x_{peak}$ to $24x_{peak}$.

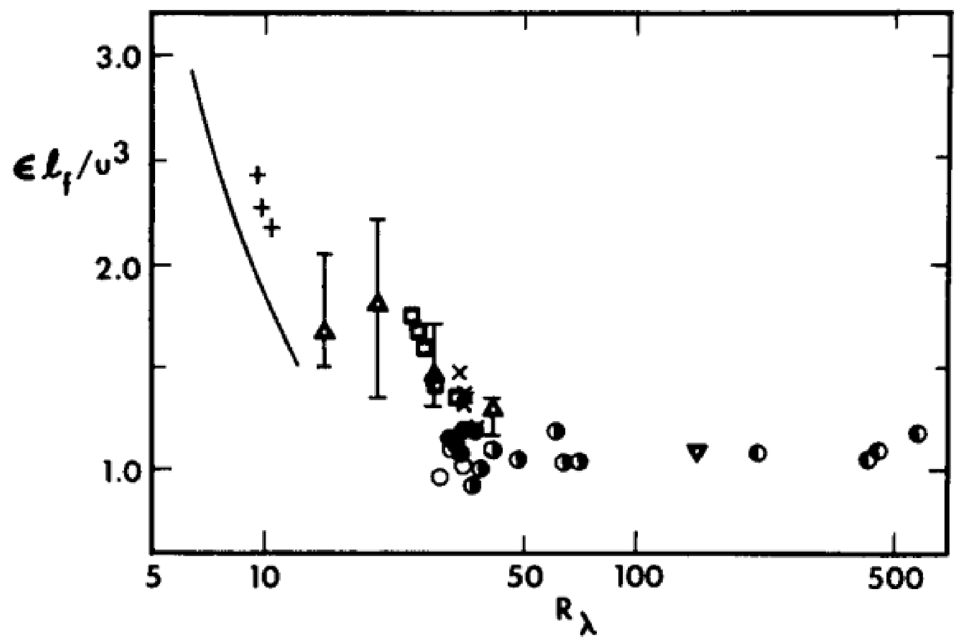
However wide near $-5/3$ at $x/x_* \approx 0.6$

The first region can be quite significant in length and is definitely the region with the (by far) best $E_{xx}(k_x) \sim k_x^{-5/3}$.



This and previous slide from Valente & V (PRL 2012)

Earlier evidence for $C_\epsilon \equiv \epsilon L/u'^3 = const$



Plot from Sreenivasan (1984): 4 highest Re_λ points are obtained by Kistler & Vebralovich (1966) at same point x by varying Re_I. $C_\epsilon \sim Re_I/Re_L$ would then not show up and give 4 points with same value as Re_L at that point would change in proportion to Re_I.

Summary of wind tunnel C_ϵ results

In decay region of fractal/regular grid turbulence and axisymmetric wakes, $C_\epsilon \sim Re_I/Re_L \sim \sqrt{Re_I}/Re_\lambda$ with very clear near $-5/3$ energy spectra over decade or more.

This occurs where $x_{peak} < x < x_e$ (and $x_e \approx 5x_{peak}$ for RG60). Further downstream where the Reynolds number has decayed further, $C_\epsilon \approx const$ for RG60.

(It has been possible to check the far downstream constancy of C_ϵ only for RG60, our regular grid wih the smallest mesh. Test sections not long enough for the other grids.)

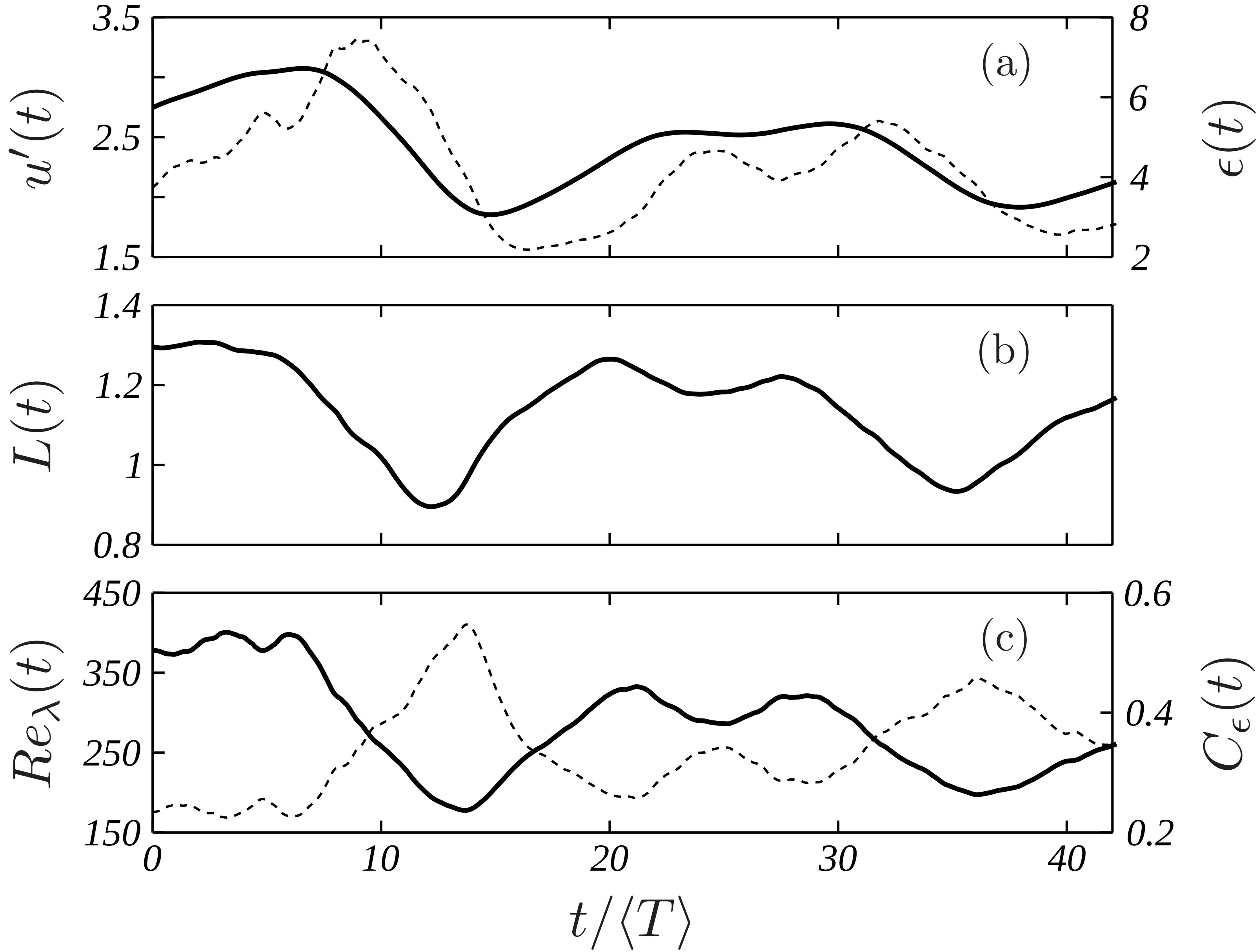
Note that the range x_{peak} to $5x_{peak}$ is about $2M$ to $10M$ in classical grids with $\sigma \approx 40$% whereas it is about $5M$ to $25M$ in our FSGs and RGs with few large meshes; there $\sigma \approx 25$% or lower

DNS of turbulence with periodic B.C.

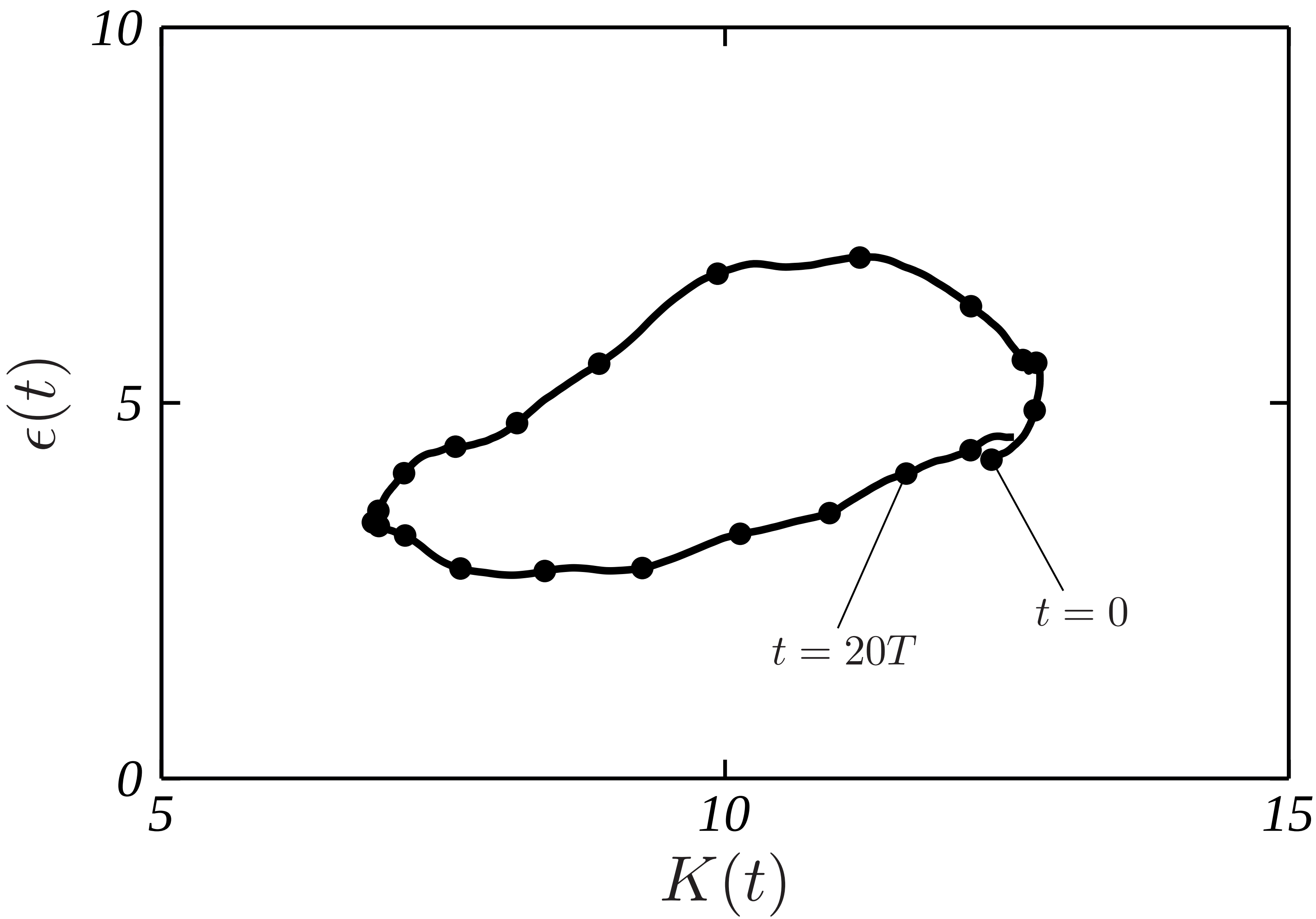
DNS OF SPATIALLY PERIODIC INCOMPRESSIBLE TURBULENCE WITH SPATIAL PERIOD l_B. BOTH DECAYING AND FLUCTUATING IN TIME.

Application of a steady force $\mathrm{f} = (\sin(2\pi mx/l_B)\cos(2\pi my/l_B), -\cos(2\pi mx/l_B)\sin(2\pi my/l_B), 0)$ (where m is an integrer) to the incompressible Navier-Stokes equations.

Cyclic turbulence



Cyclic turbulence

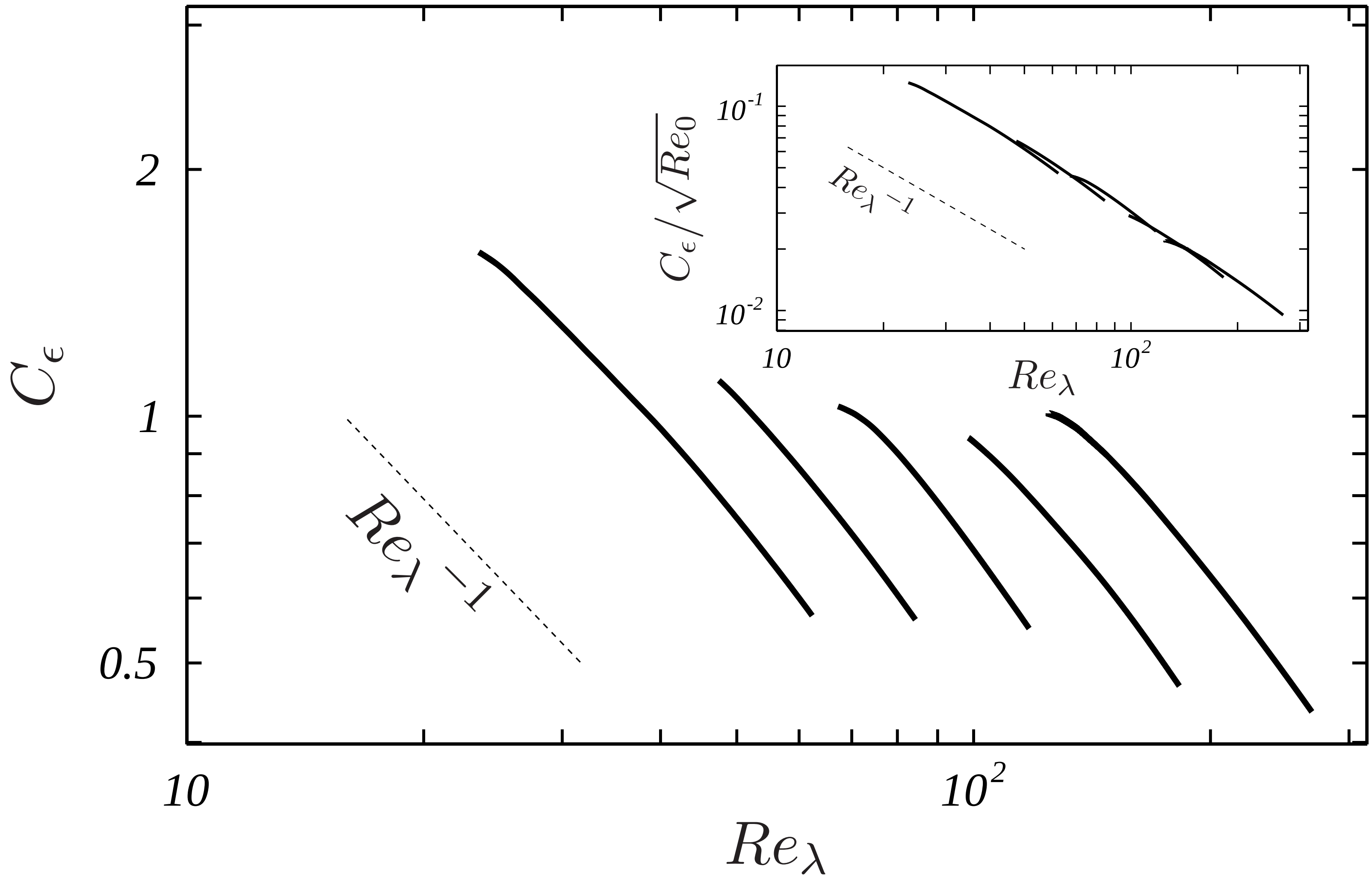


From Goto, Saito & Kawahara (2015)

Case I

Case I: Decaying turbulence
$m = 4$ and switched off forcing when $\epsilon(t)$ reached max.
$L(t) < l_B/10$ during relevant decay.
Considered 5 different values of initial Re_I corresponding to simulation sizes between 128^3 and 1024^3 for similar small-scale resolutions.

Decaying turbulence



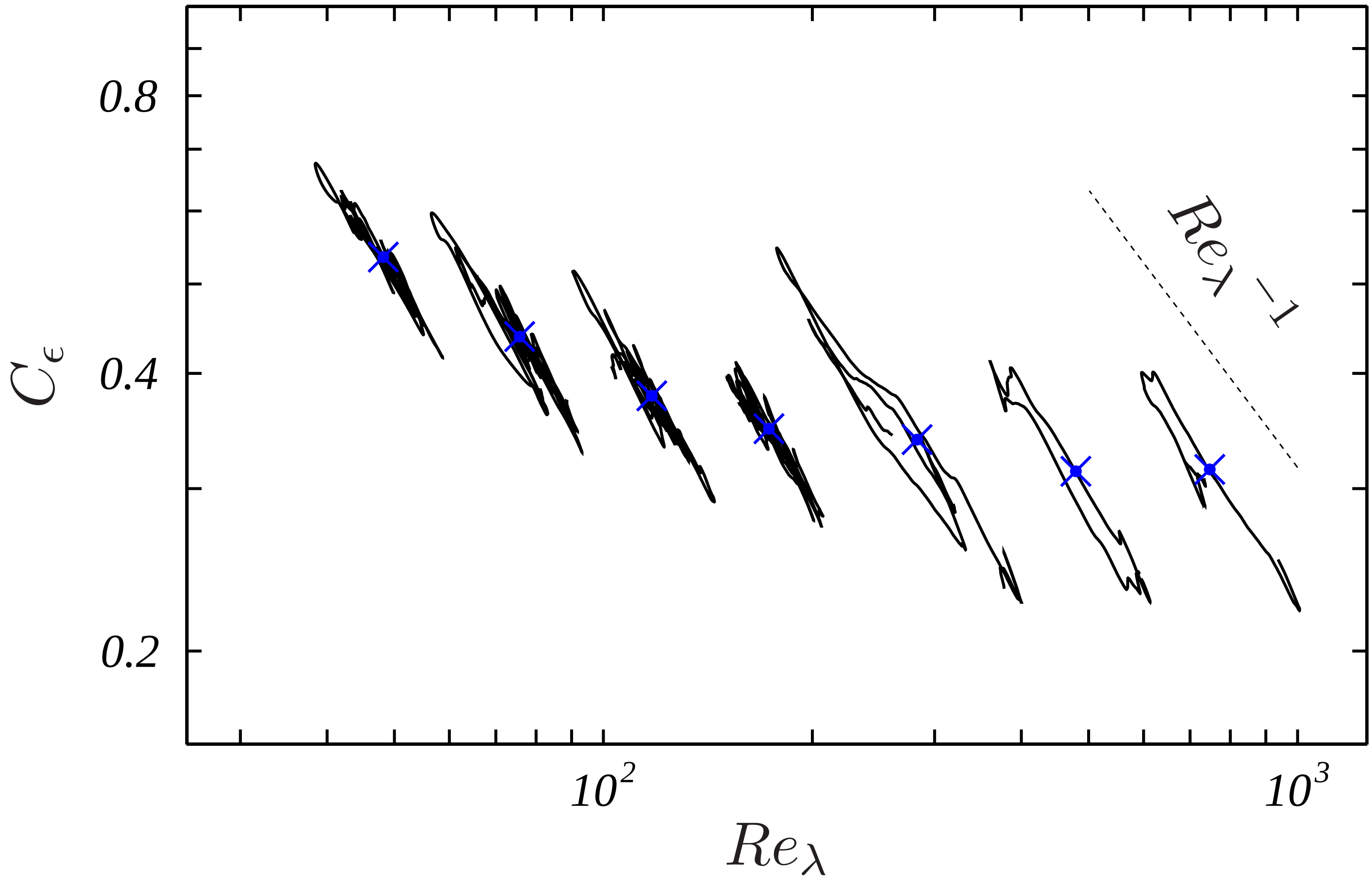
$$C_\epsilon \sim Re_I/Re_L \sim \sqrt{Re_I}/Re_\lambda$$

Case II

Higher Reynolds numbers: $m = 1$ and keep the forcing on throughout.

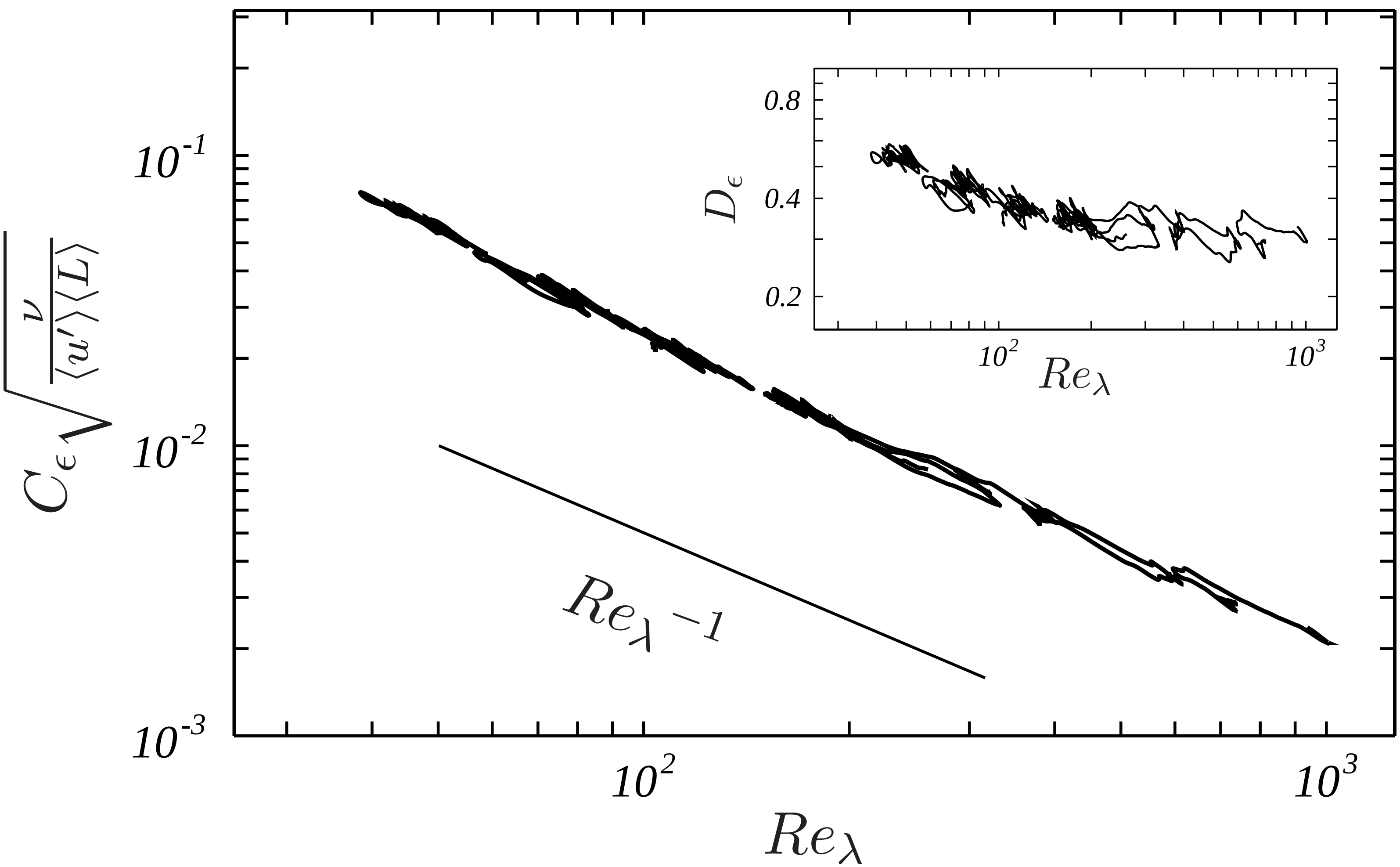
Considered 7 different values of global Re_I (based on the long-time averages of $u'(t)$ and $L(t)$) corresponding to simulation sizes between 64^3 and 2048^3 for similar small-scale resolutions.

Forced turbulence



$C_\epsilon(t) \sim Re_\lambda(t)^{-1}$ but $< \epsilon > \sim < u' >^3 / < L >$ as $Re_I \to \infty$

$C_\epsilon(t) \sim \sqrt{Re_I}/Re_\lambda$ again



$D_\epsilon \equiv C_\epsilon(t) Re_\lambda(t)/\sqrt{Re_I}$ tends to vary around a constant as Reynolds number tends to ∞

Interscale energy flux

$$\frac{\partial}{\partial t}E(k,t) = -\frac{\partial}{\partial k}\Pi(k,t) - 2\nu k^2 E(k,t)$$

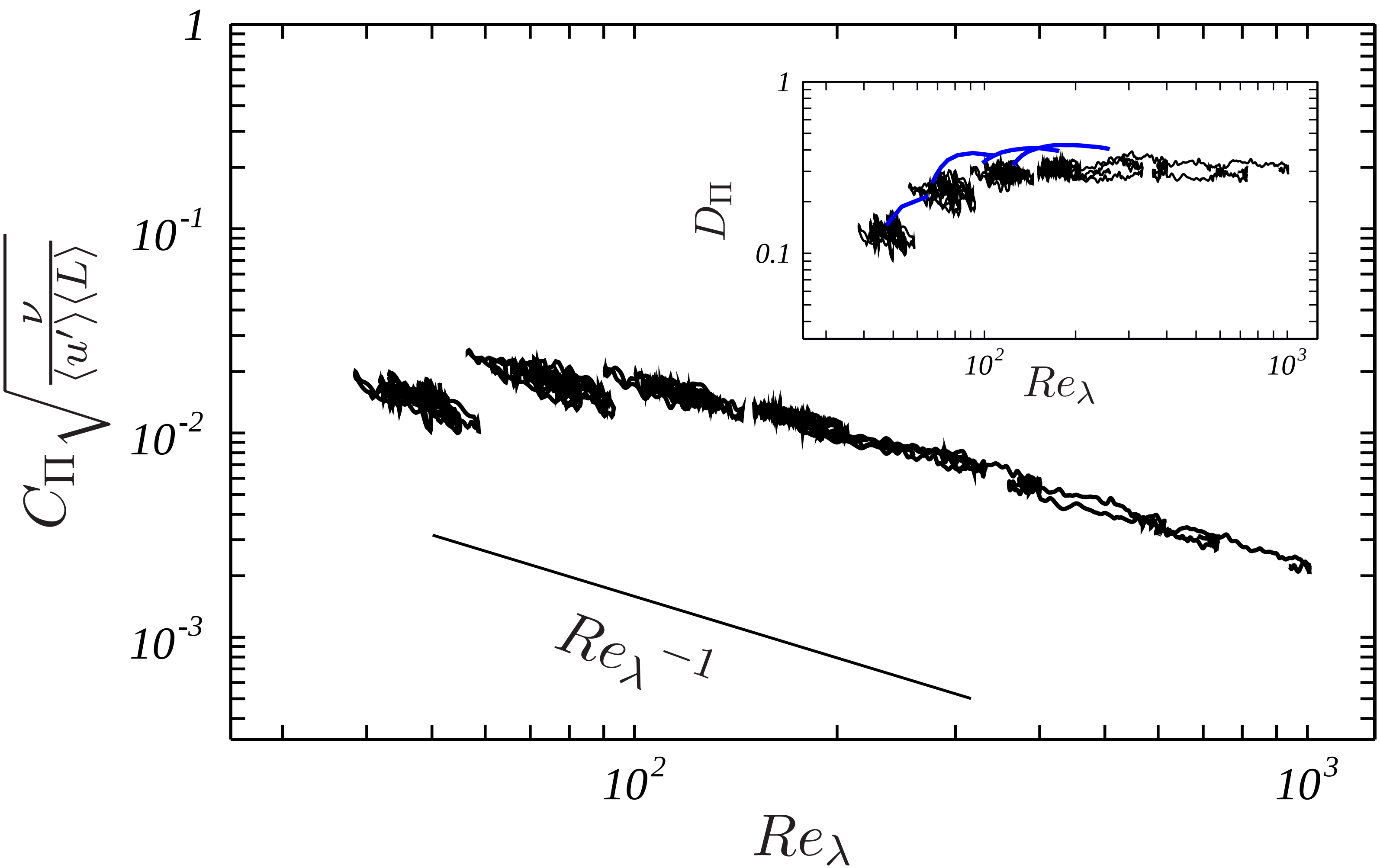
i.e.

$$\int_k^\infty \frac{\partial}{\partial t}E(k,t) \approx \Pi(k,t) - \epsilon(t)$$

for $1/L \ll k \ll 1/\lambda$.

Define $C_\Pi(k,t)$ by $\Pi(k,t) = C_\Pi(k,t)u'(t)^3/L(t)$ and calculate it for various values of k larger than k_f and smaller than $1/\lambda$.

Flux at $k = 5k_f$ ($k/k_f = 10, 20$ similar)



$$C_\Pi(k,t) = D_\Pi(k)\sqrt{Re_I}/Re_\lambda$$
$$\Pi(k,t) = D_\Pi(k)(\nu Re_I)u'(t)^2/L(t)^2$$

Conclusion for Periodic DNS study

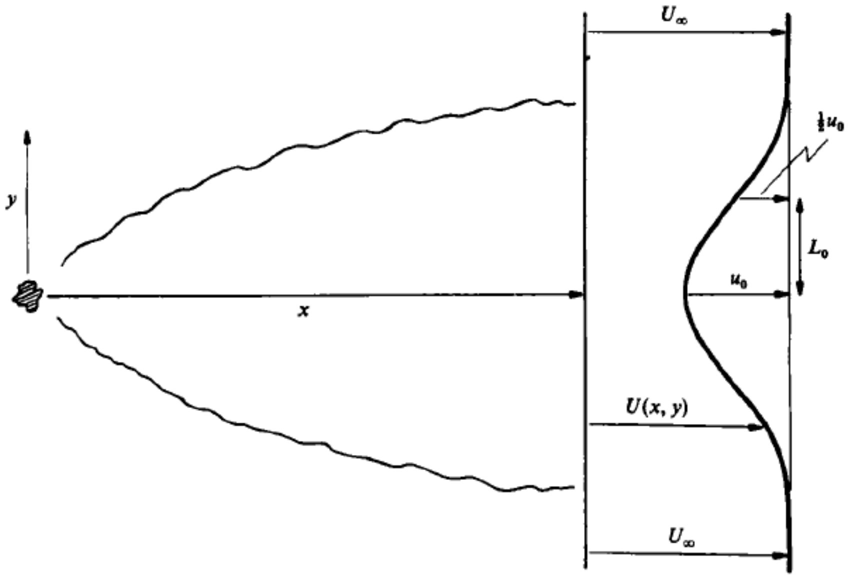
1. $\epsilon \sim (\nu Re_I) u'^2 / L^2$ i.e. $C_\epsilon \sim Re_I / Re_L \sim \sqrt{Re_I} / Re_\lambda$ also present in DNS of spatially periodic unsteady turbulence, as it is in turbulence generated by various types of grids and in axisymmetric turbulent wakes. DNS of forced periodic turbulence shows that this dissipation scaling does not only hold when the turbulence is decaying during the forced cycle but even when the turbulence is building up!

2. In these DNS, the interscale energy flux for intermediate wavenumbers scales in the same way, i.e. $\Pi(k,t) = D_\Pi(k)(\nu Re_I) u'(t)^2 / L(t)^2$

See Goto & V Phys. Lett. A **379**, 1144 (2015)

Consequences for self-preserving axisymmetric wakes?

Self-preserving turbulent wakes



$$U_\infty - U(x,r) = u_0(x) f[r/\delta(x)]$$

Usually one assumes that the wake becomes self-preserving at some point donwstream... and axisymmetry also helps...

Tennekes & Lumley 1972

The Reynolds-averaged streamwise momentum equation for an axisymmetric wake in a uniform and constant stream,

$$U_\infty \frac{\partial}{\partial x}(U_\infty - U) = -\frac{1}{r}\frac{\partial}{\partial r} r < u_x u_r >$$

has the general self-preserving solution

$$U_\infty - U = u_0(x) f[r/\delta(x)]$$

and

$$< u_x u_r >= u_0^2 g[r/\delta(x)]$$

under the conditions

$$\frac{d}{dx}\delta(x) \sim \frac{u_0}{U_\infty} \text{ and } u_0 \delta^2 \sim U_\infty \theta^2 = Const$$

where θ is the conserved momentum thickness.

2 conditions for 2 unknowns (δ, u_0), so get

$$u_0/U_\infty \sim (\frac{x - x_0}{\theta})^{-2/3} \text{ \& } \delta/\theta \sim (\frac{x - x_0}{\theta})^{1/3}$$

Townsend 1976

The Reynolds-averaged streamwise momentum equation for an axisymmetric wake in a uniform and constant stream,

$$U_\infty \frac{\partial}{\partial x}(U_\infty - U) = -\frac{1}{r}\frac{\partial}{\partial r} r < u_x u_r >$$

has the general self-preserving solution

$$U_\infty - U = u_0(x) f[r/\delta(x)]$$

but

$$< u_x u_r >= R_0(x) g[r/\delta(x)]$$

under the conditions

$$\frac{d}{dx}\delta(x) \sim \frac{R_0}{U_\infty u_0} \text{ and } u_0 \delta^2 \sim U_\infty \theta^2 = Const$$

where θ is the conserved momentum thickness.

2 conditions for 3 unknowns (δ, u_0, R_0), so make use of kinetic energy equation.

Townsend 1976, George 1989

The Reynolds-averaged kinetic energy equation for an axisymmetric wake in a uniform and constant stream,

$$U_\infty \frac{\partial}{\partial x} K = - < u_x u_r > \frac{\partial U}{\partial r} + Transport - \epsilon$$

has the general self-preserving solution

$$K(x,r) = K_0(x) k[r/\delta(x)],$$

$$Transport = T_0(x) t[r/\delta(x)]$$

and

$$\epsilon = D_0(x) e[r/\delta(x)]$$

under the additional conditions

$$\frac{d}{dx}\delta(x) \sim \frac{T_0 \delta}{K_0 U_\infty} \sim \frac{D_0 \delta}{K_0 U_\infty} \text{ and } K_0 \sim u_0^2$$

IN TOTAL: 5 conditions for 6 unknowns δ, u_0, R_0, K_0, T_o and D_0. So need one more relation:

$$D_0 \sim K_0^{3/2} / \delta$$

Taylor 1935, Kolmogorov 1941

$D_0 \sim K_0^{3/2}/\delta$ is effectively $\epsilon \sim K^{3/2}/L$ (introduced in a single off the cuff sentence by Taylor in 1935 and given a theoretical basis by Kolmogorov's equilibrium cascade in 1941) but adapted to self-preserving turbulent shear flows with the assumption that $\delta \sim L$

The Townsend-George approach implies

$$u_0/U_\infty \sim \left(\frac{x-x_0}{\theta}\right)^{-2/3}$$

&

$$\delta/\theta \sim \left(\frac{x-x_0}{\theta}\right)^{1/3}$$

like Tennekes & Lumley (1972).

Equivalent theories?

$$u_0/U_\infty \sim (\frac{x-x_0}{\theta})^{-2/3} \text{ \& } \delta/\theta \sim (\frac{x-x_0}{\theta})^{1/3}$$

follow from both approaches. Does it mean that the two approaches are effectively equivalent except that Townsend (1976) and George (1989) predict $K_0 \sim u_0^2$ whereas Tennekes & Lumley (1972) have no say on K_0?

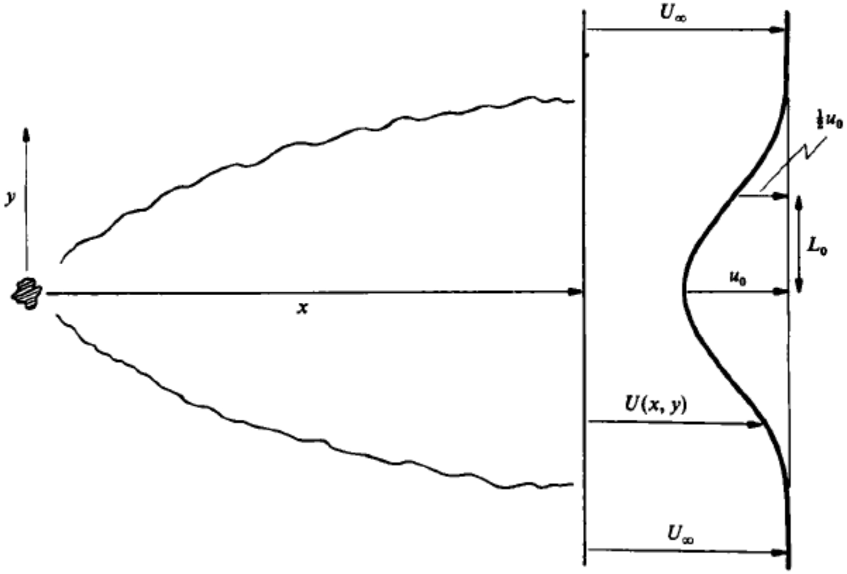
Well....not quite because $R_0 \sim u_0^2$ for Tennekes & Lumley (1972) but actually George (1989) predicts

$$R_0 \sim U_\infty u_0 \frac{d}{dx}\delta$$

But so what? Given the scalings of u_0 and δ there is no way to distinguish between $R_0 \sim u_0^2$ and $R_0 \sim U_\infty u_0 \frac{d}{dx}\delta$.

Unless the turbulence dissipation does not scale as in the Richarson-Kolmogorov equilibrium theory...

Self-preserving turbulent wakes



$$U_\infty - U(x,r) = u_0(x) f[r/\delta(x)]$$

Self-preserving axisymmetric turbulent wakes beyond some point downstream.

Townsend 1976

The Reynolds-averaged streamwise momentum equation for an axisymmetric wake in a uniform and constant stream,

$$U_\infty \frac{\partial}{\partial x}(U_\infty - U) = -\frac{1}{r}\frac{\partial}{\partial r} r < u_x u_r >$$

has the general self-preserving solution

$$U_\infty - U = u_0(x) f[r/\delta(x)]$$

but

$$< u_x u_r >= R_0(x) g[r/\delta(x)]$$

under the conditions

$$\frac{d}{dx}\delta(x) \sim \frac{R_0}{U_\infty u_0} \text{ and } u_0 \delta^2 \sim U_\infty \theta^2 = Const$$

where θ is the conserved momentum thickness.

2 conditions for 3 unknowns (δ, u_0, R_0), so make use of kinetic energy equation.

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$$U_\infty \frac{\partial}{\partial x} K = - < u_x u_r > \frac{\partial U}{\partial r} + Transport - \epsilon$$

has the general self-preserving solution

$$K(x,r) = K_0(x) k[r/\delta(x)],$$

$$Transport = T_0(x) t[r/\delta(x)]$$

and

$$\epsilon = D_0(x) e[r/\delta(x)]$$

under the additional conditions

$$\frac{d}{dx}\delta(x) \sim \frac{T_0 \delta}{K_0 U_\infty} \sim \frac{D_0 \delta}{K_0 U_\infty} \text{ and } K_0 \sim u_0^2$$

IN TOTAL: 5 conditions for 6 unknowns δ, u_0, R_0, K_0, T_o and D_0. So need one more relation:

$$D_0 \sim K_0^{3/2}/\delta \text{ or } D_0 \sim U_\infty L_b K_0/\delta^2?$$

Which dissipation scaling to apply?

$C_\epsilon = const$, i.e. $D_0 \sim K_0^{3/2}/\delta$
implies $u_0/U_\infty \sim (\frac{x-x_0}{\theta})^{-2/3}$ & $\delta/\theta \sim (\frac{x-x_0}{\theta})^{1/3}$

and CANNOT distinguish between $R_0 \sim u_0^2$ and
$R_0 \sim U_\infty u_0 \frac{d}{dx}\delta$.

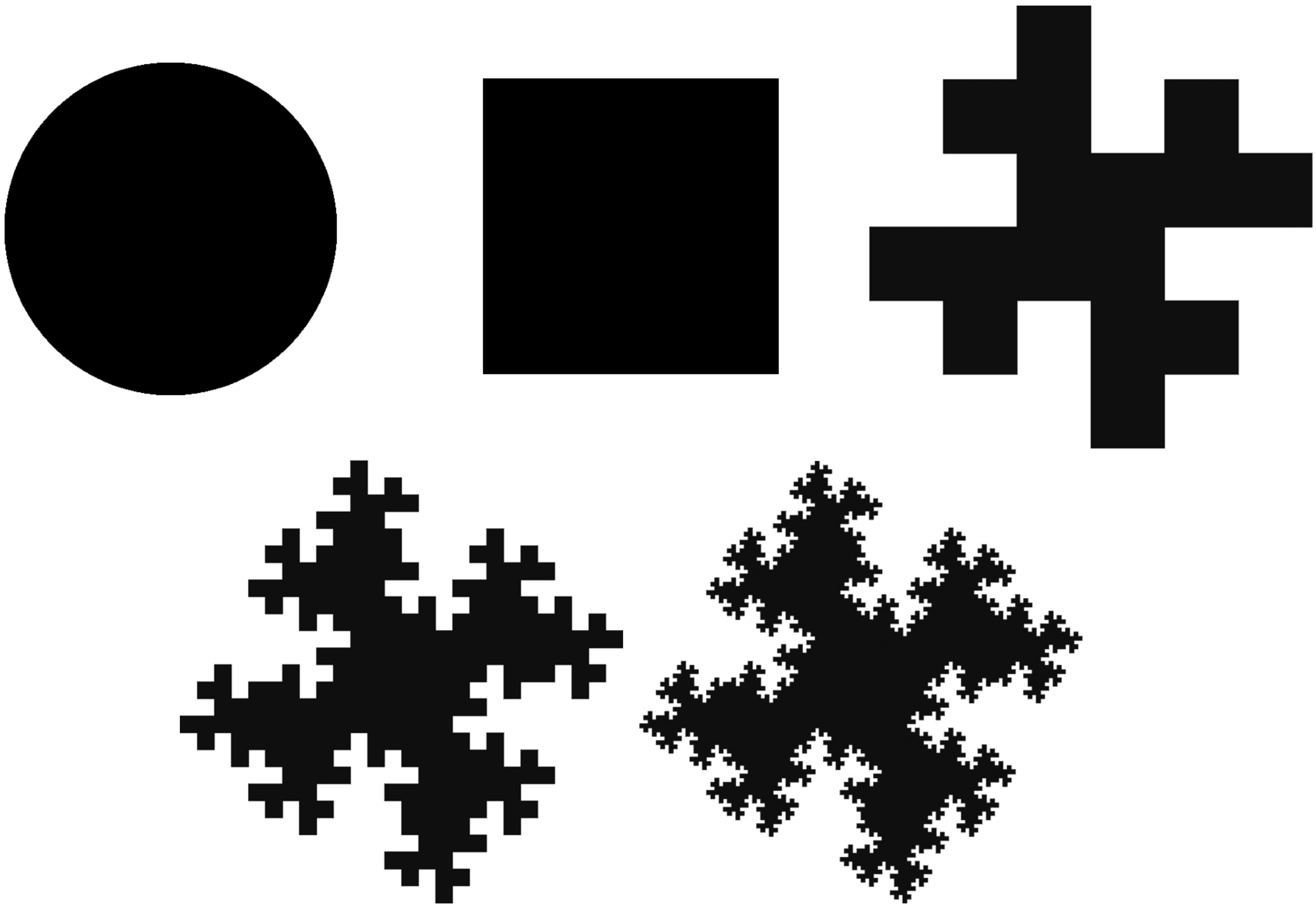
$C_\epsilon \approx Re_I/Re_L$, i.e. $D_0 \sim U_\infty L_b K_0/\delta^2$
implies
$u_0/U_\infty \sim (\frac{x-x_0}{\theta})^{-1}(L_b/\theta)^{-1}$ & $\delta/\theta \sim (\frac{x-x_0}{\theta})^{1/2}(L_b/\theta)^{1/2}$

and CAN distinguish between $R_0 \sim u_0^2$ and $R_0 \sim U_\infty u_0 \frac{d}{dx}\delta$.

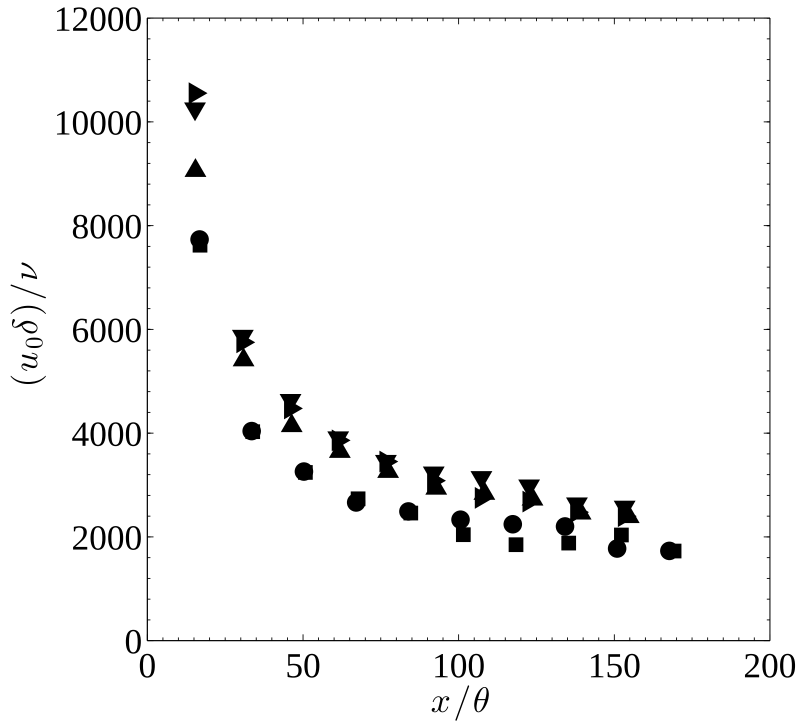
Note: δ is a measure of the wake width and is taken to be δ where $\delta^2 = \int_0^\infty \frac{U_\infty - U}{u_0} r dr$.

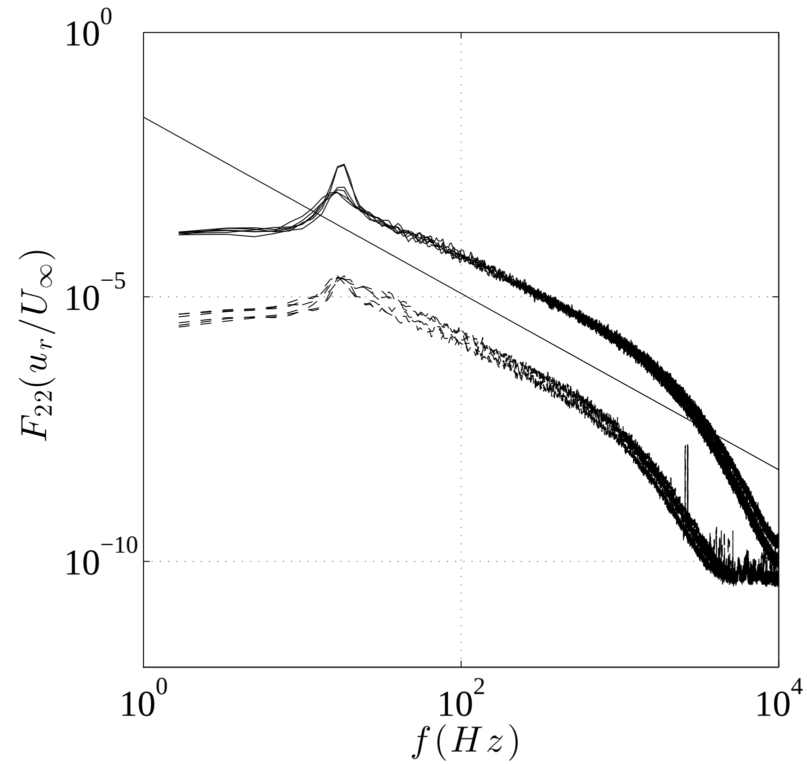
Wakes of flat plates normal to U_∞

all with equal surface area A. Here $L_b \equiv \sqrt{A}$.



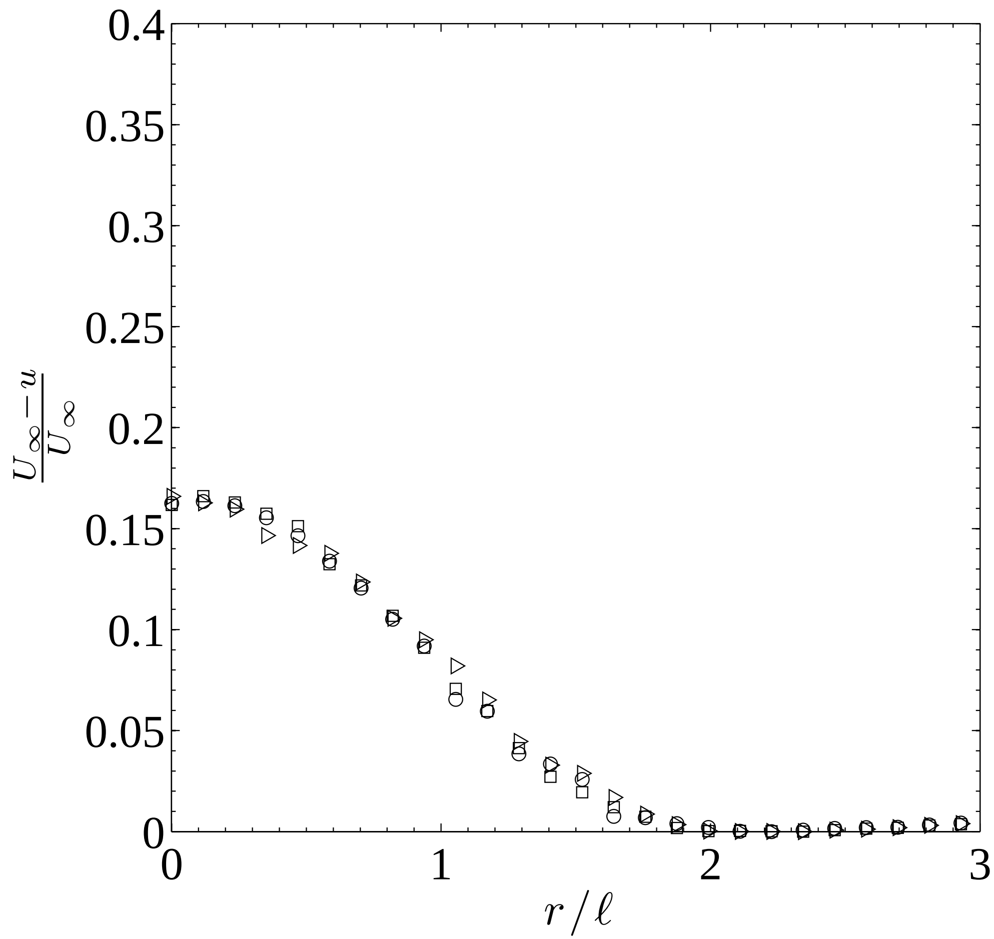
Reynolds numbers and spectra





Along centreline.

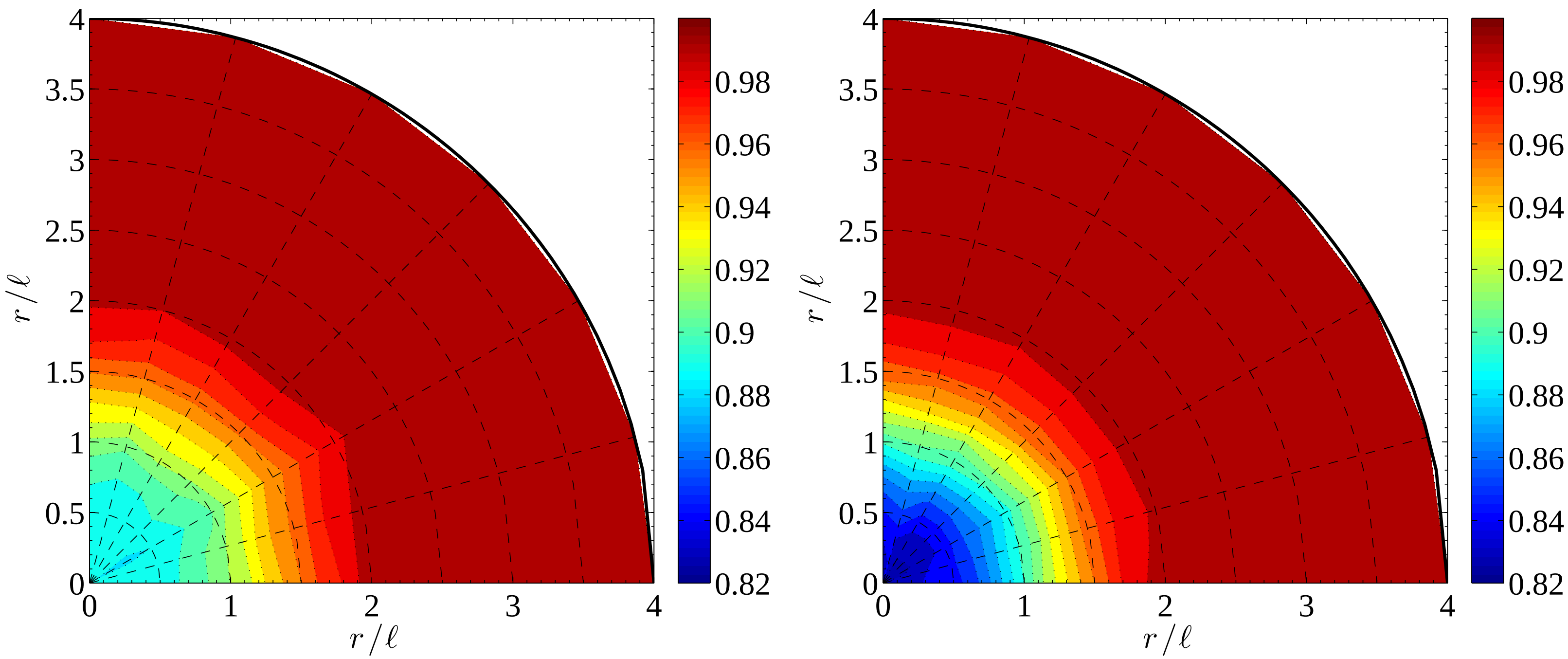
Approximate axisymmetry at $x \geq 10L_b$



2nd iteration “fractal” plate

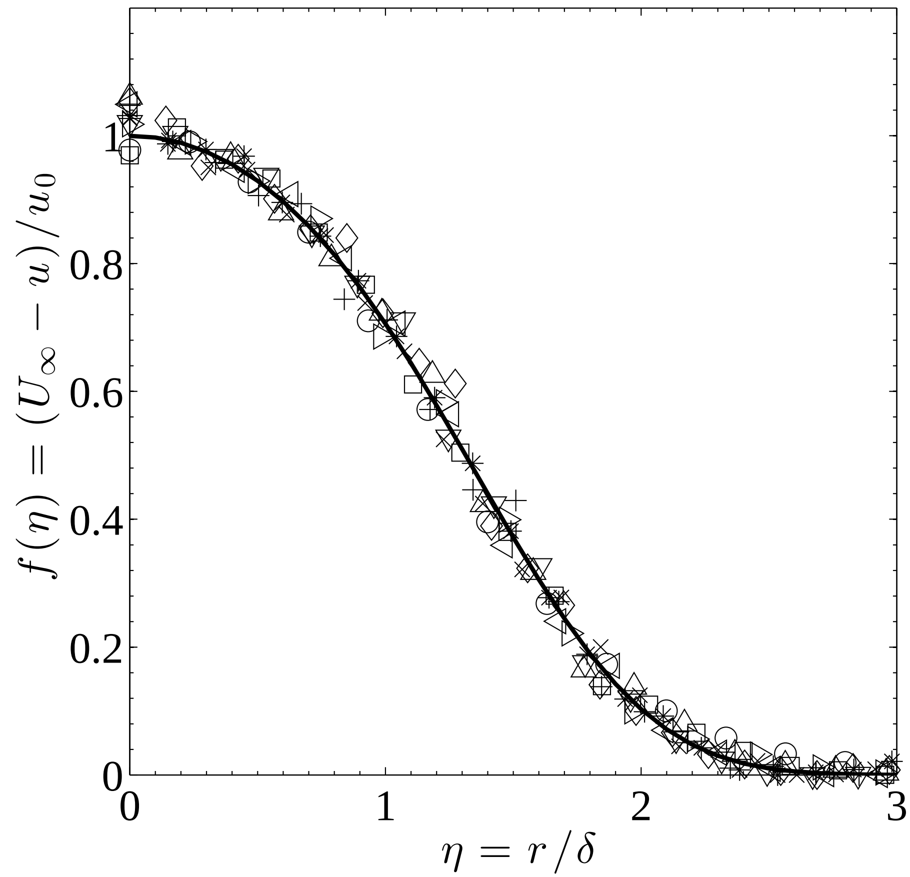
Profiles taken at 0^o, 30^o and 60^0 angles.

Approximate axisymmetry at $x \geq 10L_b$



Mean flow U/U_∞ profile at $x = 10L_b$ for disk (left) and 2nd iteration “fractal” plate (right)

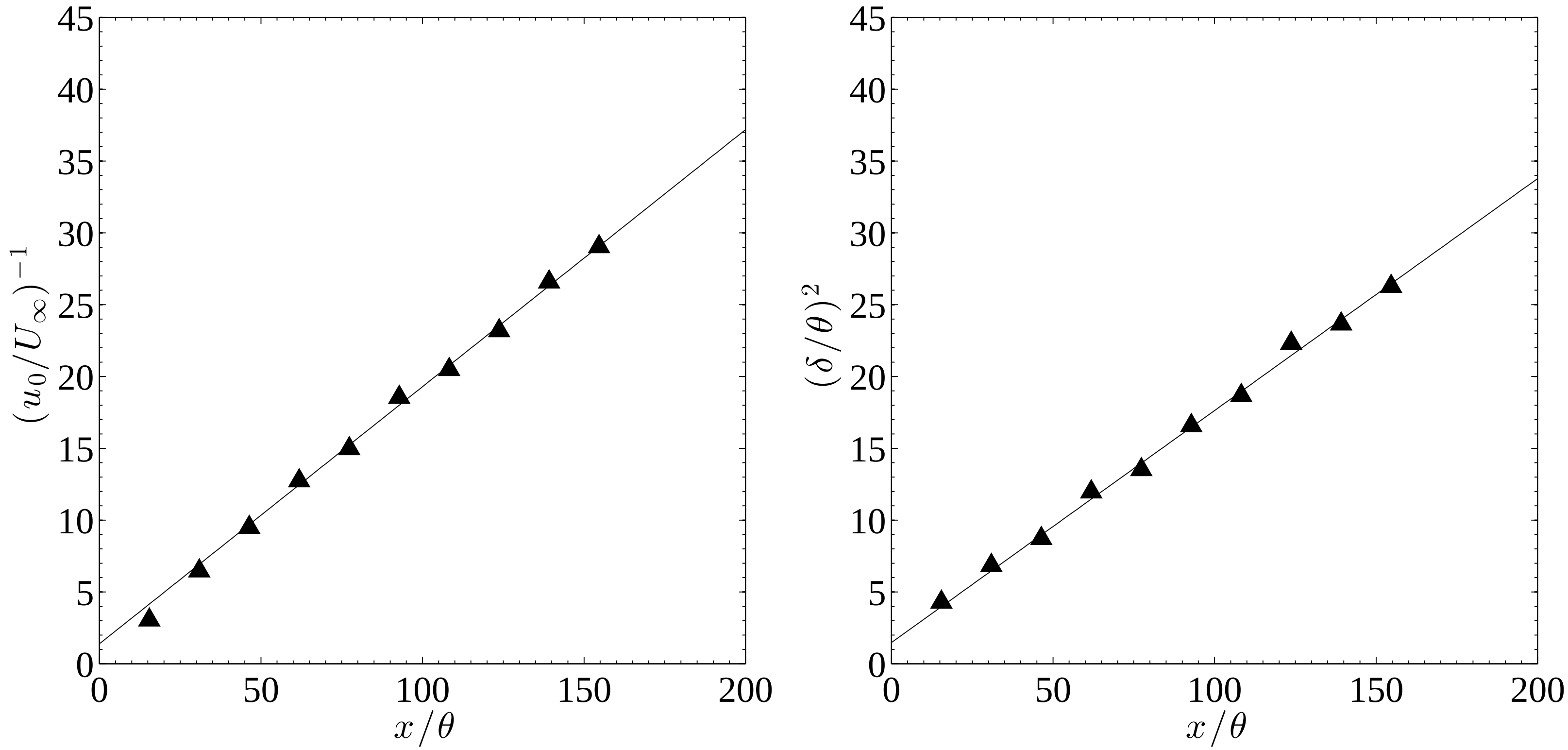
Self-similarity with $\delta^2 \equiv \int_0^\infty \frac{U_\infty - U}{u_0} r dr$



Data for 2nd iteration "fractal" plate at

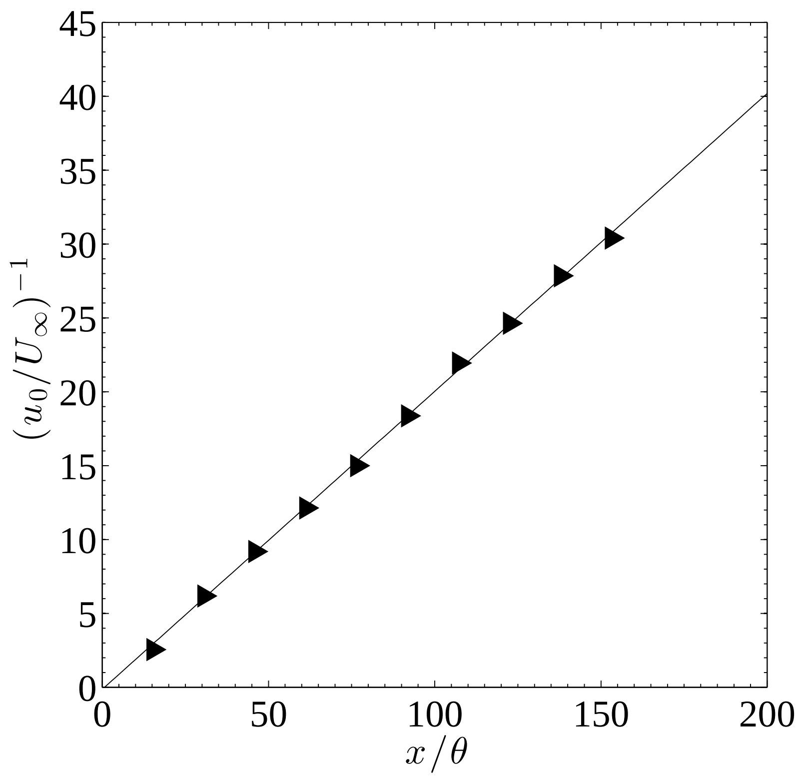
$x = 5L_b, 10L_b, 15L_b, 20L_b, 25L_b, 30L_b, 35L_b, 40L_b, 45L_b, 50L_b$

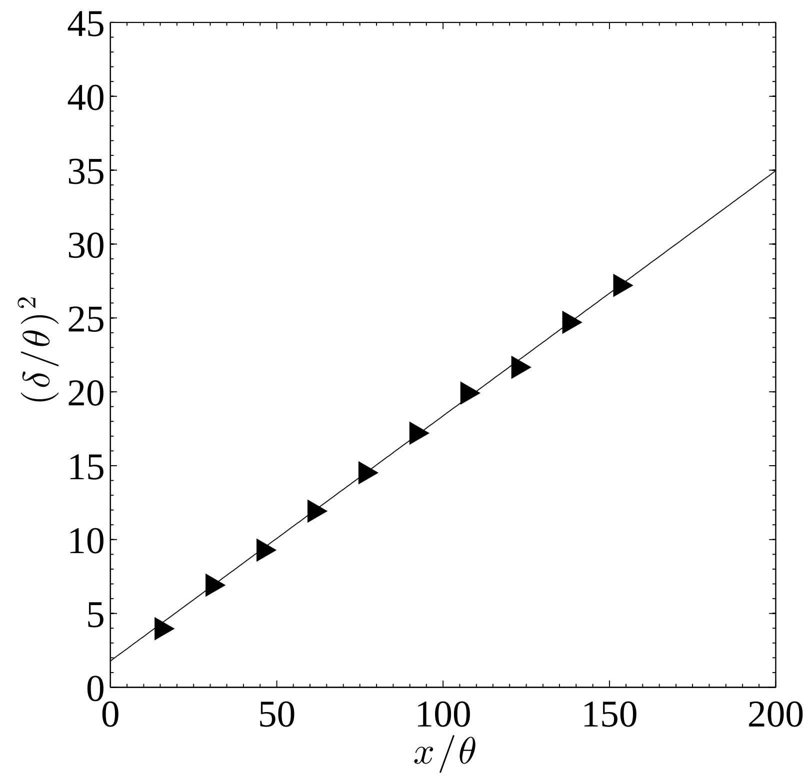
1st iteration "fractal" plate



$(u_0/U_\infty)^{-1}$ and $(\delta/\theta)^2$ vary linearly with streamwise distance x from plate (over x-range considered) in agreement with $u_0/U_\infty \sim (\frac{x-x_0}{\theta})^{-1}$ & $\delta/\theta \sim (\frac{x-x_0}{\theta})^{1/2}$ which follows from the new dissipation law.

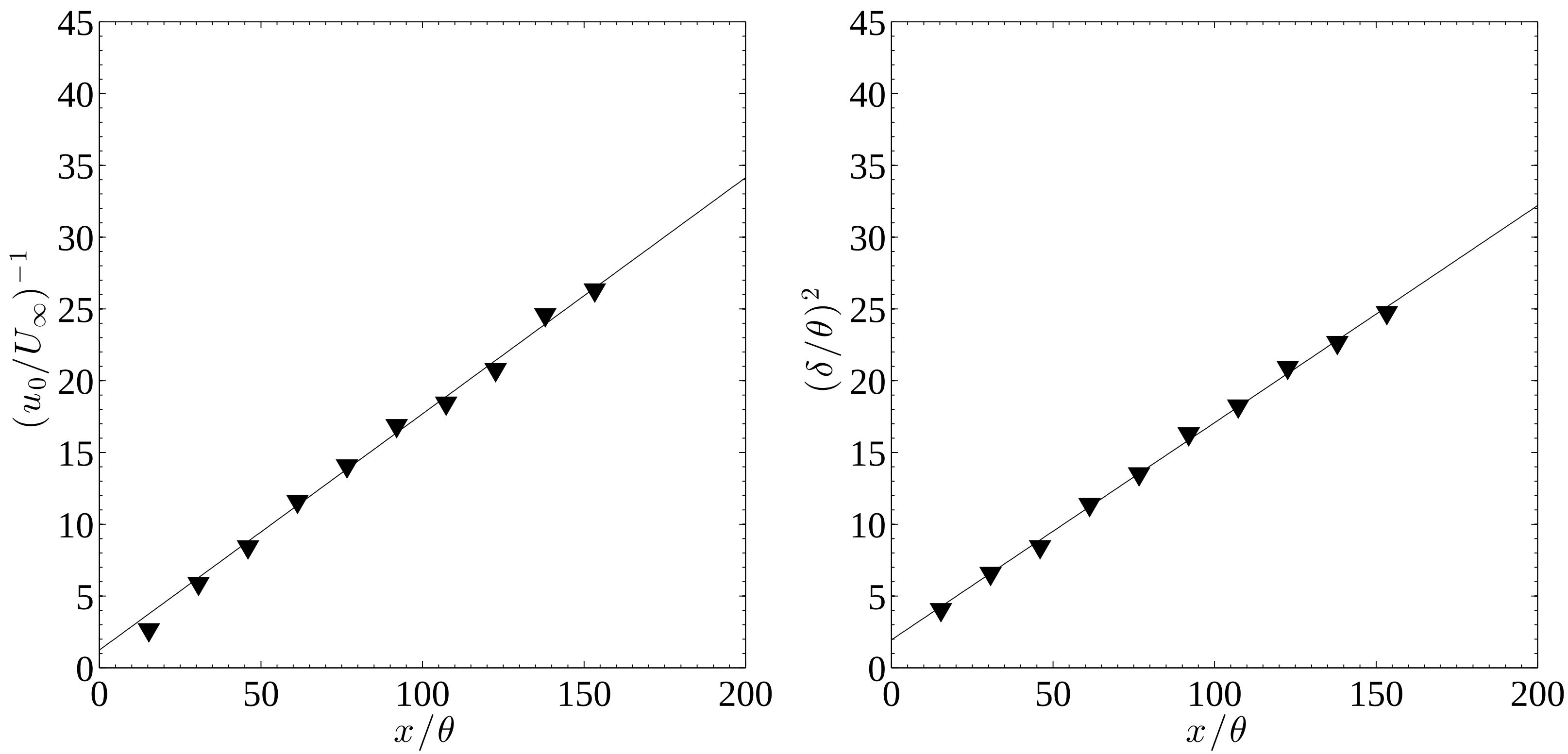
2nd iteration "fractal" plate





$(u_0/U_\infty)^{-1}$ and $(\delta/\theta)^2$ vary linearly with streamwise distance x from plate (over x-range considered) in agreement with $u_0/U_\infty \sim (\frac{x-x_0}{\theta})^{-1}$ & $\delta/\theta \sim (\frac{x-x_0}{\theta})^{1/2}$ which follows from the new dissipation law.

3d iteration “fractal” plate



$(u_0/U_\infty)^{-1}$ and $(\delta/\theta)^2$ vary linearly with streamwise distance x from plate (over x-range considered) in agreement with $u_0/U_\infty \sim (\frac{x-x_0}{\theta})^{-1}$ & $\delta/\theta \sim (\frac{x-x_0}{\theta})^{1/2}$ which follows from the new dissipation law.

$$u_0/U_\infty = A(\frac{x-x_0}{\theta})^{-\alpha} \textbf{ and } \delta/\theta = B(\frac{x-x_0}{\theta})^{\beta}$$

Use a MATLAB^{TM} nonlinear least-squares regression algorithm to determine coefficients, exponents and virtual origins.

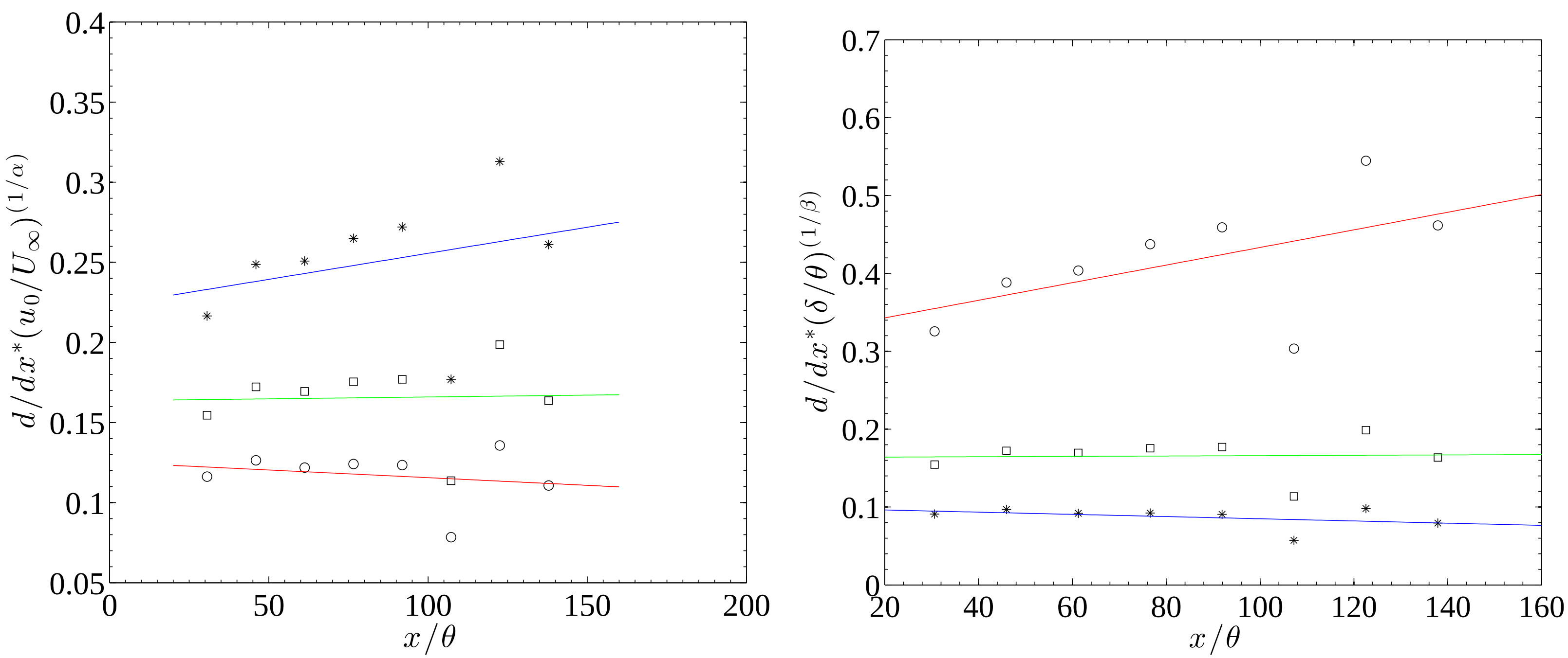
	A	$-x_{0A}/\theta$	α	B	$-x_{0B}/\theta$	β
1.5(1)	21.53	-28.31	1.22	0.34	-15.76	0.53
1.5(2)	8.15	-14.77	1.06	0.42	-9.69	0.49
1.5(3)	12.70	-20.44	1.12	0.49	-5.16	0.46

$\alpha = 2/3$ and $\beta = 1/3$ if $D_0 \sim u_0^3/L_0$ ($C_\epsilon = const$).

$\alpha = 1$ and $\beta = 1/2$ if $D_0 \sim (\frac{U_\infty L_b}{\nu})(\frac{u_0 L_0}{\nu})^{-1} u_0^3/L_0$ ($C_\epsilon \approx Re_I/Re_L$).

$u_0/U_\infty = A(\frac{x-x_0}{\theta})^{-\alpha}$ and $\delta/\theta = B(\frac{x-x_0}{\theta})^{\beta}$

1. Calculate $\frac{d}{dx}(u_0/U_\infty)^{-1/\alpha}$ and $\frac{d}{dx}(\delta/\theta)^{1/\beta}$ for a range of values of α and β



2. Chose the values of α and β for which a linear fit $c_1 x/\theta + c_2$ of above plots is such that $c_1 = 0$.
3. Then estimate $A^{-1/\alpha}$ and $B^{1/\beta}$ from c_2 in each plot.
4. Having A, α, B and β, then estimate x_0 for each plot.

$$u_0/U_\infty = A(\frac{x-x_0}{\theta})^{-\alpha}$$ **and** $$\delta/\theta = B(\frac{x-x_0}{\theta})^{\beta}$$

	A	$-x_{0A}/\theta$	α	B	$-x_{0B}/\theta$	β
1.5(1)	7.67	13.65	1.03	0.36	13.56	0.52
1.5(2)	6.53	12.13	1.01	0.39	11.96	0.51
1.5(3)	3.61	2.62	0.89	0.53	2.53	0.44

$\alpha = 2/3$ and $\beta = 1/3$ if $D_0 \sim u_0^3/L_0$ ($C_\epsilon = const$).

$\alpha = 1$ and $\beta = 1/2$ if $D_0 \sim (\frac{U_\infty L_b}{\nu})(\frac{u_0 L_0}{\nu})^{-1} u_0^3/L_0$ ($C_\epsilon \approx Re_I/Re_L$).

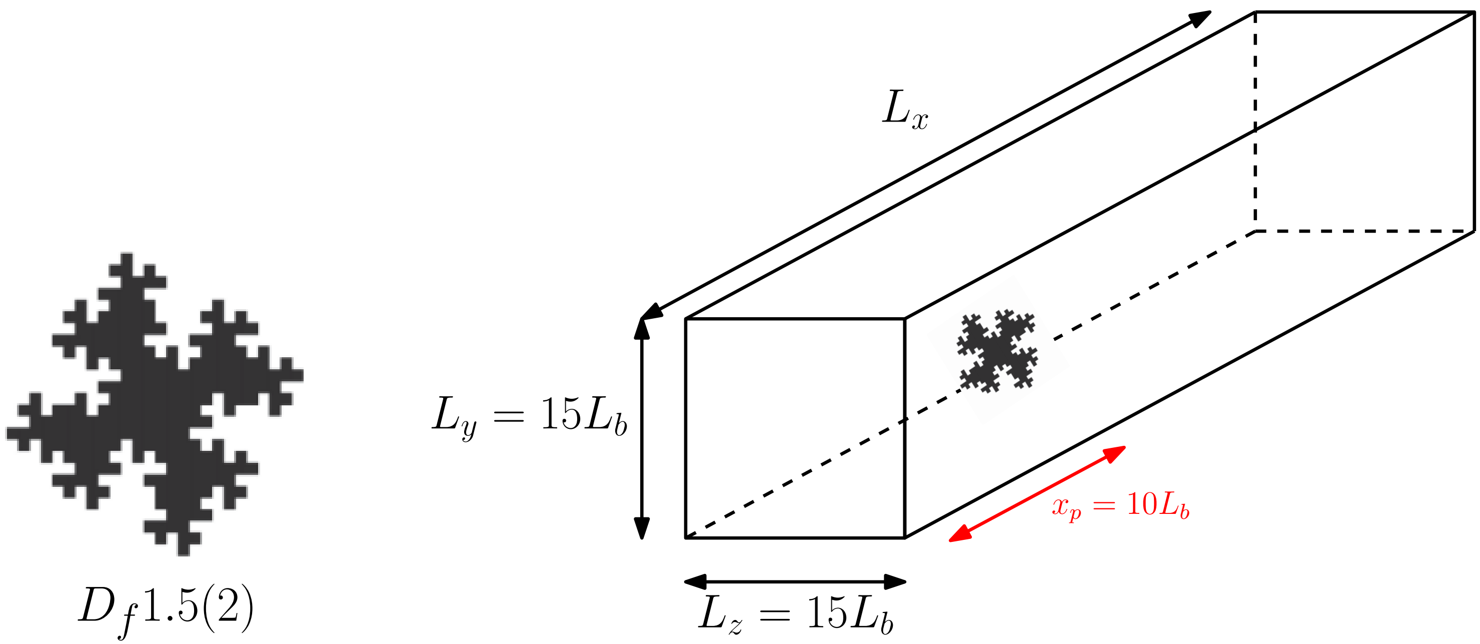
Conclusion: turbulent wake scalings

Consequence of $C_\epsilon \approx Re_I/Re_L$, i.e. of $D_0 \sim (\frac{U_\infty L_b}{\nu})(\frac{u_0 L_0}{\nu})^{-1} u_0^3/L_0$, validated in the wake of fractal-like plates in range $5L_b \leq x \leq 50L_b$ where $L_b = \sqrt{A}$.

This consequence is that $(u_0/U_\infty)^{-1}$ and $(\delta/\theta)^2$ vary linearly with streamwise distance x from plate (over x-range considered). No such behaviour has been detected yet for any other turbulent free shear flow, but it could be present in many if one knows where to look.

We should therefore be able to distinguish between $R_0 \sim u_0^2$ and $R_0 \sim U_\infty u_0 \frac{d}{dx}\delta$. Can we?

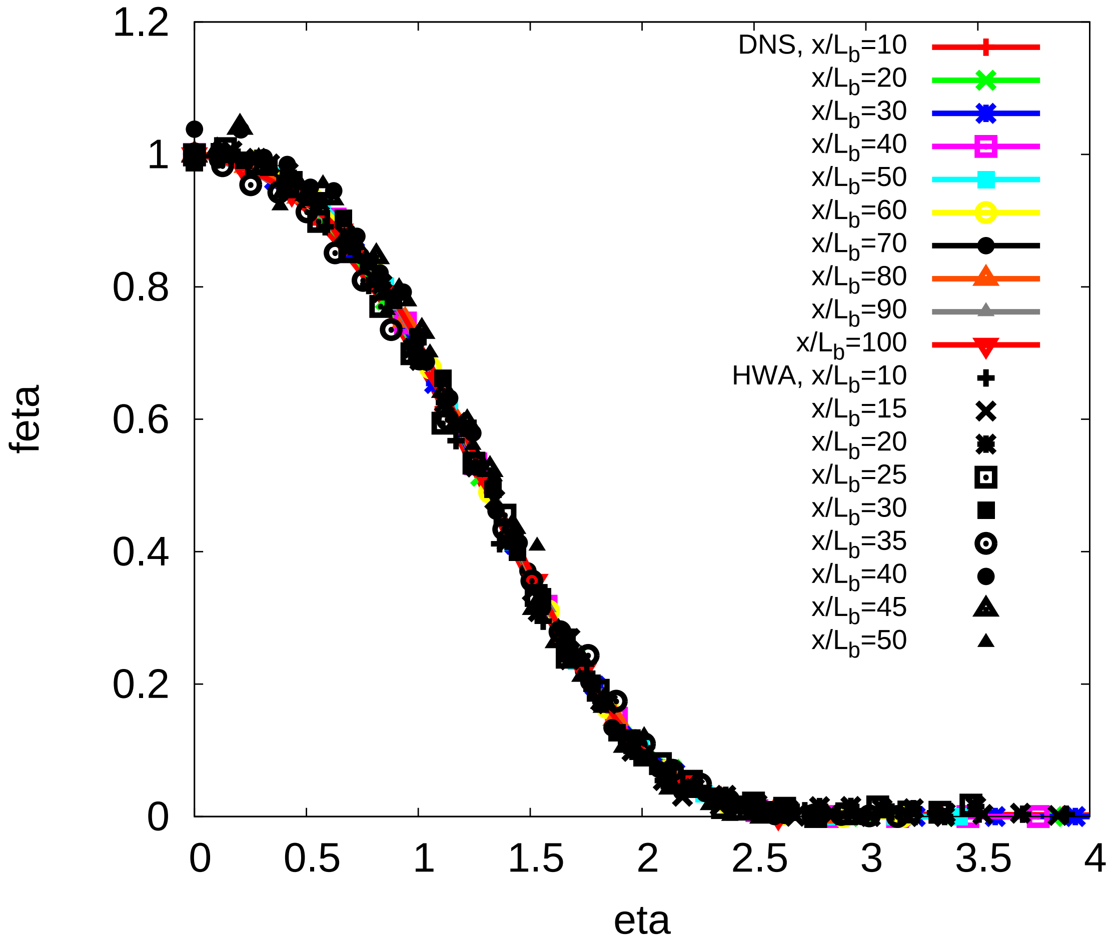
Wakes of flat plates normal to U_∞



HWA ($U_\infty L_b/\nu = 40000$) and DNS ($U_\infty L_b/\nu = 5000$) with Incompact3D (Laizet & Lamballais 2009 and http://code.google.com/p/incompact3d/)

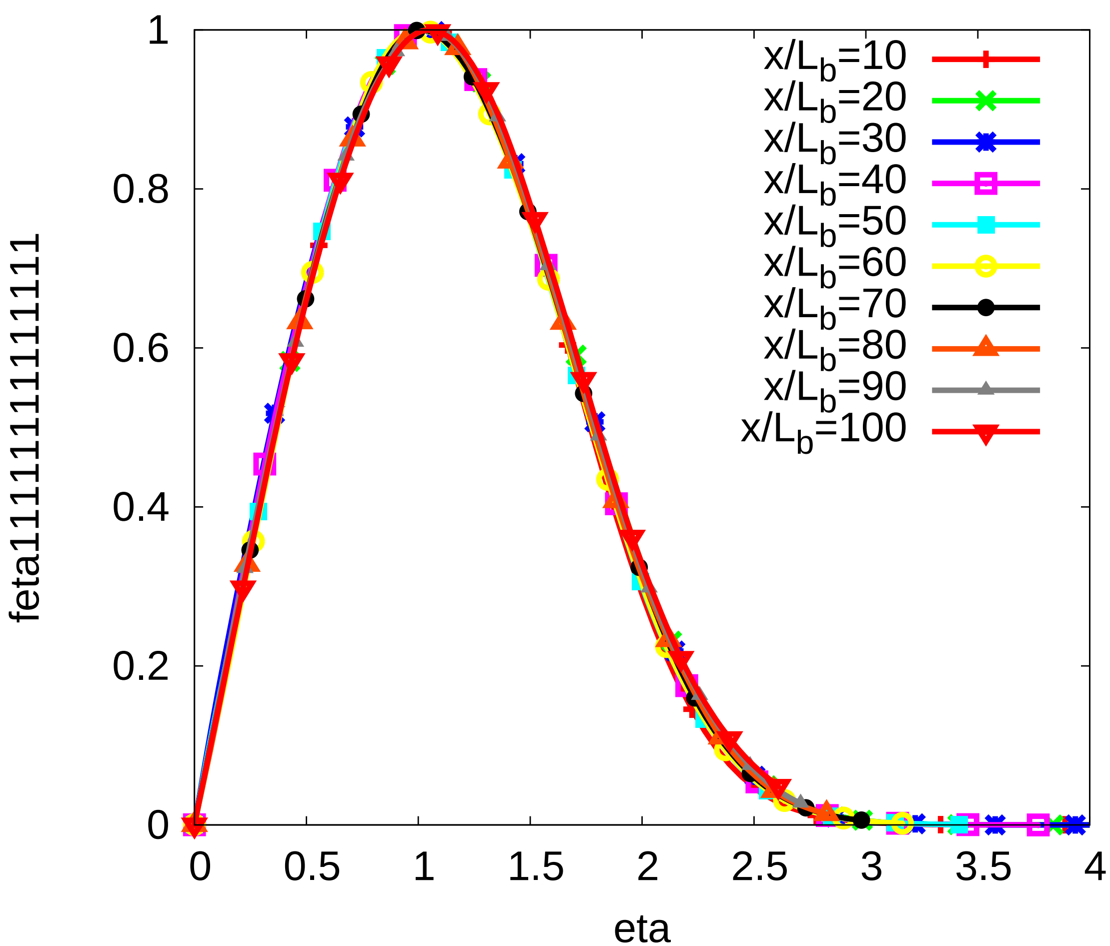
Wakes statistically axisymmetric at streamwise distances greater than $10L_b$ from plate

$$U_\infty - U(x,r) = u_0(x) f(\eta), \eta \equiv r/\delta$$



Self-preserving Reynols shear stress

$< u_x u_r > / max_r(< u_x u_r >)$ versus $\eta \equiv r/\delta(x)$



Reynolds shear stress scaling

Tennekes & Lumley (1972): $R_0 \sim u_0^2$

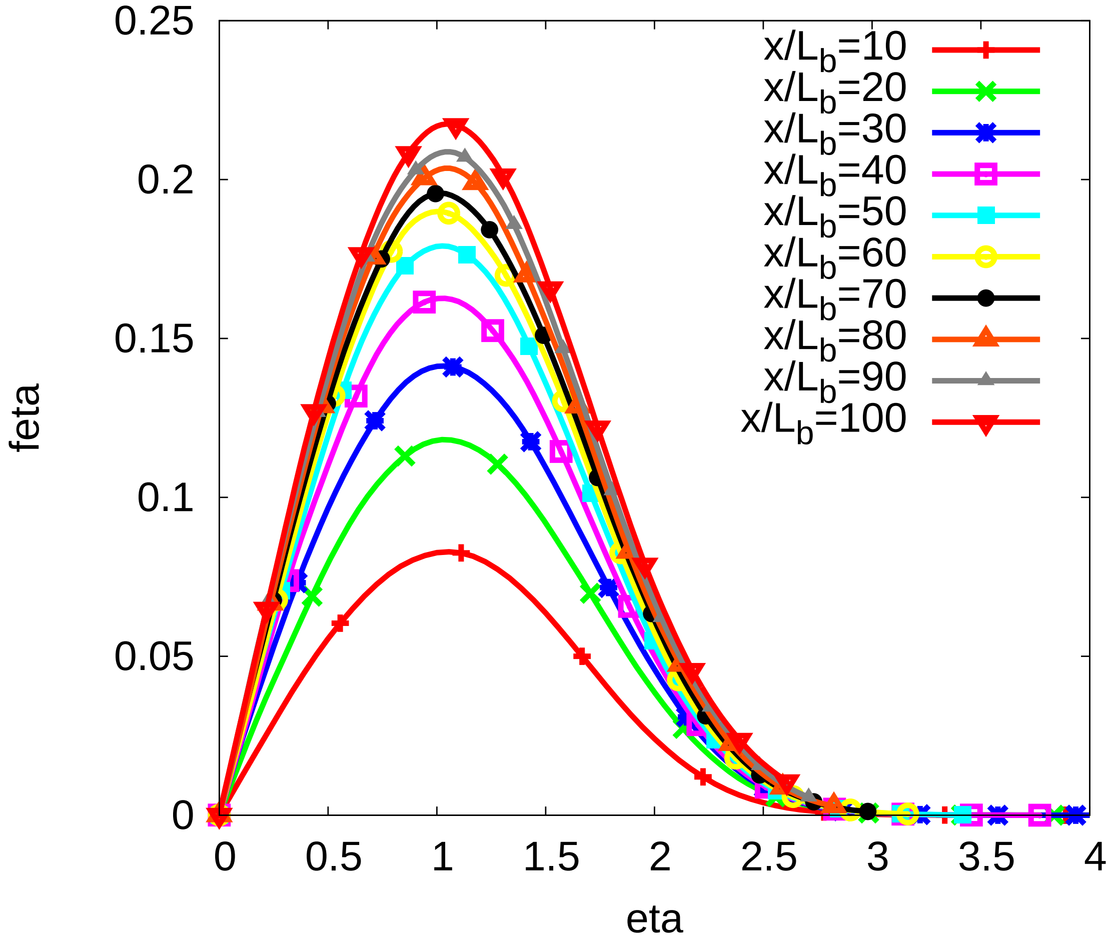
George (1989): $R_0 \sim U_\infty u_0 \frac{d}{dx}\delta$

No real difference if $D_0 \sim K_0^{3/2}/\delta$.

But the two R_0 scalings are different if $D_0 \sim U_\infty L_b K_0/\delta^2$, in which case it should be possible to distinguish between them.

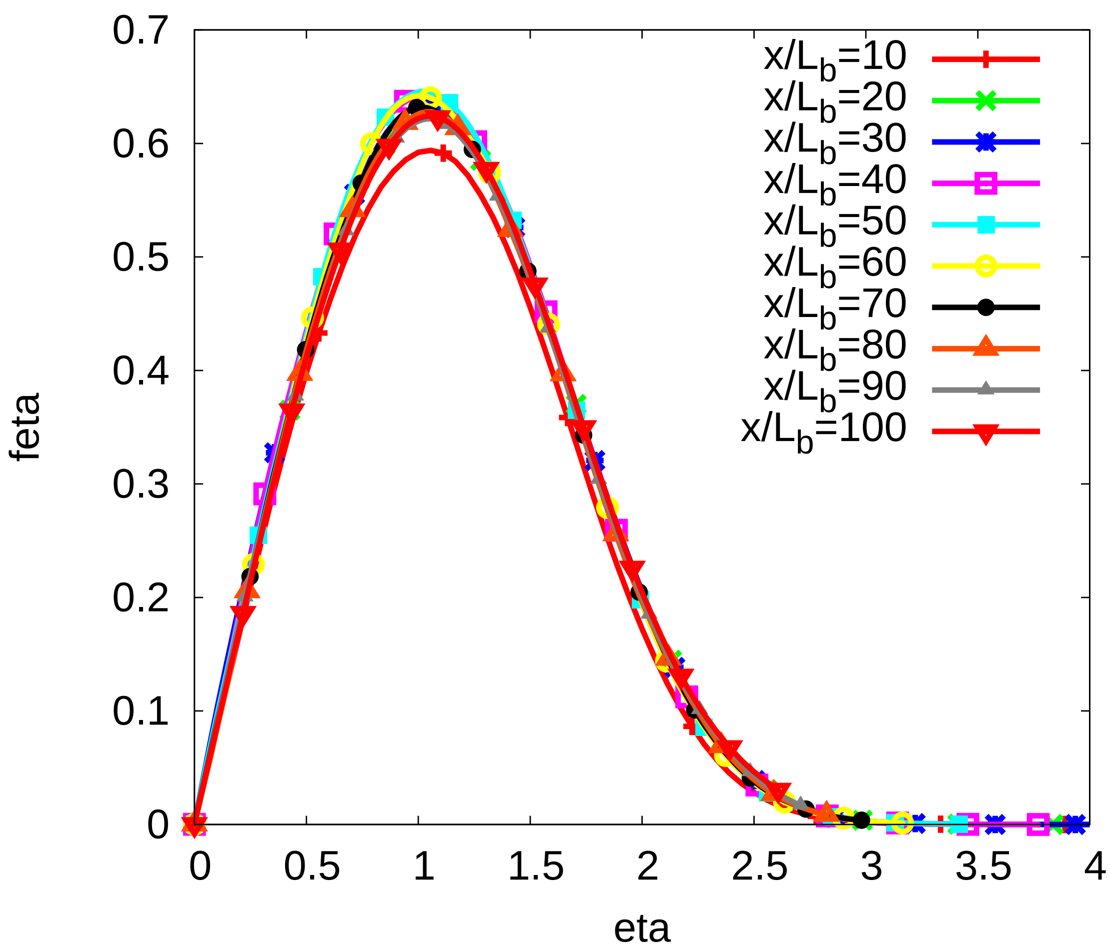
Tennekes & Lumley (1972)

$< u_x u_r > / u_0^2$ versus $\eta \equiv r/\delta(x)$



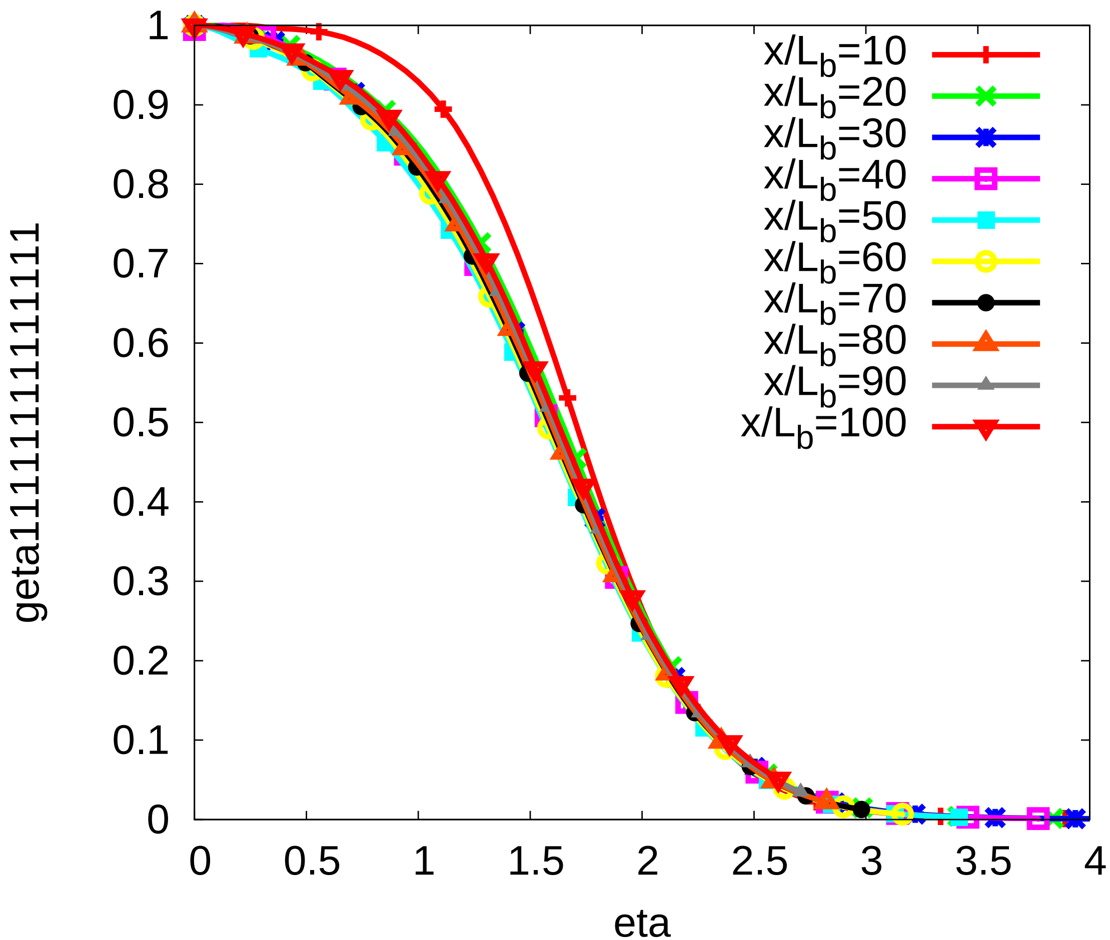
George (1989) at $x/L_b \geq 20$

$< u_x u_r > /(U_\infty u_0 d\delta/dx)$ versus $\eta \equiv r/\delta(x)$



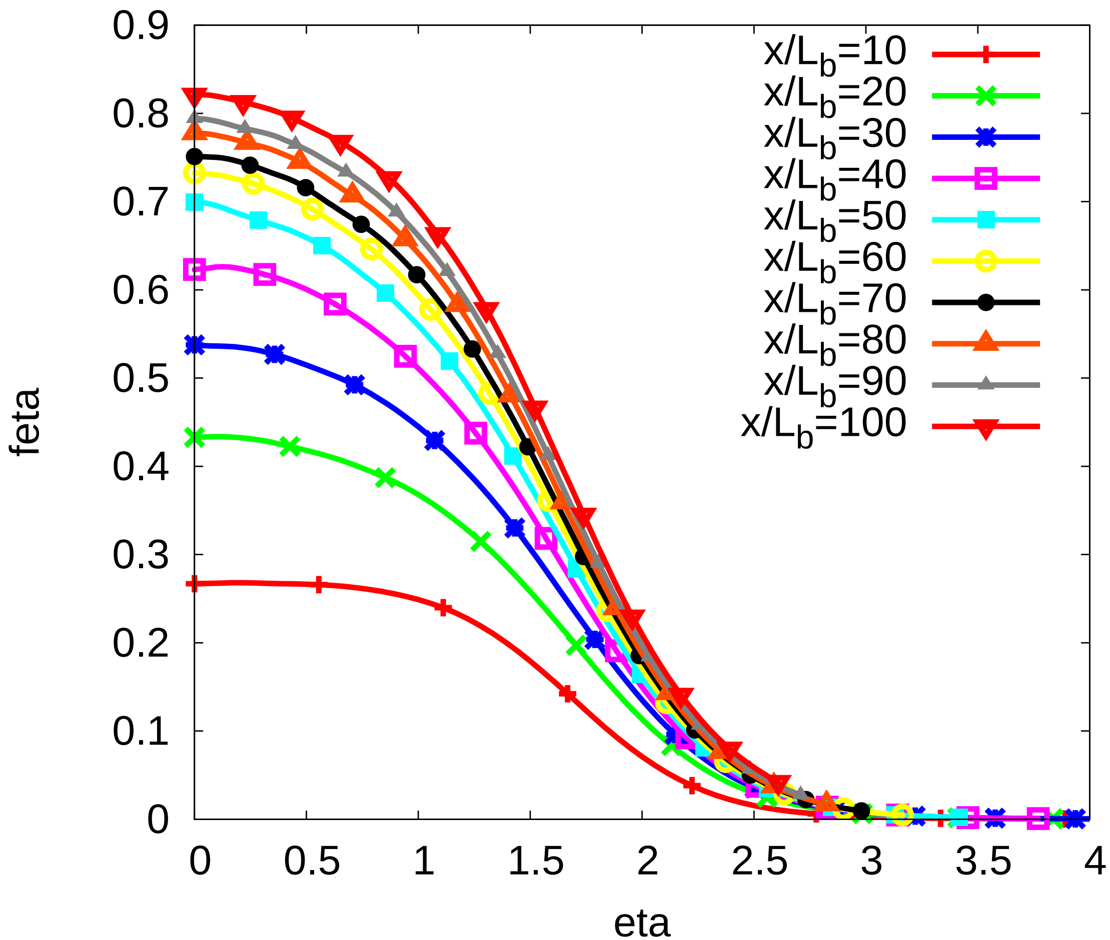
Self-preserving TKE at $x/L_b \geq 20$

$K(x,r)/max_r(K)$ versus $\eta \equiv r/\delta(x)$



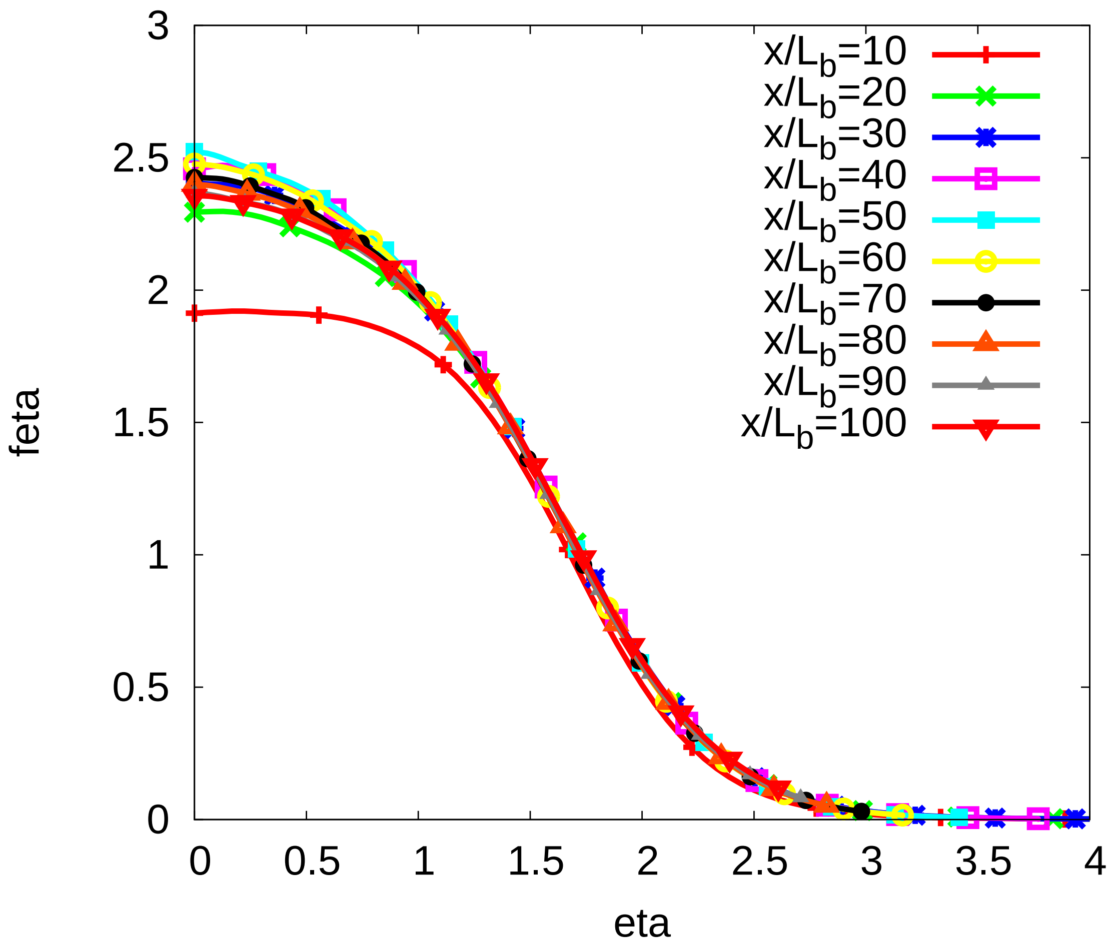
Townsend 1976 and George 1989

$K(x,r)/u_0^2$ versus $\eta \equiv r/\delta(x)$



Different TKE scaling

$K(x,r)/(U_\infty u_0 d\delta/dx)$ versus $\eta \equiv r/\delta(x)$



$K_0(x) \sim (U_\infty u_0 d\delta/dx)$ also supported by HWA (Nedic 2013)

$K_0(x) \sim (U_\infty u_0 d\delta/dx)$ **for** $x \geq 20L_b$

The failure of $K_0 \sim u_0^2$ points to a failure of
$U_\infty \frac{\partial}{\partial x} K = - < u_x u_r > \frac{\partial U}{\partial r} + Transport - \epsilon$
because Production $\approx - \langle u_x u_r \rangle \frac{\partial U}{\partial r}$ is essential for obtaining $K_0 \sim u_0^2$.

Our DNS shows that the production term is dominated by normal stress terms on and around the centreline and that these normal stress terms are not negligible off centreline either. They are also not quite self-preserving,
(See Dairay, Obligado & V, JFM **781**, 2015)

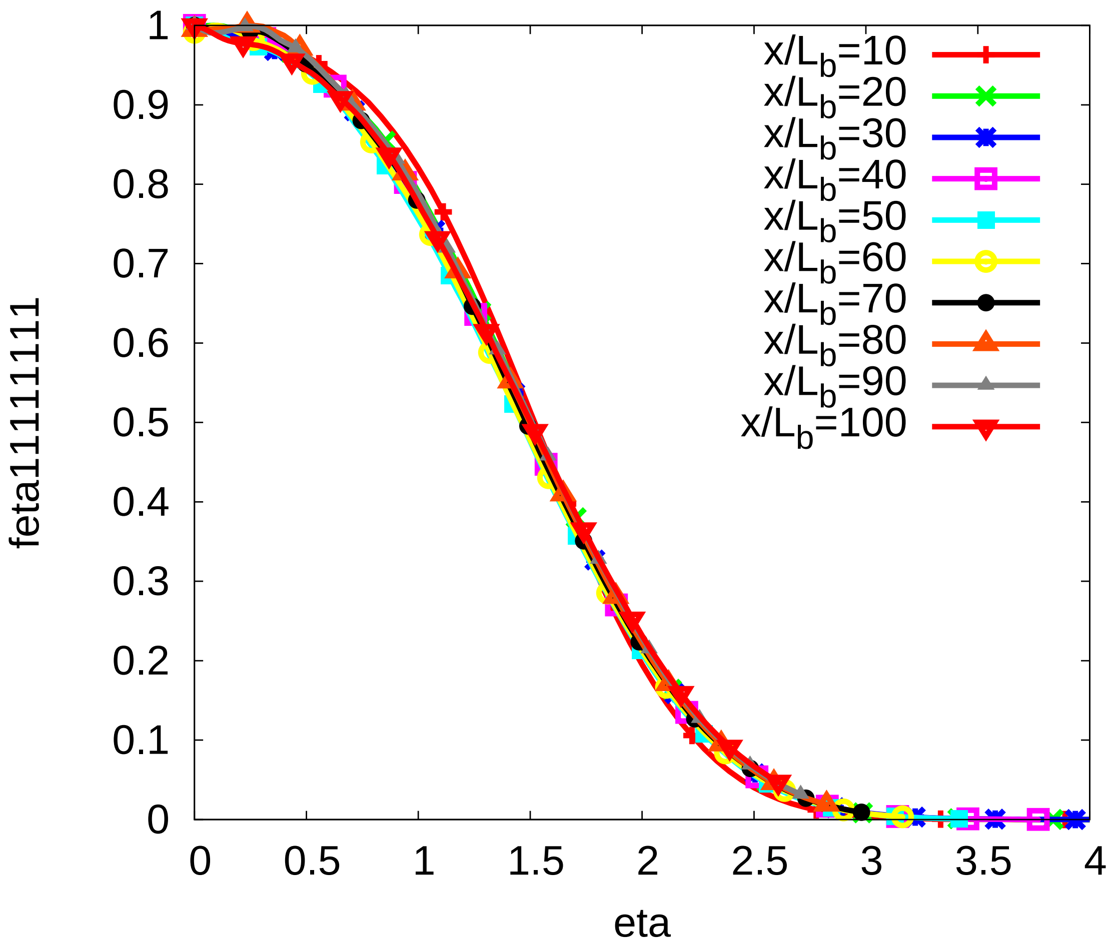
SOLUTION: use the TKE equation in the general form

$$U_\infty \frac{\partial}{\partial x} K = P + T - \epsilon$$

and do NOT assume self-preservation for P and T.

Dissipation is self-preserving at $x \geq 20L_b$

$\epsilon(x,r)/max_r(\epsilon)$ versus $\eta \equiv r/\delta(x)$



Revised Townsend-George theory

The Reynolds-averaged streamwise momentum equation for an axisymmetric wake in a uniform and constant stream,

$$U_\infty \frac{\partial}{\partial x}(U_\infty - U) = -\frac{1}{r}\frac{\partial}{\partial r} r < u_x u_r >$$

has the general self-preserving solution

$$U_\infty - U = u_0(x) f[r/\delta(x)]$$

and

$$< u_x u_r >= R_0(x) g[r/\delta(x)]$$

under the conditions

$$\frac{d}{dx}\delta(x) \sim \frac{R_0}{U_\infty u_0} \text{ and } u_0 \delta^2 \sim U_\infty \theta^2 = Const$$

where θ is the conserved momentum thickness.

2 conditions for 3 unknowns (δ, u_0, R_0), so make use of kinetic energy equation.

Revised Townsend-George theory

The Reynolds-averaged kinetic energy equation for an axisymmetric wake in a uniform and constant stream,

$$U_\infty \frac{\partial}{\partial x} K = Production + Transport - \epsilon$$

has the general self-preserving solution

$$K(x,r) = K_0(x) k[r/\delta(x)],$$

$$Production + Transport = PT_0(x) t[r/\delta(x)]$$

and

$$\epsilon = D_0(x) e[r/\delta(x)]$$

under the additional conditions

$$U_\infty \frac{dK_0}{dx} \sim \frac{U_\infty K_0}{\delta}\frac{d\delta}{dx} \sim D_0$$

IN TOTAL: 4 conditions for 6 unknowns δ, u_0, R_0, K_0 and D_0. Not enough to use

$$D_0 \sim K_0^{3/2}/\delta$$

BUT!

If we dispense with the idea that the small-scale turbulence is in Richardson-Kolmogorov equilibrium and therefore we remove the main basis for $D_0 \sim K_0^{3/2}/\delta$ (i.e. $\epsilon \sim K^{3/2}/L$)

and if we adopt instead the non-equilibrium dissipation law $\epsilon \sim U_\infty L_b K/L^2$, i.e. $D_0 \sim U_\infty L_b K_0/\delta^2$
–see Ann. Rev. Fluid Mech. **47**, 95-114 (2015) & Goto & V Phys. Lett. A **379**, 1144 (2015)–

then the conditions can be solved and we get:
$R_0 \sim U_\infty u_0 \frac{d\delta}{dx}$ as in George (1989)
and
$\frac{\delta(x)}{\theta} \sim (\frac{x-x_0}{\theta})^{1/2}(L_b/\theta)^{1/2}$ and $\frac{u_0}{U_\infty} \sim (\frac{x-x_0}{\theta})^{-1}(\theta/L_b)$
in agreement with DNS and HWA data.

Conclusion

George (1989) WITHOUT Kolmogorov equilibrium
$\epsilon \sim K^{3/2}/L$ BUT WITH non-equilibrium dissipation
$\epsilon \sim U_\infty L_b K/L^2$ AND WITH WEAK RATHER THAN STRONG self-preservation
implies
$R_0 \sim U_\infty u_0 \frac{d\delta}{dx}$ as in George (1989)
$\frac{\delta(x)}{\theta} \sim (\frac{x-x_0}{\theta})^{1/2}(L_b/\theta)^{1/2}$ and $\frac{u_0}{U_\infty} \sim (\frac{x-x_0}{\theta})^{-1}(\theta/L_b)$

New dissipation law valid in axisymmetric self-preserving turbulence wakes too (see Dairay *et al* JFM **781** (2015) and Obligado *et al* PRE (2016)). It is the only dissipation scaling for which revised Townsend-George theory can be conclusive!
What happens much further downstream? Surely not tending towards classical equilibrium as local Re_L is dropping, even if tending towards $\epsilon \sim K^{3/2}/L$?

Reminder conclusion

In various cases of unsteady turbulence (fractal grids, regular grids, axisymmetric wakes, periodic turbulence)

$C_\epsilon \approx Re_I/Re_L$ i.e. $\epsilon \sim U_\infty L_b u'^2/L^2$

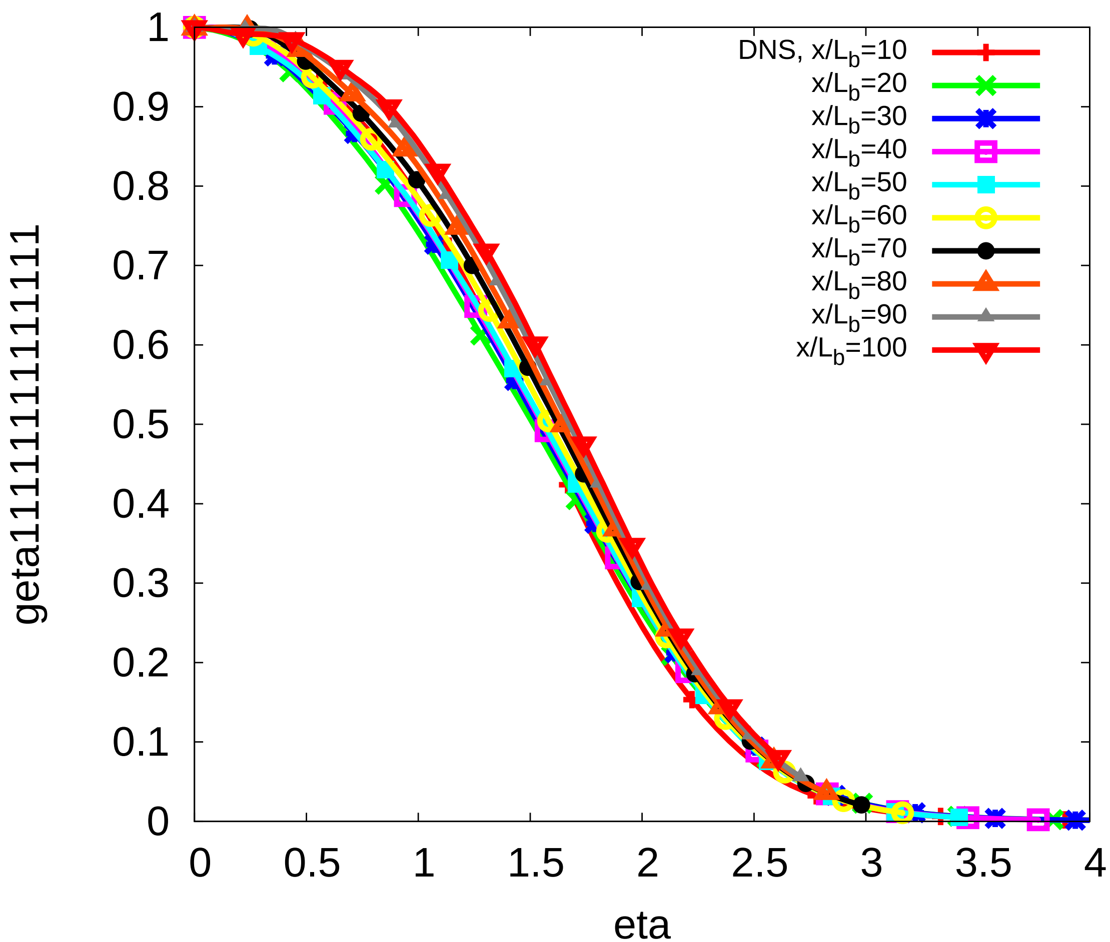
over several (initial) eddy turnover times.

There are very well defined $-5/3$ energy spectra over more than a decade in this region. But not caused by equilibrium cascade which seems absent here.

Consequence for mean flow deficit $u_0(x)$ and wake width $\delta(x)$ of axisymmetric weakly self-preserving wakes : $(u_0/U_\infty)^{-1}$ and $(\delta/\theta)^2$ vary linearly with streamwise distance from the wake generator.

$$K_0 \sim U_\infty u_0 \frac{d\delta}{dx}\ \textbf{?}$$

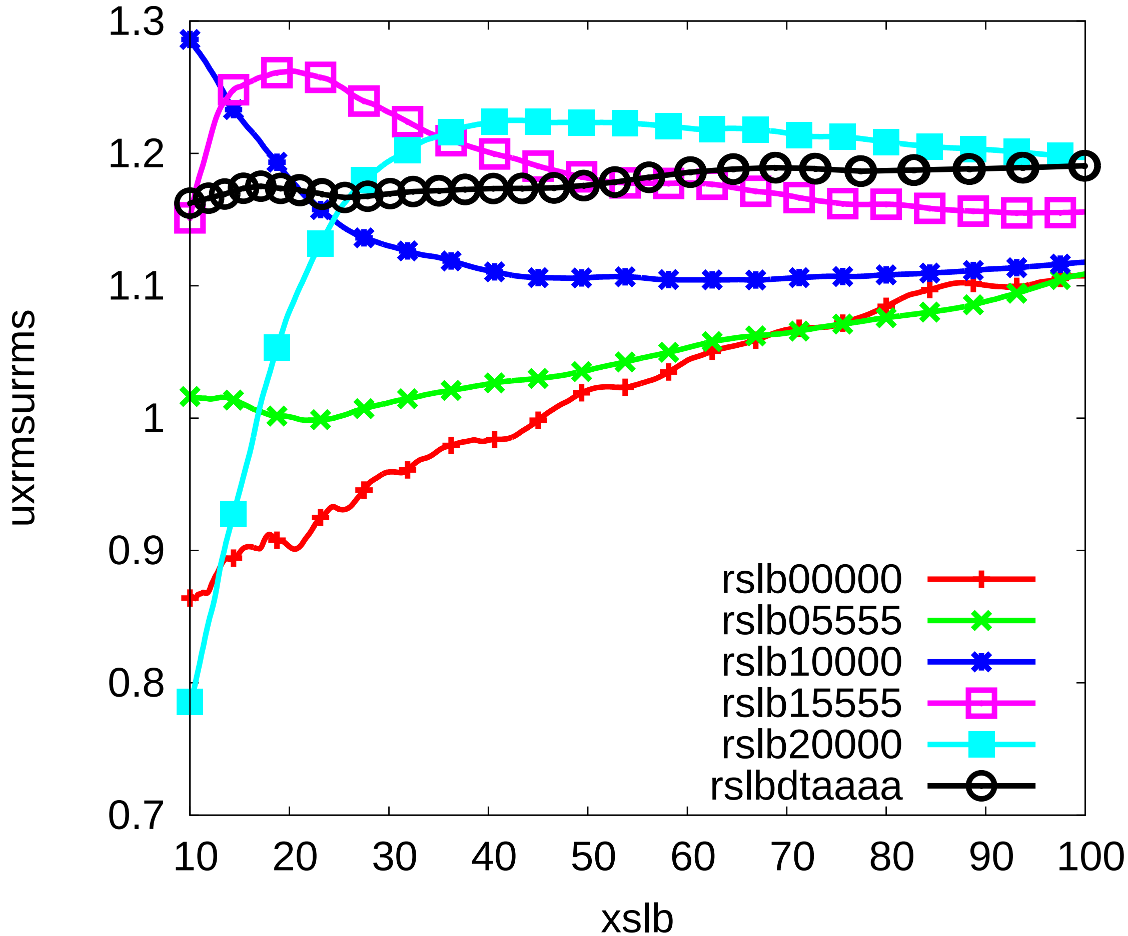
$< u_r^2 > / max_r(< u_r^2 >)$ versus $\eta \equiv r/\delta(x)$



Cannot use self-preservation assumptions for extra terms.

Assumption of constant anisotropy

$\sqrt{< u_x^2 > / < u_r^2 >}$ versus x/L_b



Assumption of constant anisotropy

The correlation function between u_x and u_r and the ratios of the r.m.s. values of u_x, u_r and u_ϕ are constant on the surface $r = \delta(x)$ defining the locations of the maximum Reynolds shear stress.

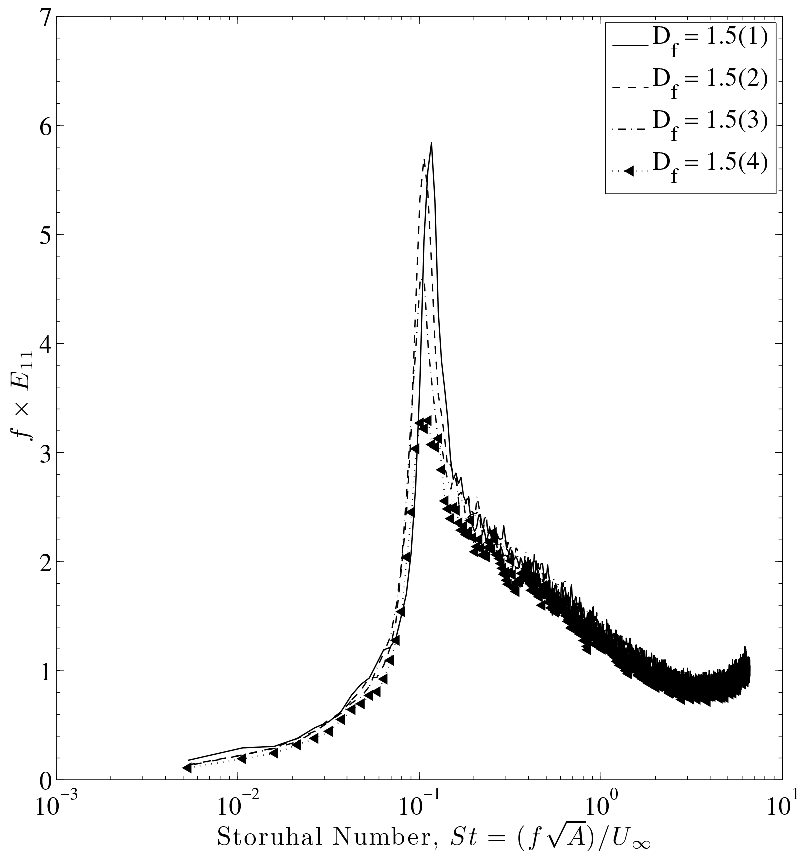
It takes a little algebra to show that the revised Townsend-George theory plus this assumption imply

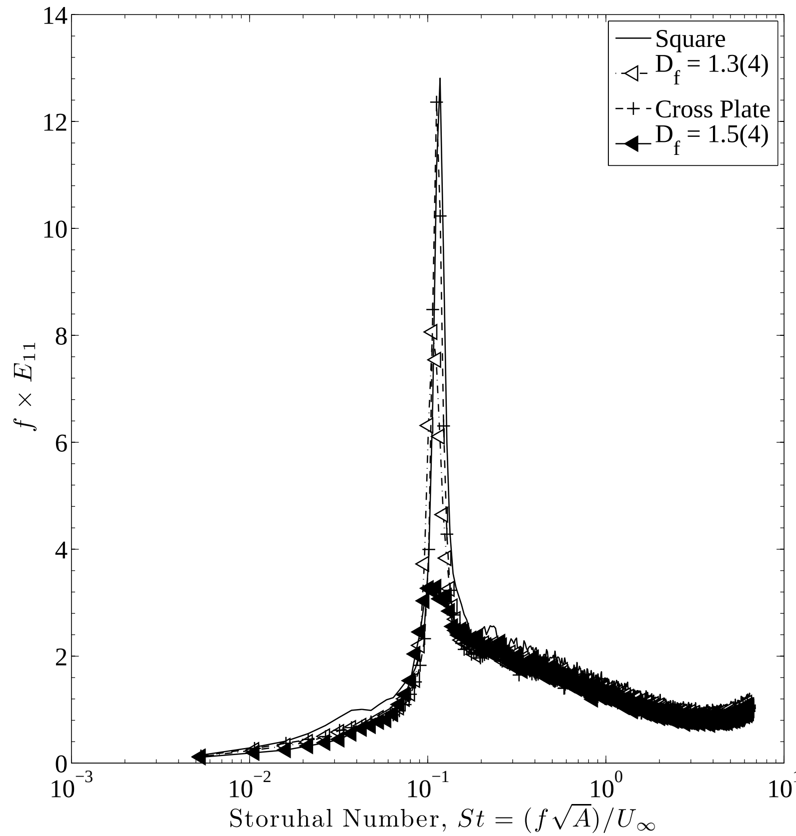
$$K_0 \sim U_\infty u_0 \frac{d\delta}{dx}$$

And if we add to this revised theory + assumption of constant anisotropy the tradtional dissipation law $\epsilon \sim K^{3/2}/L$ (instead of the non-equilibrium dissipation law $\epsilon \sim U_\infty L_b K/L^2$), then we obtain the wake laws of Tennekes & Lumley, Townsend and George:

$$u_0/U_\infty \sim (\frac{x-x_0}{\theta})^{-2/3} \text{ \& } \delta/\theta \sim (\frac{x-x_0}{\theta})^{1/3}$$

Vortex shedding and $L_b = \sqrt{A}$





$$u_0\delta^2 = U_\infty\theta^2$$