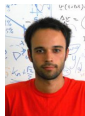


Critical Transitions in Turbulence

ALEXAKIS, Alexandros

11 April 2019

with many thanks to the students and postdocs that worked with me on this subject...



**K. Sesashayanan, S. Benavides, A. Van Kan, G. Sahoo,
V. Dallas, T. Nemoto, A. Cameron**

Most results presented in this talk are reviewed in:

A. Alexakis, L. Biferale

“Cascades and transitions in turbulent flows”

Physics Reports **767-769**, 1-101 (2018)

Physics Reports 767–769 (2018) 1–101



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Cascades and transitions in turbulent flows

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^b Department of Physics and INFN University of Rome 'Tor Vergata', Via della Ricerca Scientifica 1, 00133 Rome, Italy



I. Introduction

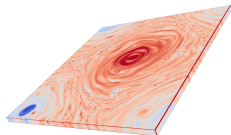
Coexistence of “coherent structures at large scales” & “small scale turbulence”



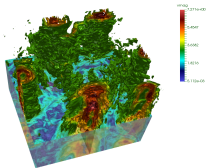
Picture from: <https://earthobservatory.nasa.gov>

Thin layers/Rotating/Stratified/Magnetic fields ...

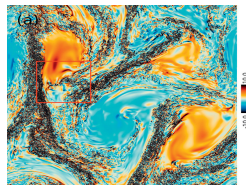
There is a variety of systems that have the ability to generate both large and small scale structures...



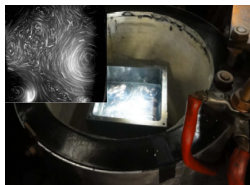
Thin Layers



Rotating Flows



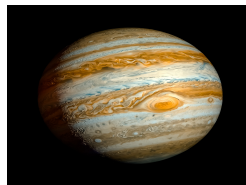
Rotating & Stratified



in the Lab



in the atmosphere



in planets

Pictures from: A^2 & LB Phys. Rep. **767-769**, 1-101 (2018), L. Biferale et al Phys. Rev. X **6** 041036 (2016), D. Rosenberg, et al Phys. Fluids **27** 055105 (2015), J. Herault, et al, Europhys. Lett. **111** 44002 (2015), <https://earthobservatory.nasa.gov>

Thin layers/Rotating/Stratified/Magnetic fields ...

Thin Layers

- L. Smith, J. Chasnov, & F. Waleffe, Phys. Rev. Lett. **77**, 2467 (1996)
- A. Celani, S. Musacchio, and D. Vincenzi, Phys. Rev. Lett. **104**, 184506 (2010)

Rotating flows

- A. Sen, et al Phys. Rev. E **86**, 036319 (2012)
- E. Deusebio, et al Phys. Rev. E **90**, 023005 (2014)

Rotating and Stratified flows

- A. Pouquet and R. Marino, Phys. Rev. Lett. **111**, 234501 (2013)
- R. Marino, et al European Phys. Lett. **102** 44006 (2013)
- A. Sozza, et al Phys. of Fluids **27**, 035112 (2014)

Magnetic fields

- A. Alexakis, Phys. Rev. E **84**, 056330 (2011)
- K. Seshasayanan, S.J. Benavides, A. Alexakis Phys. Rev. E **90** 051003(R) (2014)

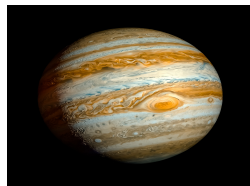
Experiments

- M. Shats, D. Byrne, H. Xia Phys. Rev. Lett. **105**, 264501 (2010)
- H. Xia, D. Byrne, G. Falkovich, M. Shats Nature Physics **7**, 321-324 (2011)
- A. Potherat, R. Klein, J. Fluid Mech. **761** 168 (2014)

And in more exotic systems...

- **atmosphere** D. Byrne et al - Geophys.R.L (2013) *"Height dependent transition from 3D to 2D turbulence in the hurricane boundary layer"*
- **ocean** G. P. King, et al - J. Geophys. Res. (2015) *"Upscale and downscale energy transfer over the tropical Pacific revealed by scatterometer winds"*
- **Venus** M. N. Izakov, Solar System Research (2013) *"Large-scale quasi-2D turbulence and a inverse spectral flux of energy in the atmosphere of Venus"*
- **Jupiter** R. Young et al Nature Physics (2017) *"Forward and inverse kinetic energy cascades in Jupiters turbulent weather layer"*
- **plasma flows** G. Miloshevich et al, Plasma Physics (2018) *"Direction of cascades in a magnetofluid model"*
- **optical turbulence** V. Malkin et al, Phys. Rev. E (2018) *"Transition between inverse and direct energy cascades in multiscale optical turbulence"*
- **acoustic turbulence** A. Ganshin, et al Phys.Rev.Lett. (2008) *"Observation of an inverse energy cascade in acoustic turbulence in superfluid helium"*
- **capillary turbulence** Abdurakhimov et al Phys.Rev.E (2015) *"Bidirectional energy cascade in surface capillary waves"*

A need for a unified treatment of these problems



- There is a large number of diverse systems that display a simultaneous cascade of energy in the large and in the small scales (split cascades).
- Such split cascades are present due to different mechanisms (confinement, rotation, magnetic fields, ...).
- **is there a unified treatment of these problems?**

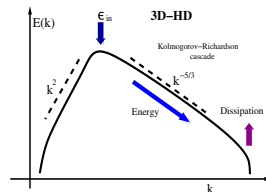
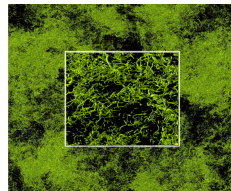
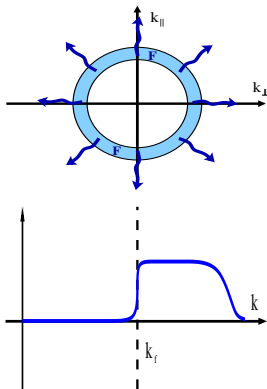
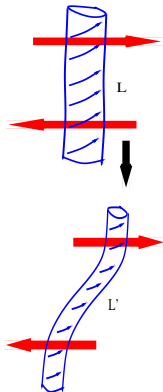
What this talk is about

- Define as precisely as I can the problem.
- Present all possible scenarios of “cascade transitions”.
- Demonstrate with examples each possibility.
- Describe our current state of understanding of these systems.
- Present open problems!

II. Setting up the problem

Forward Cascade: 3D turbulence

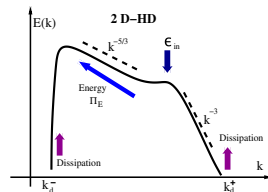
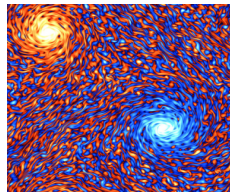
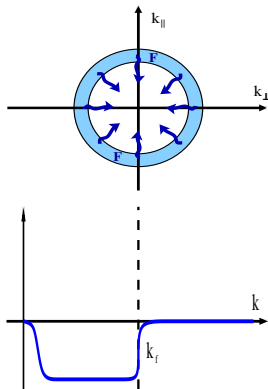
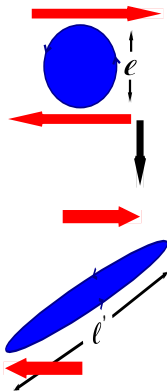
Vortex tube stretching $\Rightarrow$ Forward cascade



Picture from: M. Yokokawa et al. Proceedings of the 2002 ACM/IEEE Conference on Supercomputing

Inverse Cascade: 2D turbulence

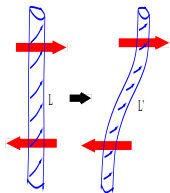
Vortex patch shearing $\Rightarrow$ Inverse cascade



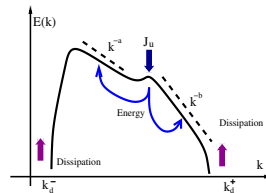
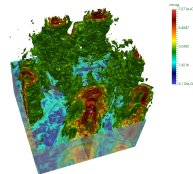
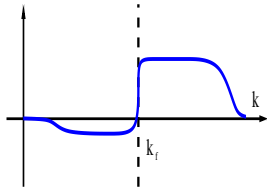
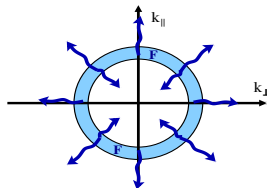
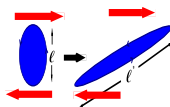
Picture from: C-K Chan Phys. Rev. E 85, 036315

Split Cascade: Thin layers, Rotating turbulence, ...

A balance between the two $\Rightarrow$ Split cascade?

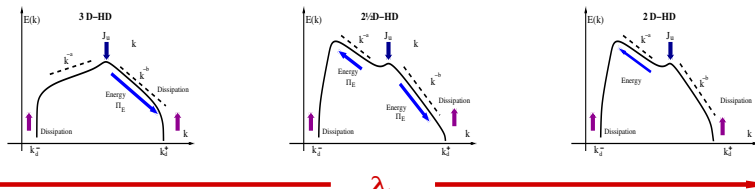


+



Picture from: L. Biferale et al Phys. Rev. X **6** 041036 (2016)

A turbulence to turbulence transition ...

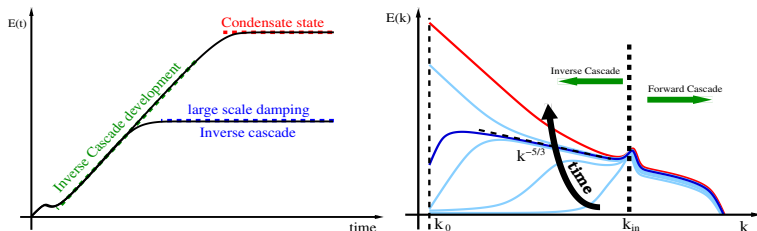


- the system transitions from one turbulent state (forward cascading) to an other (inverse cascading) varying a parameter λ .
- the transition occurs in the presence of turbulence ($\lambda \neq Re$).
- through a state that cascades energy both forward and inversely:

Split Cascade!

Two stages of an inverse cascade

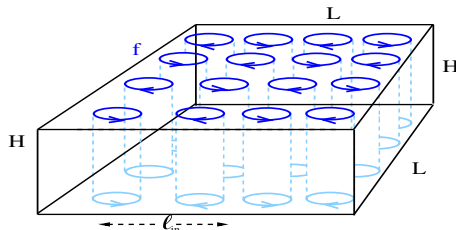
Inverse cascade and condensates



- At early stages energy cascades inversely to the large scales.
- In the presence of a **sufficiently strong large scale dissipation** this process will saturate at a scale ℓ_α smaller than the box size L and a **cascade** from the forcing scales ℓ_f to ℓ_α is build.
- For **weak large scale dissipation or in its absence** energy will pile up in the largest scales forming a **condensate**.

In this talk we will focus on the former case!

A General System



$$\partial_t \mathbf{V} = \omega \mathcal{L}[\mathbf{V}] + \mathcal{N} \mathcal{L}[\mathbf{V}, \mathbf{V}] - \nu (-\Delta)^n \mathbf{V} - \alpha (-\Delta)^{-m} \mathbf{V} + \mathbf{F}$$

ℓ_{in} = energy injection scale, ϵ_{in} = energy injection rate,
 ν = hyper-viscosity ($n = 1$), α = hypo-viscosity ($m = 0$),
 ω = wave frequency (eg rotation rate, Brunt-Väisälä frequency, etc)

$$\epsilon_\alpha = \alpha \langle |\nabla^{-m} \mathbf{V}|^2 \rangle, \quad \epsilon_\nu = \nu \langle |\nabla^n \mathbf{V}|^2 \rangle$$

$$\epsilon_{in} = \epsilon_\alpha + \epsilon_\nu$$

Control and order Parameters

Forcing scale Control parameters:

$$\lambda_1 = \frac{\ell_{in}}{H}, \quad \lambda_2 = \frac{\epsilon_{in}^{1/3} \ell_{in}^{-2/3}}{\omega}, \quad \dots$$

Viscous & Domain size parameters

$$Re = \frac{\epsilon_{in}^{1/3} \ell_{in}^{4/3}}{\nu} \rightarrow \infty, \quad R_\alpha = \frac{\epsilon_{in}^{1/3} \ell_{in}^{-2/3}}{\alpha} \rightarrow \infty, \quad \Lambda = \frac{L}{\ell_{in}} \rightarrow \infty$$

Order parameters

$$Q_\alpha = \frac{\epsilon_\alpha}{\epsilon_{in}}, \quad Q_\nu = \frac{\epsilon_\nu}{\epsilon_{in}} \quad \text{with} \quad Q_\alpha + Q_\nu = 1$$

Control and order Parameters

Forcing scale Control parameters:

$$\lambda_1 = \frac{\ell_{in}}{H}, \quad \lambda_2 = \frac{\epsilon_{in}^{1/3} \ell_{in}^{-2/3}}{\omega}, \quad \dots$$

Viscous & Domain size parameters

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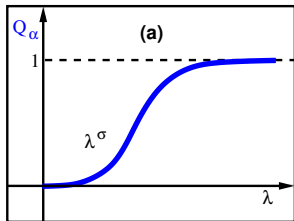
Order parameters

$$Q_\alpha = \frac{\epsilon_\alpha}{\epsilon_{in}}, \quad Q_\nu = \frac{\epsilon_\nu}{\epsilon_{in}} \quad \text{with} \quad Q_\alpha + Q_\nu = 1$$

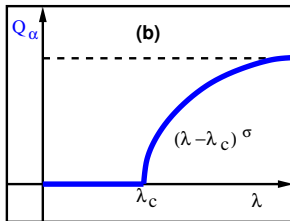
We would like to know how Q_α, Q_ν change as $\lambda_1, \lambda_2, \dots$ vary, in the limit $Re, R_\alpha, \Lambda \rightarrow \infty$

Classification:

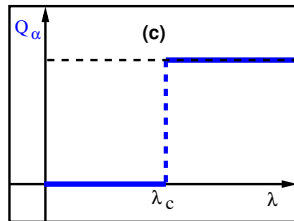
Smooth, 2nd order and 1st order transitions



Smooth,



2nd order
phase transition,

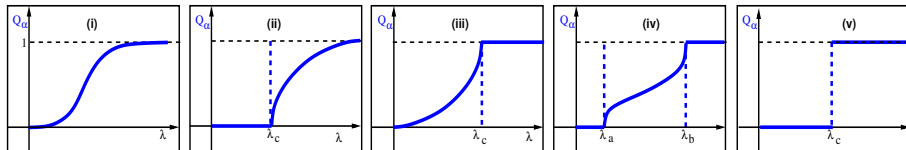


1st order
phase transition

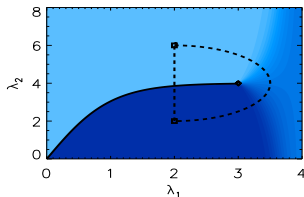
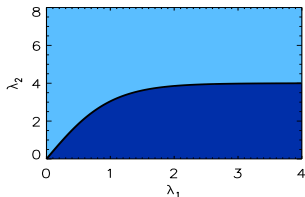
Cases (b) & (c) will be referred as “**critical**”

Phase Space Diagrams

One parameter systems: λ

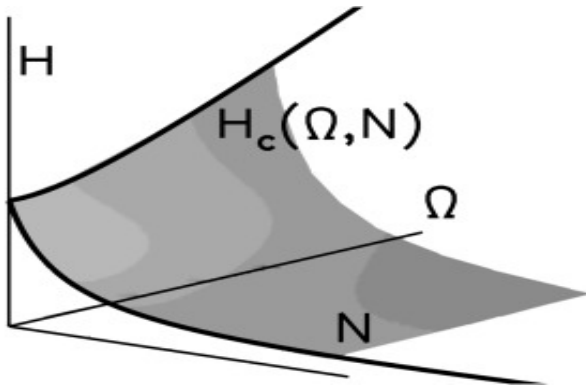


Two parameter systems: λ_1, λ_2



Phase space Diagrams

Three parameter systems: $\lambda_1 = H, \lambda_2 = \Omega, \lambda_3 = N$
eg Rotating and stratified turbulence

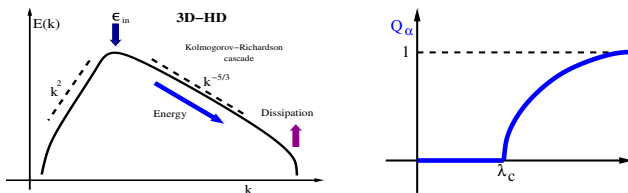


Questions

- ① In what sense can these transitions be critical?
(*ie*, what does having exactly zero flux means?)
- ② How can split cascades even exist?
(*ie*, how can a system cascade energy both to large and small scales at the same time?)
- ③ When the transition is:
 - (i) smooth?
 - (ii) 1st order (discontinuous)?
 - (iii) 2nd order (continuous with discontinuous derivatives)?

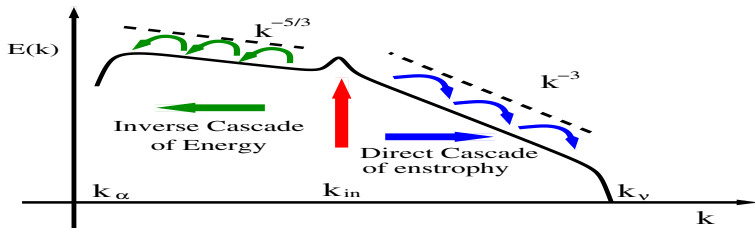
Criticality

- 1 In what sense can these transitions be critical?
(ie, what does having exactly zero flux means?)



$$\partial_t \mathbf{V} = \omega \mathcal{L}[\mathbf{V}] + \mathcal{N} \mathcal{L}[\mathbf{V}, \mathbf{V}] + \nu \Delta \mathbf{V} - \alpha \mathbf{V} + \mathbf{F}$$

2D Turbulence



$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla \mathbf{P} + \mathbf{f} - \nu_n (-\Delta)^n \mathbf{u} - \alpha_m (-\Delta)^{-m} \mathbf{u}$$

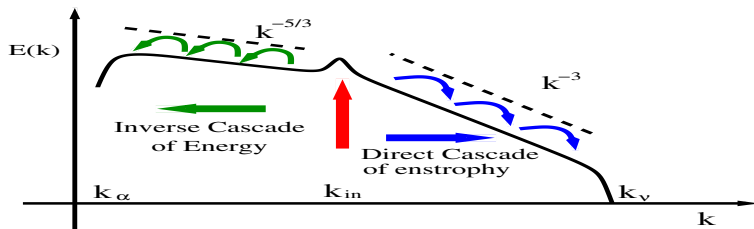
Ideal invariants:

$$\mathcal{E} = \frac{1}{2} \langle |\mathbf{u}|^2 \rangle, \quad \mathcal{Z} = \frac{1}{2} \langle |\boldsymbol{\omega}|^2 \rangle \quad \text{with} \quad \boldsymbol{\omega} = \nabla \times \mathbf{u}$$

Injection and Dissipation rates:

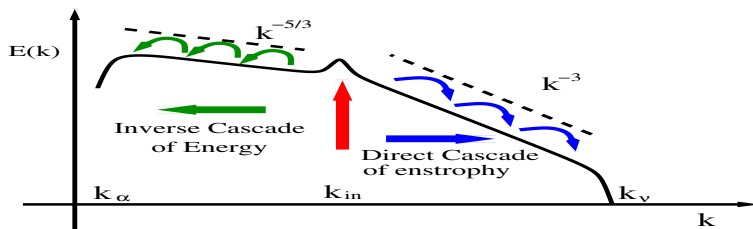
$$\begin{aligned}\epsilon_{in} &= \langle \mathbf{u} \cdot \mathbf{f} \rangle, & \epsilon_\nu &= \nu \langle |\nabla^n \mathbf{u}|^2 \rangle, & \epsilon_\alpha &= \alpha \langle |\nabla^{-m} \mathbf{u}|^2 \rangle, \\ \eta_{in} &= \langle \boldsymbol{\omega} \cdot \nabla \times \mathbf{f} \rangle, & \eta_\nu &= \nu \langle |\nabla^n \boldsymbol{\omega}|^2 \rangle, & \eta_\alpha &= \alpha \langle |\nabla^{-m} \boldsymbol{\omega}|^2 \rangle,\end{aligned}$$

2D Turbulence



$$E(k) = \begin{cases} \epsilon_{in}^{2/3} k^{-5/3} & k_\alpha \ll k \ll k_{in} \\ \eta_{in}^{2/3} k^{-3} & k_{in} \ll k \ll k_\nu \end{cases}$$

2D Turbulence



$$\eta_{in} = \epsilon_{in} k_{in}^2$$

$$\epsilon_\alpha \simeq \epsilon_{in}$$

$$\eta_\alpha \simeq \epsilon_\alpha k_\alpha^2$$

$$\eta_\alpha \simeq \eta_{in} \left(\frac{k_\alpha}{k_{in}} \right)^2$$

$$\eta_\alpha \simeq \eta_{in} R_\alpha^{-3}$$

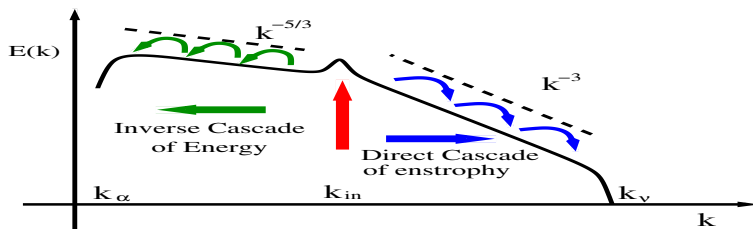
$$\eta_\nu \simeq \eta_{in}$$

$$\epsilon_\nu \simeq \eta_\nu k_\nu^{-2}$$

$$\epsilon_\nu \simeq \epsilon_{in} \left(\frac{k_{in}}{k_\nu} \right)^2$$

$$\epsilon_\nu \simeq \epsilon_{in} Re^{-1}$$

2D Turbulence



$$\eta_{in} = \epsilon_{in} k_{in}^2$$

$$\epsilon_\alpha \simeq \epsilon_{in}$$

$$\eta_\alpha \simeq \epsilon_\alpha k_\alpha^2$$

$$\eta_\alpha \simeq \eta_{in} \left(\frac{k_\alpha}{k_{in}} \right)^2$$

$$\eta_\alpha \simeq \eta_{in} R_\alpha^{-3}$$

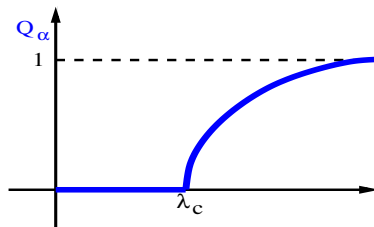
$$\eta_\nu \simeq \eta_{in}$$

$$\epsilon_\nu \simeq \eta_\nu k_\nu^{-2}$$

$$\epsilon_\nu \simeq \epsilon_{in} \left(\frac{k_{in}}{k_\nu} \right)^2$$

$$\epsilon_\nu \simeq \epsilon_{in} Re^{-1}$$

- Zero inverse/forward flux is realized only in the large box, zero ν and zero α limit.



$$Q_\alpha = \lim_{Re \rightarrow \infty} \lim_{R_\alpha \rightarrow \infty} \lim_{L/\ell_{in} \rightarrow \infty} \left(\frac{\epsilon_\alpha}{\epsilon_{in}} \right)$$

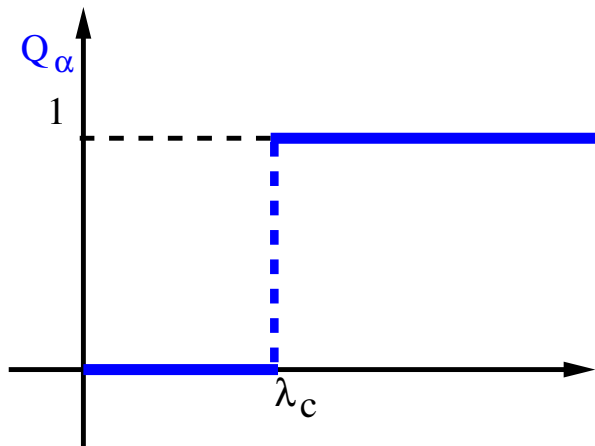
$$Q_\nu = \lim_{Re \rightarrow \infty} \lim_{R_\alpha \rightarrow \infty} \lim_{L/\ell_{in} \rightarrow \infty} \left(\frac{\epsilon_\nu}{\epsilon_{in}} \right)$$

To conclude criticality the $L/\ell_{in}, Re, R_\alpha \rightarrow \infty$ limits have to be taken first.

III. Examples

Example I

1st order transition



Example I: Helical Decomposition

$$\tilde{\mathbf{u}}(\mathbf{k}) = \frac{1}{(2\pi L)^3} \int e^{i\mathbf{k}\mathbf{x}} \mathbf{u} d\mathbf{x}^3, \quad \mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}} \tilde{\mathbf{u}}(\mathbf{k})$$

$$\tilde{\mathbf{u}}(\mathbf{k}) = u^+(\mathbf{k}) \mathbf{h}_{\mathbf{k}}^+ + u^-(\mathbf{k}) \mathbf{h}_{\mathbf{k}}^-$$

$$\mathbf{h}_{\mathbf{k}}^{\pm} = \frac{\mathbf{k} \times (\mathbf{k} \times \hat{\mathbf{e}})}{\sqrt{2}|\mathbf{k} \times (\mathbf{k} \times \hat{\mathbf{e}})|} \pm i \frac{\mathbf{k} \times \hat{\mathbf{e}}}{\sqrt{2}|\mathbf{k} \times \hat{\mathbf{e}}|}$$

$$i\mathbf{k} \times \mathbf{h}_{\mathbf{k}}^{\pm} = \pm k \mathbf{h}_{\mathbf{k}}^{\pm}, \quad \mathbf{h}_{\mathbf{k}}^s \cdot \mathbf{h}_{\mathbf{q}}^{-s'} = \delta_{\mathbf{k},\mathbf{q}} \delta_{s,s'}$$

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- M. Lesieur, Revue 'Turbulence', Observatoire de Nice (1972)
- J. R. Herring, Phys. Fluids 17, 859 (1974)
- P. Constantin, & A. Majda, Commun. Math. Phys. 115, 435-456 (1988)
- C. Cambon, & L. Jacquin, J. Fluid Mech. 202, 295-317 (1989)
- F. Waleffe, Phys. Fluids 4, 350 (1992)

Example I: Helical Decomposition

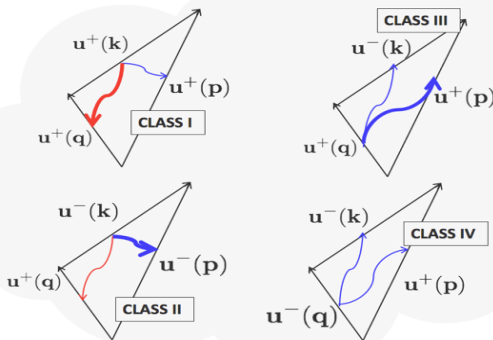
The nature of triad interactions in homogeneous turbulence

Fabian Waleffe

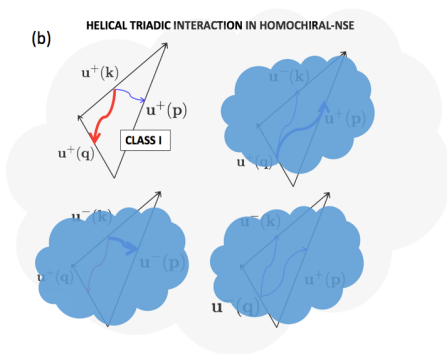
Center for Turbulence Research, Stanford University–NASA Ames, Building 500,
Stanford, California 94305-3030

(Received 24 July 1991; accepted 22 October 1991)

(a) HELICAL TRIADIC INTERACTION IN NSE



Example I: Helical Decomposition

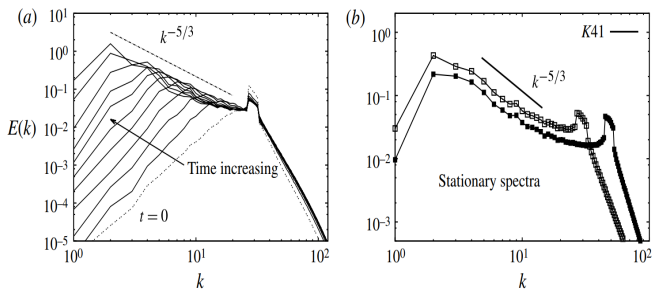


Ideal Invariants:

$$E^{\pm} = \frac{1}{2} \langle \mathbf{u}^{\pm} \cdot \mathbf{u}^{\pm} \rangle, \quad H^{\pm} = \pm \frac{1}{2} \langle \mathbf{w}^{\pm} \cdot \mathbf{u}^{\pm} \rangle > 0$$

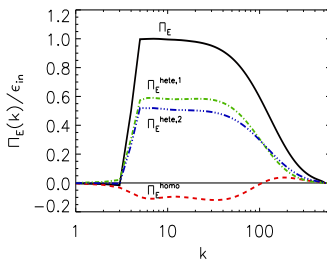
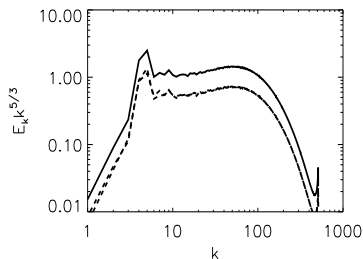
Example I: Helical Decomposition

Homochiral turbulence



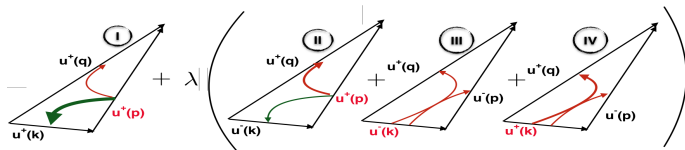
When interactions are restricted to homochiral the energy cascades inversely to large scales

Example I: Helical Decomposition



Even in the full Navier-Stokes equations homochiral interactions cascade energy inversely! (but are sub-dominant)

Example I: Helical Decomposition



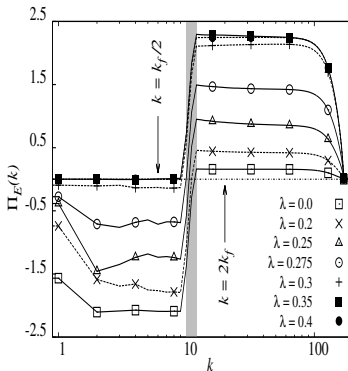
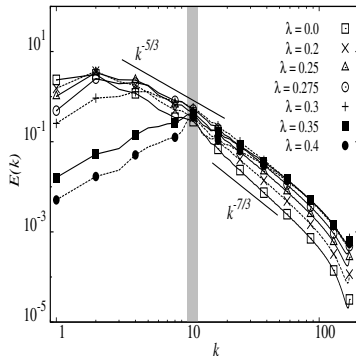
$$\partial_t \mathbf{u} = [\mathcal{NL}] - \nu \Delta^4 \mathbf{u} - \mu \Delta^{-2} \mathbf{u} + \mathbf{F}$$

$$\mathcal{NL} = \lambda \mathbb{P}(\mathbf{u} \times \mathbf{w}) + (1 - \lambda) [\mathbb{P}^+(\mathbf{u}^+ \times \mathbf{w}^+) + \mathbb{P}^-(\mathbf{u}^- \times \mathbf{w}^-)]$$

$\lambda = 1$: Navier-Stokes, $\lambda = 0$ homochiral NS :

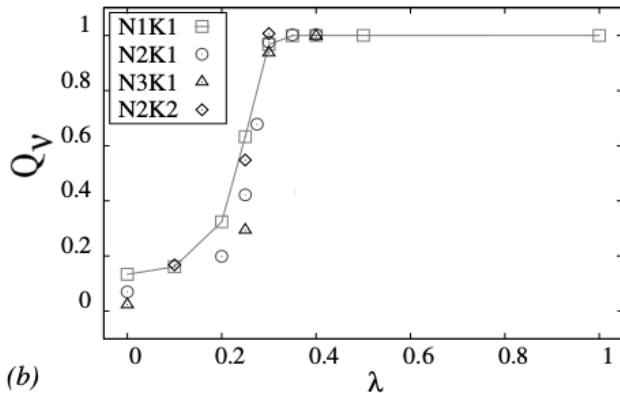
- the equations conserve E, H for any value of λ .
- the nonlinearity is self similar under scale transformations

Example I: Helical Decomposition



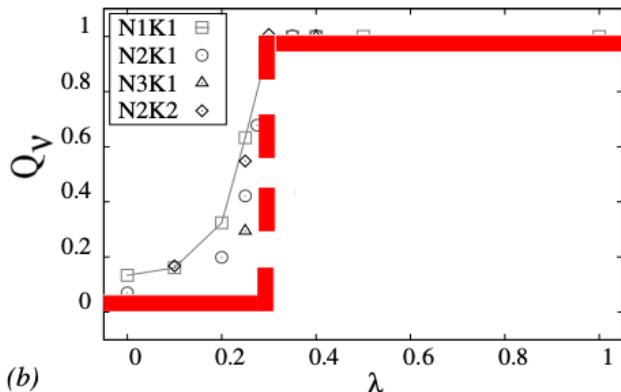
Example I: Helical Decomposition

$$Q_\nu = \epsilon_\nu / \epsilon_{in}$$



Example I: Helical Decomposition

$$Q_\nu = \epsilon_\nu / \epsilon_{in}$$



The transition becomes critical at $Re, R_\alpha, L/\ell_{in} \rightarrow \infty$.

Why is the transition 1st order?

- The model is scale invariant
- Same physics control both the large and the small scales
- If there is enough scale separation so that forcing and damping effects can be neglected the system can cascade energy either forward or inverse not both!

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Why is the transition 1st order?

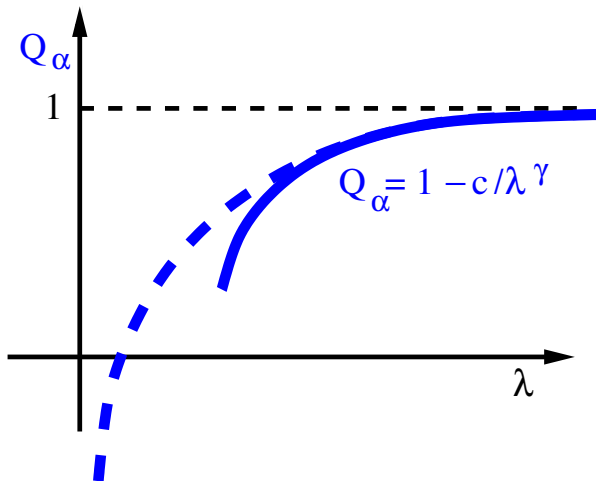
- The model is scale invariant
- Same physics control both the large and the small scales
- If there is enough scale separation so that forcing and damping effects can be neglected the system can cascade energy either forward or inverse not both!
- **Thus the transition (if present) has to be 1st order!**

Implication:

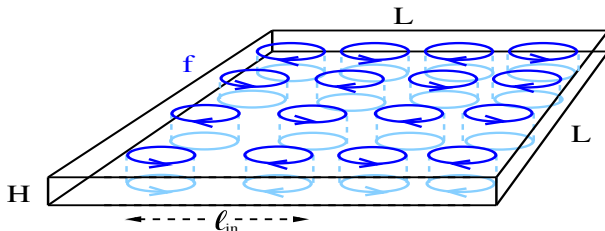
For a **split cascade** to be present the large scales have to follow different physics than the small scales!

Example II

smooth transition



Example II: Very thin layer $H \ll \ell_{in}$



$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P - \alpha \Delta^{-m} \mathbf{u} - \nu \Delta^n \mathbf{u} + \mathbf{f}$$

$$\epsilon_\alpha = \alpha \langle |\nabla^{-m} \mathbf{u}|^2 \rangle, \quad \epsilon_{in} = \langle \mathbf{f} \cdot \mathbf{u} \rangle, \quad \epsilon_\nu = \nu \langle |\nabla^n \mathbf{u}|^2 \rangle,$$

$$\epsilon_{in} = \epsilon_\alpha + \epsilon_\nu$$

$$\lambda = \frac{\ell_{in}}{H}, \quad Q_\nu = \frac{\epsilon_\nu}{\epsilon_{in}} \quad \& \quad Q_\alpha = \frac{\epsilon_\alpha}{\epsilon_{in}}$$

Example II: Very thin layer $H \ll \ell_{in}$

PRL **104**, 184506 (2010)

PHYSICAL REVIEW LETTERS

week ending
7 MAY 2010

Turbulence in More than Two and Less than Three Dimensions

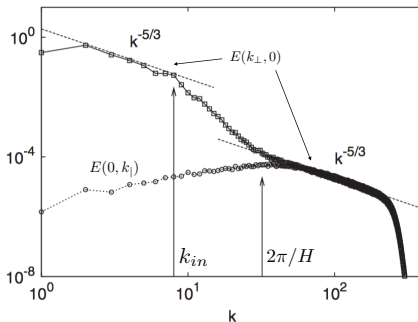
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¹CNRS URA 2171, Institut Pasteur, 28 rue du docteur Roux, 75724 Paris Cedex 15, France

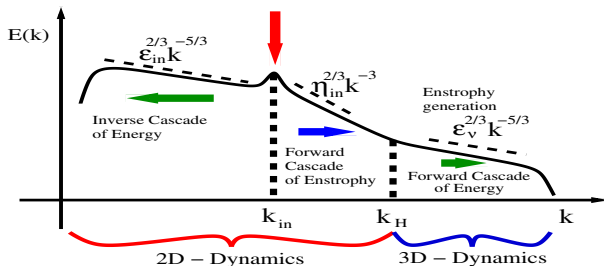
²Dipartimento di Fisica Generale and INFN, Università di Torino, via P. Giuria 1, 10125 Torino, Italy

³CNRS UMR 6621, Laboratoire J.A. Dieudonné, Université de Nice Sophia Antipolis, Parc Valrose, 06108 Nice, France

(Received 12 January 2010; published 7 May 2010)

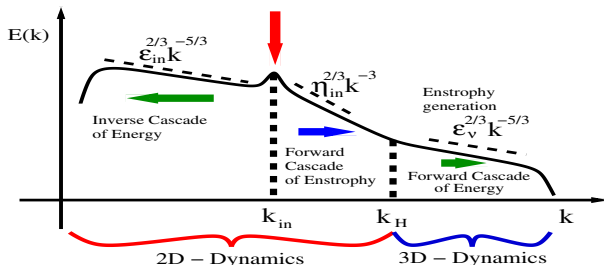


Example II: Very thin layer $H \ll \ell_{in}$



- 2D dynamics for $k \ll k_H$ ($\ell \gg H$) $\Rightarrow$ inverse energy cascade.
- 3D dynamics for $k \gg k_H$ ($\ell \ll H$) $\Rightarrow$ forward energy cascade

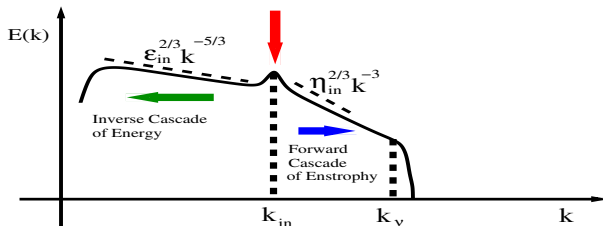
Example II: Very thin layer $H \ll \ell_{in}$



- 2D dynamics for $k \ll k_H$ ($\ell \gg H$) $\Rightarrow$ inverse energy cascade.
- 3D dynamics for $k \gg k_H$ ($\ell \ll H$) $\Rightarrow$ forward energy cascade

How can a forward energy cascade build up at $k > k_H$?

Example II: Very thin layer ($H = 0$, ie 2D)



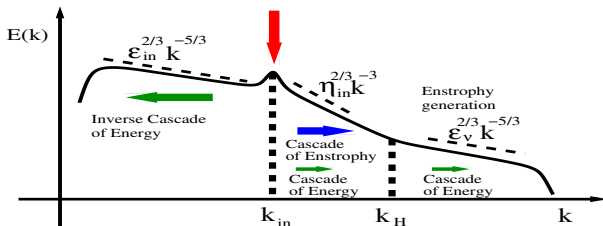
Enstrophy arrives at k_v at a rate

$$\eta_{in} = \eta_v = \epsilon_{in} k_{in}^2$$

Energy arrives at k_v at a rate

$$\epsilon_v \propto \epsilon_{in} \left(\frac{k_{in}}{k_v} \right)^2$$

Example II: Very thin layer $H \ll \ell_{in}$, ($\lambda = \ell_{in}/H$)

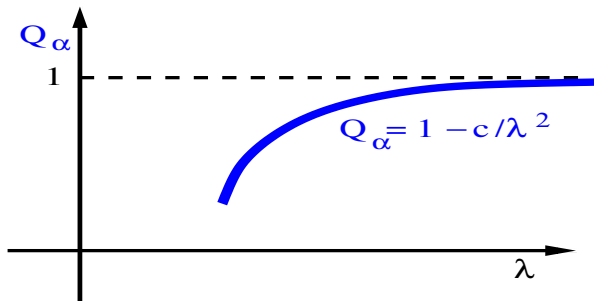


- Replace k_ν by k_H

Energy that arrives at small scales is given by:

$$\epsilon_\nu = \epsilon_{in} \left(\frac{k_{in}}{k_H} \right)^2 = \epsilon_{in} \left(\frac{H}{\ell_{in}} \right)^2 = \epsilon_{in} \frac{1}{\lambda^2}$$

Example II: Very thin layer $H \ll \ell_{in}$, ($\lambda = \ell_{in}/H$)

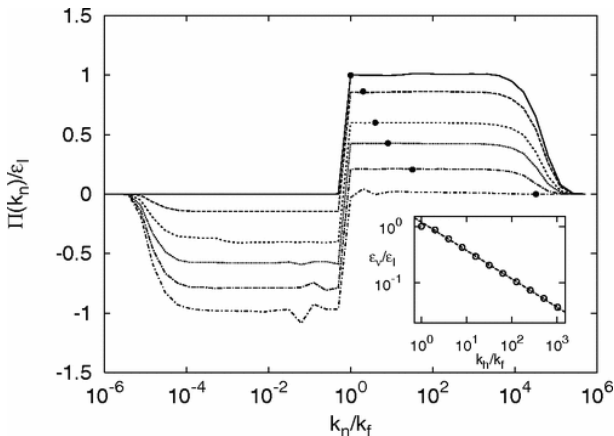


$$Q_\nu = \epsilon_\nu / \epsilon_{in} \propto 1/\lambda^2 \quad \Rightarrow \quad Q_\alpha = (1 - Q_\nu) \simeq 1 - c/\lambda^2$$

This provides an example for a '*smooth*' transition from a split cascade to a strictly forward cascade.

Example II: Very thin layer $H \ll \ell_{in}$, ($\lambda = \ell_{in}/H$)

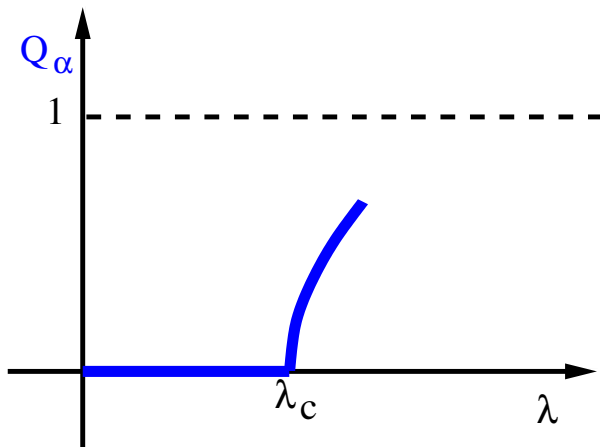
Scaling tested (up to my knowledge) only using shell models!



$$Q_\nu = \epsilon_\nu/\epsilon_{in} \propto 1/\lambda^2$$

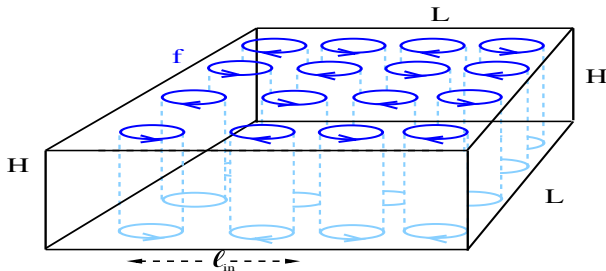
Example III

Second order phase transitions



Example III: Thick layer Turbulence $H \sim \ell_{in}$

From a strictly forward to an inverse cascade



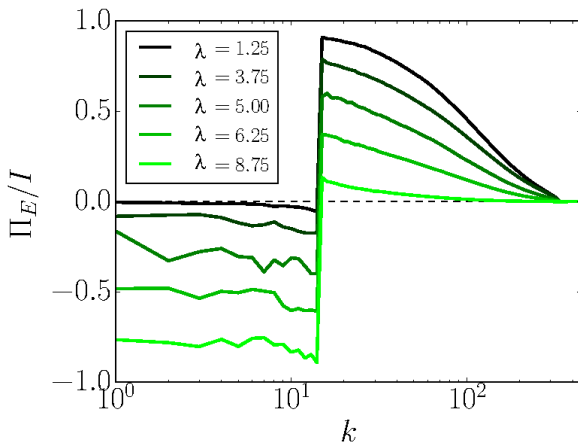
$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P - \alpha \Delta^{-m} \mathbf{u} - \nu \Delta^n \mathbf{u} + \mathbf{f}$$

$$\epsilon_\alpha = \alpha \langle |\nabla^{-m} \mathbf{u}|^2 \rangle, \quad \epsilon_{in} = \langle \mathbf{f} \cdot \mathbf{u} \rangle, \quad \epsilon_\nu = \nu \langle |\nabla^n \mathbf{u}|^2 \rangle,$$

$$\lambda = \frac{\ell}{H}, \quad Q_\nu = \frac{\epsilon_\nu}{\epsilon_{in}} \quad \& \quad Q_\alpha = \frac{\epsilon_\alpha}{\epsilon_{in}}$$

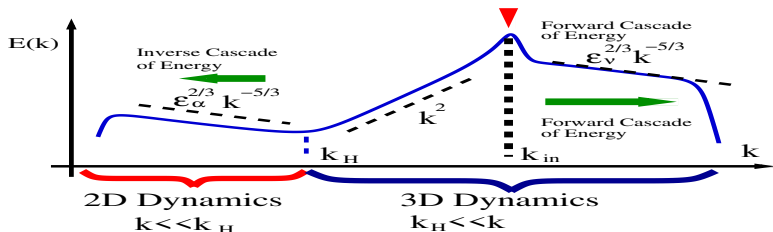
Example III: Thick layer Turbulence $H \sim \ell_{in}$

Fluxes for different values of $\lambda = \ell_{in}/H$



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Example III: Thick layer Turbulence ($\lambda = \ell_{in}/H$)



$$E(k) \propto \epsilon_\alpha^{2/3} k^{-5/3} \quad k \ll k_H \quad (1)$$

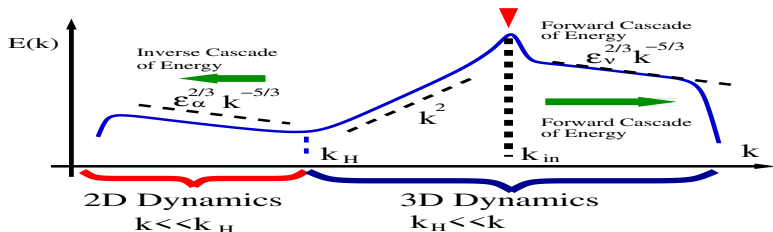
$$E(k) \propto (\epsilon_{in}^{2/3} k_{in}^{-11/3}) k^2 \quad k_H \ll k \ll k_{in} \quad (2)$$

$$E(k) \propto \epsilon_{in}^{2/3} k^{-5/3} \quad k_{in} \ll k \quad (3)$$

Equating $E(k)$ at $k = k_H$ we obtain

$$\epsilon_\alpha = \epsilon_{in} \left(\frac{k_H}{k_{in}} \right)^{11/2} = \epsilon_{in} \lambda^{11/2} \quad \text{smooth?}$$

Example III: Thick layer Turbulence ($\lambda = \ell_{in}/H$)



$$E(k) \propto \epsilon_\alpha^{2/3} k^{-5/3} \quad k \ll k_H \quad (1)$$

$$E(k) \propto (\epsilon_{in}^{2/3} k_{in}^{-11/3}) k^2 \quad k_H \ll k \ll k_{in} \quad (2)$$

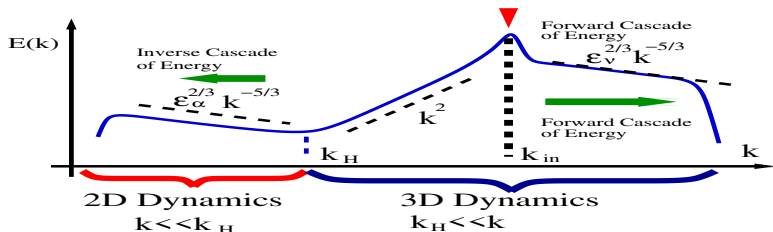
$$E(k) \propto \epsilon_{in}^{2/3} k^{-5/3} \quad k_{in} \ll k \quad (3)$$

Equating $E(k)$ at $k = k_H$ we obtain

$$\epsilon_\alpha = \epsilon_{in} \left(\frac{k_H}{k_{in}} \right)^{11/2} = \epsilon_{in} \lambda^{11/2} \quad \text{smooth?}$$

However...

Example III: Thick layer Turbulence ($\lambda = \ell_{in}/H$)



at $\ell \simeq H$:

Inverse Flux due to
local 2D interactions

Forward flux due to
a turbulent eddy viscosity

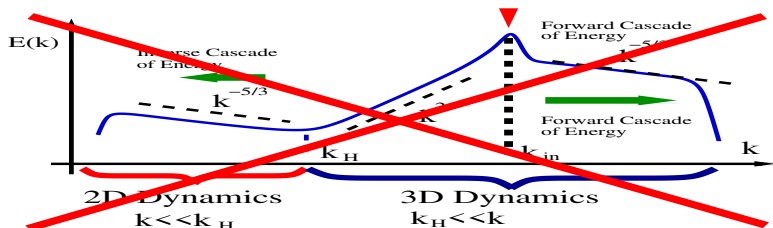
$$\epsilon_{in} \left(\frac{k_H}{k_{in}} \right)^{11/2}$$

$$\nu_{eddy} \frac{U_H^2}{H^2}, \quad (\nu_{eddy} \sim u_{in} \ell_{in})$$

$$\epsilon_{in} \left(\frac{k_H}{k_{in}} \right)^5$$

The effect of the eddy viscosity is larger by a factor of $(k_{in}/k_H)^{1/2}$!

Example III: Thick layer Turbulence ($\lambda = \ell_{in}/H$)

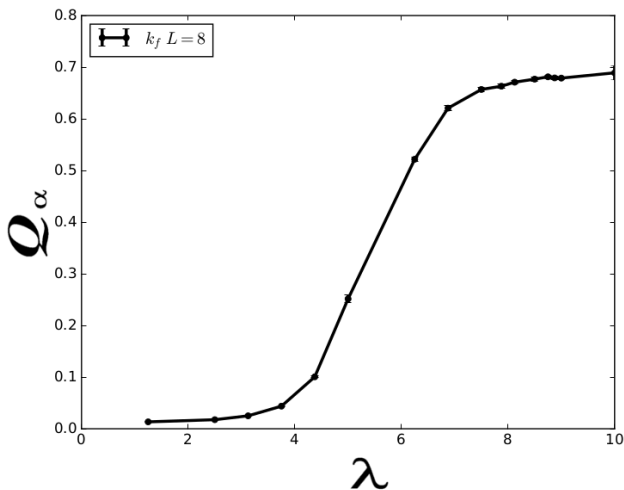


- Nonlocal interactions become important!
- Eddy-viscosity depends on the properties of the forcing scales
- if the forcing scales are sufficiently 2D (3D)
there is (is not) an inverse cascade
- the transition occurs at

$$H_c \propto \ell_{in}$$

Example III: Thick layer Turbulence ($\lambda = \ell_{in}/H$)

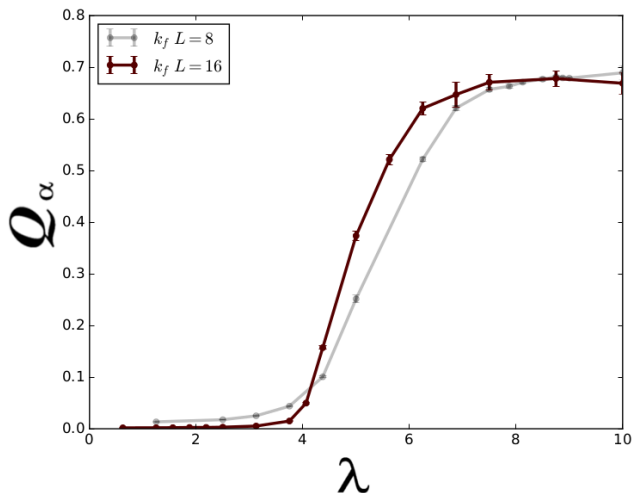
Inverse flux ($Q_\alpha = \epsilon_\alpha/\epsilon_{in}$) as a function of ($\lambda = \ell_{in}/H$)



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Example III: Thick layer Turbulence ($\lambda = \ell_{in}/H$)

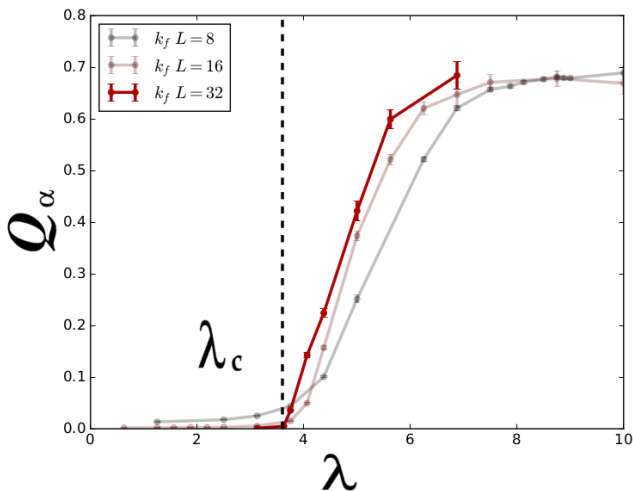
Inverse flux ($Q_\alpha = \epsilon_\alpha/\epsilon_{in}$) as a function of ($\lambda = \ell_{in}/H$)



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Example III: Thick layer Turbulence ($\lambda = \ell_{in}/H$)

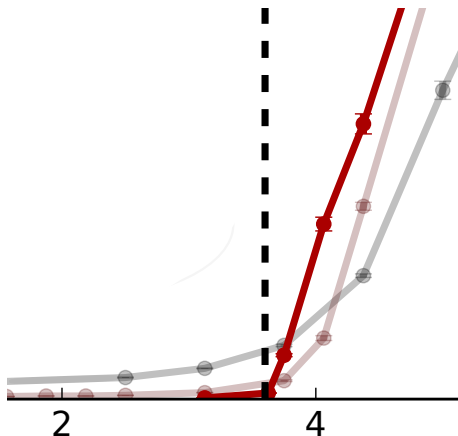
Inverse flux ($Q_\alpha = \epsilon_\alpha / \epsilon_{in}$) as a function of ($\lambda = \ell_{in}/H$)



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

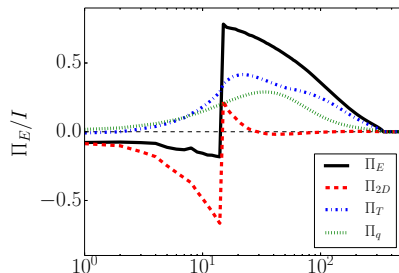
Example III: Thick layer Turbulence ($\lambda = \ell_{in}/H$)

Inverse flux ($Q_\alpha = \epsilon_\alpha/\epsilon_{in}$) as a function of ($\lambda = \ell_{in}/H$)



Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Example III: Decomposed fluxes

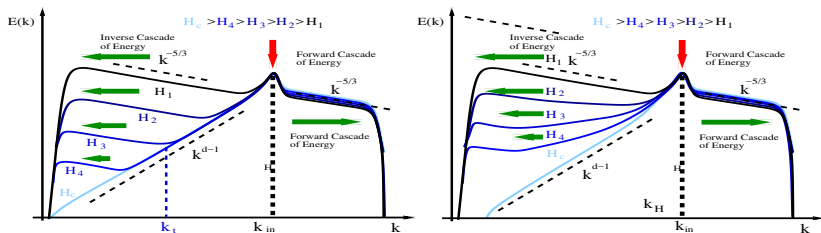


— Total Flux	$\Pi(k) = -\langle \mathbf{u}^{<k} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \rangle,$
- - - 2D Flux	$\Pi_{2D}(k) = -\langle \mathbf{u}_{2D}^{<k} \cdot (\mathbf{u}_{2D} \cdot \nabla \mathbf{u}_{2D}) \rangle,$
- · - · - 2D-3D Flux	$\Pi_q(k) = -\langle \mathbf{u}_{2D}^{<k} \cdot (\mathbf{v}_{3D} \cdot \nabla \mathbf{v}_{3D}) \rangle - \langle \mathbf{v}_{3D}^{<k} \cdot (\mathbf{v}_{3D} \cdot \nabla \mathbf{u}_{2D}) \rangle$
· · · 3D Flux	$\Pi_q(k) = -\langle \mathbf{v}_{3D}^{<k} \cdot (\mathbf{u} \cdot \nabla \mathbf{v}_{3D}) \rangle$

Some processes move energy to large scales
and some processes move energy to small scales!

Example III: Spectrum transition

Transition from a 'thermal' to a 'Kolmogorov' spectrum



- Either the spectrum changes from a thermal spectrum

$$E(k) \propto (\epsilon_{in}^{2/3} k_{in}^{-8/3}) k$$

to a Kolmogorov spectrum

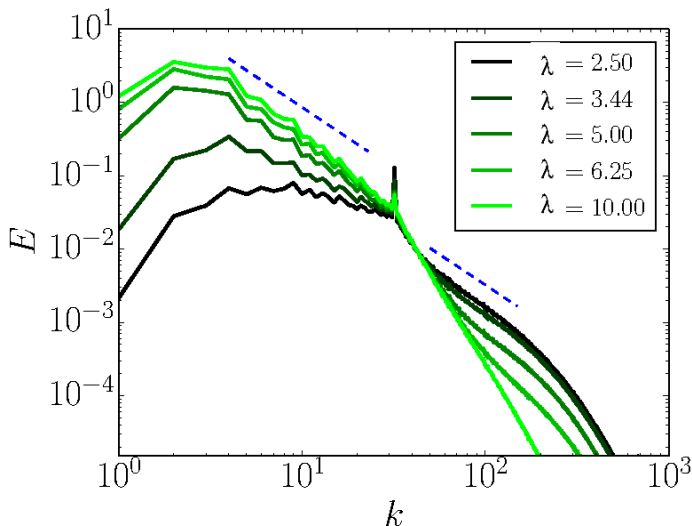
$$E(k) \propto \epsilon_{\alpha}^{2/3} k^{-5/3}$$

at $k_t = k_{in}(\epsilon_{\alpha}/\epsilon_{in})^{1/4}$

or

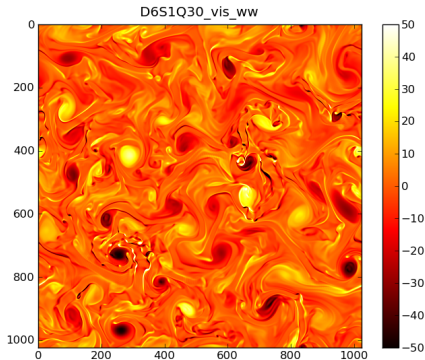
- New power-laws (and new physics) appear!

Example III: Spectrum transition



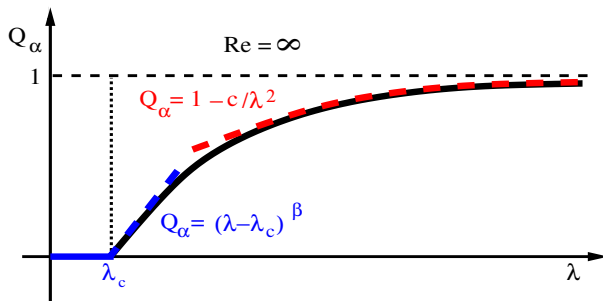
Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

Example III: Physical space, 2D and 3D Dynamics



Predator-prey dynamics?

Cascade transitions in thin layers (at infinite Re)



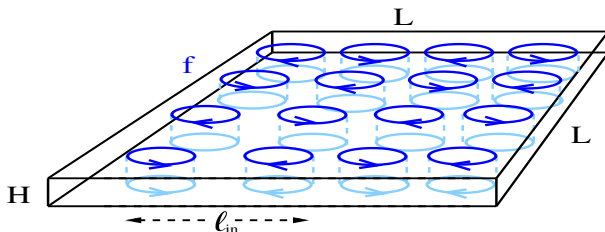
For infinite Reynolds displayed

- a “2nd order” transition (from a forward cascade to a split cascade)
- a “smooth” transition (from a split cascade to an inverse cascade)

VI.

Finite Re and Finite size effects

Finite Re: Transition to exactly 2D Dynamics

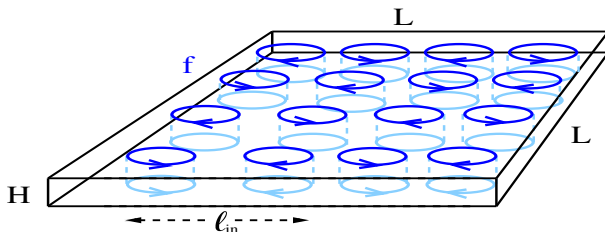


$$\mathbf{u}_{2D} = \bar{\mathbf{u}}, \quad \mathbf{v}_{3D} = \mathbf{u} - \mathbf{u}_{2D}$$

$$\partial_t \mathbf{u}_{2D} + \mathbf{u}_{2D} \cdot \nabla \mathbf{u}_{2D} = -\overline{\mathbf{v}_{3D} \cdot \nabla \mathbf{v}_{3D}} - \overline{\nabla P} + \nu \Delta \mathbf{u}_{2D} + \eta \Delta^{-2} \mathbf{u}_{2D} + \mathbf{F}_{2D}$$

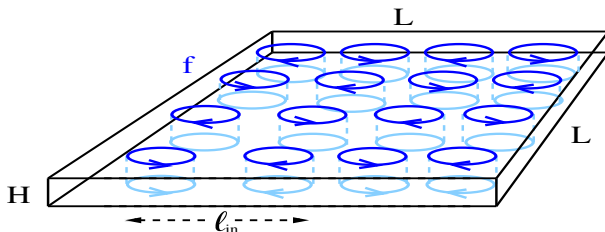
$$\partial_t \mathbf{v}_{3D} + \mathbf{u}_{2D} \cdot \nabla \mathbf{v}_{3D} + \mathbf{v}_{3D} \cdot \nabla \mathbf{v}_{3D} = -\mathbf{v}_{3D} \cdot \nabla \mathbf{u}_{2D} + \nabla p' + \nu \Delta \mathbf{v}_{3D}$$

Finite Re: Transition to exactly 2D Dynamics ($\lambda = \ell_{in}/H$)



$$\frac{d}{dt}\langle \mathbf{v}_{3D}^2 \rangle = -\langle \mathbf{v}_{3D} \cdot (\nabla \mathbf{u}_{2D}) \cdot \mathbf{v}_{3D} \rangle - \nu \langle (\nabla_{2D} \mathbf{v}_{3D})^2 \rangle - \nu \langle (\partial_z \mathbf{v}_{3D})^2 \rangle$$

Finite Re: Transition to exactly 2D Dynamics ($\lambda = \ell_{in}/H$)



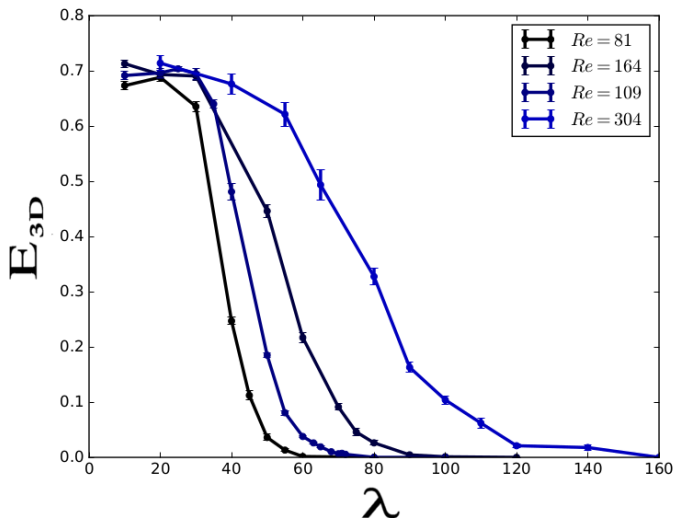
$$\frac{d}{dt} \langle \mathbf{v}_{3D}^2 \rangle = - \langle \mathbf{v}_{3D} \cdot (\nabla \mathbf{u}_{2D}) \cdot \mathbf{v}_{3D} \rangle - \nu \langle (\nabla_{2D} \mathbf{v}_{3D})^2 \rangle - \nu \langle (\partial_z \mathbf{v}_{3D})^2 \rangle$$

$$\text{if } \nu \left(\frac{2\pi}{H} \right)^2 > \|\nabla \mathbf{u}_{2D}\|_{\infty} \text{ then } \langle \mathbf{v}_{3D}^2 \rangle \rightarrow 0$$

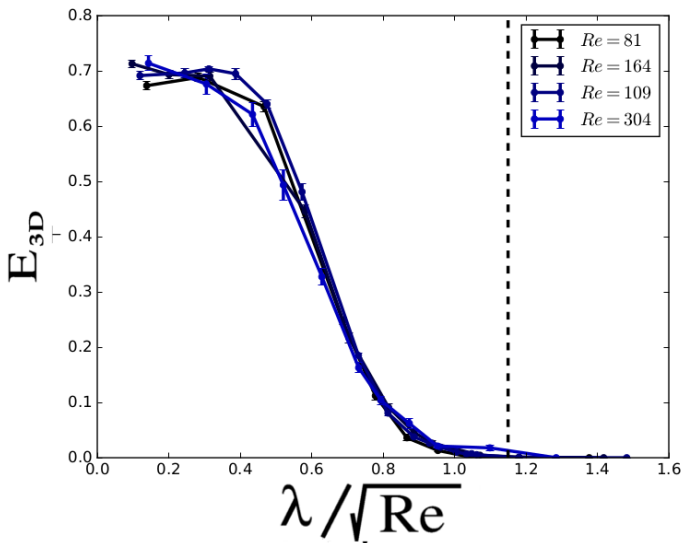
$$H_c \sim \ell_{\nu} \sim \ell_f / \sqrt{Re}$$

[see Gallet & Doering J. Fluid Mech. **773**, 154 (2015) for proof]

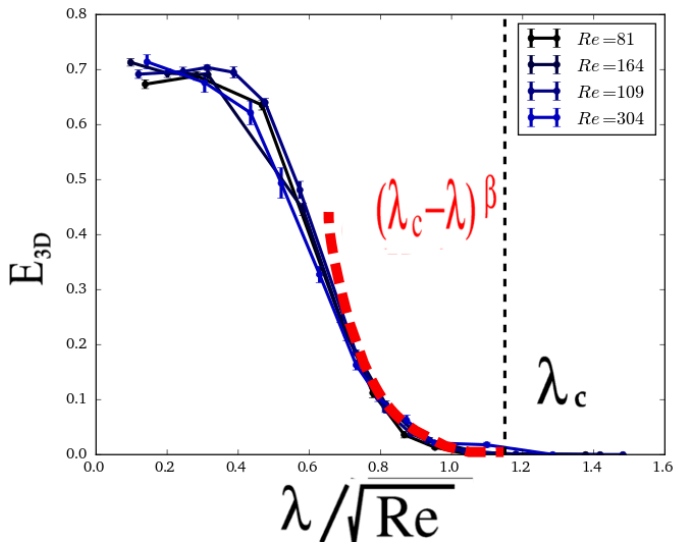
Finite Re: Transition to exactly 2D Dynamics ($\lambda = \ell_{in}/H$)



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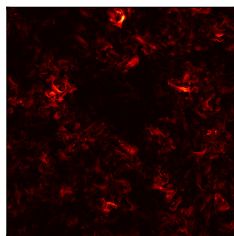


Finite Re: Transition to exactly 2D Dynamics ($\lambda = \ell_{in}/H$)

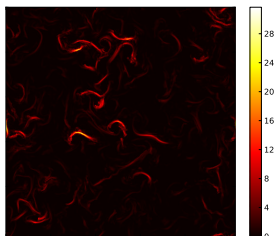


Finite Re: Transition to exactly 2D Dynamics ($\lambda = \ell_{in}/H$)

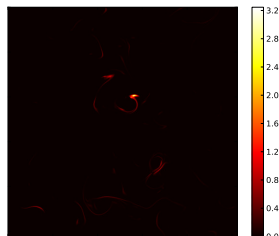
Spatial intermittency 3D energy density



$$\frac{(\lambda_c - \lambda)}{\lambda_c} = 0.70,$$



$$\frac{(\lambda_c - \lambda)}{\lambda_c} = 0.40,$$



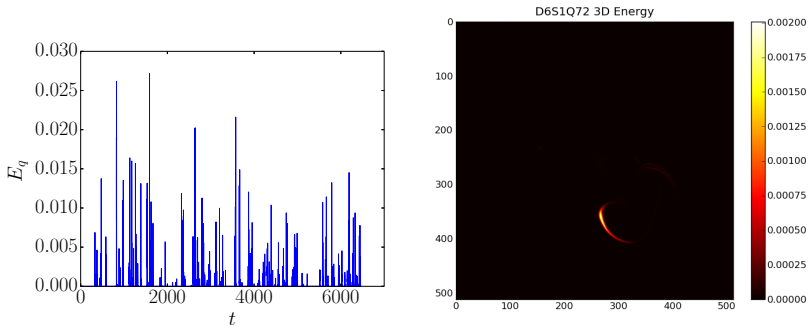
$$\frac{(\lambda_c - \lambda)}{\lambda_c} = 0.10$$

$$\langle E_{3D} \rangle = \frac{\varepsilon_{NL} V_{on}}{(V_{off} + V_{on})} \quad \text{where} \quad \varepsilon_{NL} \propto (\lambda_c - \lambda)^{\beta_1} \quad \text{and} \quad V_{on} \propto (\lambda_c - \lambda)^{\beta_2}$$

Picture from: Benavides, et al J. Fluid Mech. 822, 364-385 (2017)

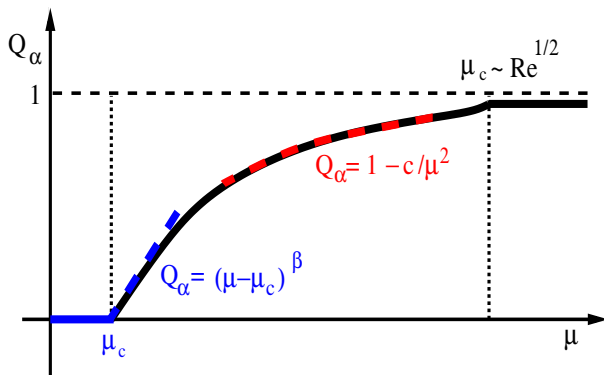
Finite Re: Transition to exactly 2D Dynamics ($\lambda = \ell_{in}/H$)

Spatio-Temporal intermittency

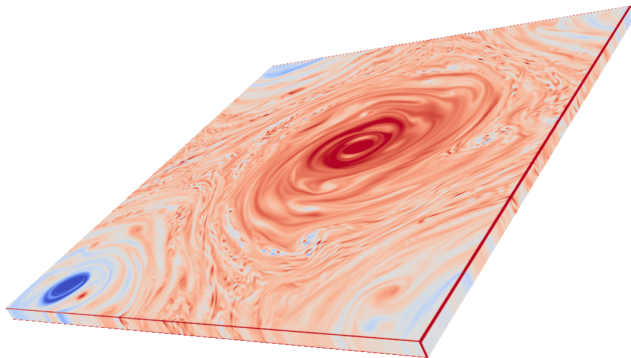


$$\langle E_{3D} \rangle = \frac{\varepsilon_{NL} V_{on} T_{on}}{(V_{off} + V_{on})(T_{on} + T_{off})}$$

Cascade transitions in thin layers (at finite Re)



Finite Size: Condensates



2D turbulence

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- Frishman, A. & Herbert, C. Phys. Rev. Lett. 120, 204505. (2018)
- ...

Finite Size: Condensates

Thin Layers

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Rotating flows

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Rotating Convection

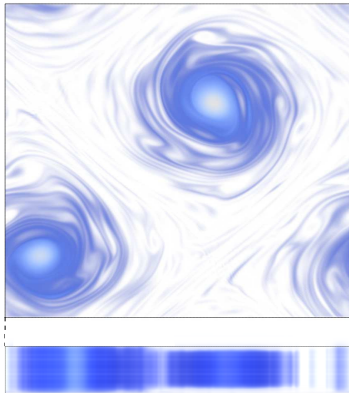
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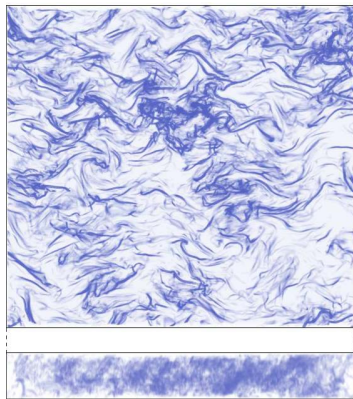
Finite Size: Condensates (Thin Layers $\lambda = \ell_f/H$)

Two distinct states



**Hi-Energy
Organized state**

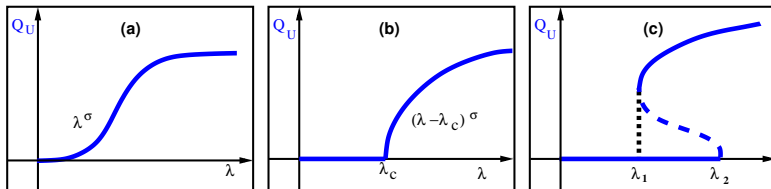
$$\lambda > \lambda_c$$



**Low-Energy
Disorganized state**

$$\lambda < \lambda_c$$

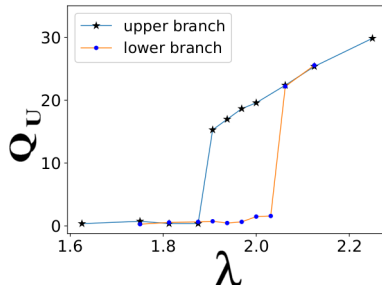
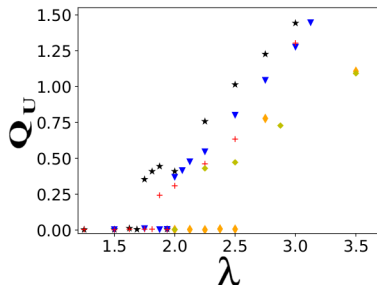
Finite Size: Condensates



$$Q_U = \frac{E}{(\epsilon_{in} \ell_{in})^{2/3}}$$

- A more relevant order parameter is the energy of the condensate
- Same classification as with the cascade rate (Q_α, Q_ν).
- Continuity of Q_α, Q_ν does not imply continuity of Q_U

Thin layer turbulence

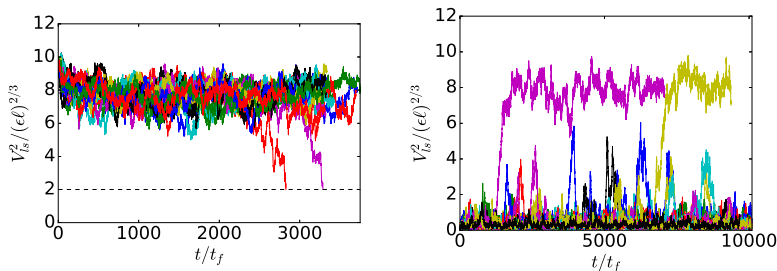


Same behavior observed in:

Rotating turbulence: N. Yokoyama et al Phys. Rev. Fluids 2, 092602 (2017)

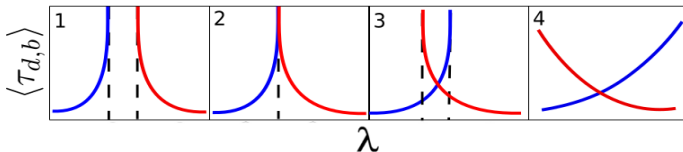
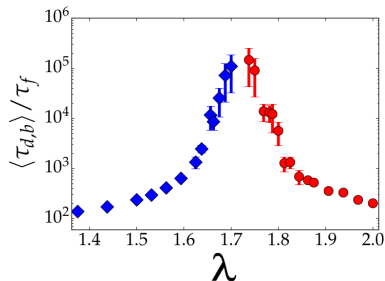
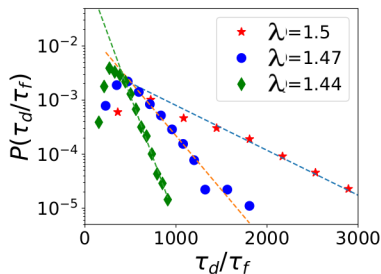
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Trajectories close to the critical points



There are rare events that transition the flow from one state to an other

Mean transition time



The transition to a condensate state is controlled by rare events!

V.

Conclusions

Conclusions

- 1 Criticality of the energy flux develops in the large L, Re, R_α limit.
- 2 A split cascade is possible if large scales follow different physics than the small scales.
- 3 1st order (discontinuous) transitions are expected in self similar models.
- 4 Smooth transitions are met when the local interactions dominate and change from inverse cascading to forward cascading.
- 5 2nd order transitions are the least understood transitions.
- 6 2nd order transition are met when local and non-local interactions compete for the direction of cascade. It might be controlled by Predator - Prey dynamics.
- 7 For finite Re spatio-temporal intermittency appears
- 8 For finite size condensates controlled by rare events are present.

- **Critical transitions:**

Criticality of the transitions has only been demonstrated in numerical simulations of moderate scale separations (moderate Λ , Re , R_α) and models and is still conjectural.

There is very little theory and very few experiments on the subject.

- **Phase space diagram of rotating and stratified turbulence:**

Understanding rotating and stratified turbulence in its different limits is crucial for understanding the atmosphere and the ocean.

- **Transition to exactly 2D flows and spatio-temporal intermittency:** Appearance of new exponents.

- **Condensates and thermal equilibrium states:** Rare event sampling methods could help.

Future work

- There is a lot more we need to understand!
- There is need for new experiments!
- There is need for new numerical simulations!
- There is need for a theory!
- **There is need for young researchers!**

**Thank you very much
for your patience!**

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