• **FIRST PART** : On the local structure of incompressible homogeneous and isotropic turbulence: Balance of kinetic energy, axiomatics of Kolmogorov's phenomenology, and some illustrations.

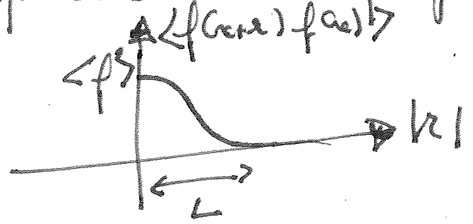
• Very particular case of fluid turbulence
  ↳ schematical
  ↳ typically approach adopted in physics and mathematics

• Call L the "large scale"

in experiments, numerics,

size of the mesh grid used in "wind tunnels" and/or diameter of the nozzle of a jet

↳ forcing term (defined later)
spatial correlation length



→ Scale L is typical of the scale at which energy is injected into the fluid.

→ Non UNIVERSAL (depends in particular on the shape of the container)

→ we will be mostly interested here in the behavior of velocity fluctuations at scales $\ell \ll L$.

- The phenomenology that will be depicted is mostly based (and developed for) on experimental observations, in wind tunnels, under the Taylor approximation, using hot-wires.
(See for instance TENNEKES AND LUMLEY)

- Notations : Take $u(x,t) \in \mathbb{R}^3$ the (vector) velocity field. $p \in \mathbb{R}^3$

and consider the Navier-Stokes equations, $\forall (x,t) \in \mathbb{R}^3 \times \mathbb{R}$

$$\boxed{\begin{cases} \dfrac{\partial u}{\partial t} + (u.\nabla)u = -\nabla p + \nu \Delta u + \underbrace{f}_{\text{divergence free forcing field.}} \\ \operatorname{div} u = 0 \end{cases}} \quad (*)$$

or component by components

$$\boxed{\begin{aligned} &\frac{\partial u_i}{\partial t} + u_k \partial_k u_i = -\partial_i p + \nu \Delta u_i + f_i \\ &\partial_k u_k = 0 \end{aligned}}$$

- remark 1 : Exp measurements, under The Taylor's hypothesis, gives access to the streamwise velocity component (along the mean flow) say, for instance, $u_1(x_1, y_0, z_0, t_0)$ for a given set (y_0, z_0, t_0) and $x_1 \in \mathbb{R}$.

- remark 2 : NS equation is a closed set of non linear partial coupled equations.
In particular, pressure p is a given non local functional of the spatial distribution of velocity gradients, solution of the Poisson equation, and has nothing to do with the interpretation of pressure in a perfect gaz.

- Similarity in fluid mechanics.

As we will see, the system NS (4) will eventually reach a statistically stationary state, in which variance of velocity fluctuations is finite.

Note $\boxed{\sigma^2 = \langle |u|^2 \rangle}$

rem: for statistically homogeneous and isotropic flows.

$$\langle u_i \rangle = 0 \quad \text{and} \quad \sigma^2 = \langle u_i u_i \rangle = 3 \langle u_1^2 \rangle$$

↑ ensemble.

→ Note $T = \dfrac{L}{\sigma}$ the implied time scale (called turnover time)

In the set of nondimensionalized variables $(x', t') = \left(\frac{x}{L}; \frac{t}{T}\right)$

define
$$u'(x';t') = \frac{1}{\sigma} u\left(\frac{x}{L}; \frac{t}{T}\right)$$
$$p'(x';t') = \frac{T}{\sigma L}\left(p\left(\frac{x}{L}; \frac{t}{T}\right)\right)$$
$$f'(x';t') = \frac{T}{\sigma} f\left(\frac{x}{L}; \frac{t}{T}\right)$$

N.S. reads

$$\boxed{\frac{\partial u'}{\partial t'} + (u' \cdot \nabla_{x'}) u' = -\nabla_{x'} p' + \frac{1}{Re} \Delta_{x'} u' + f'}$$

where $\boxed{Re = \dfrac{\sigma L}{\nu}}$ Reynolds number.

rem 1: only an illustration of the observation that in 3d h.i.t., fluids of different viscosities have similar statistical behaviours at small scales for a given Reynolds number.

rem 2: In the following, I will use in a equivalent manner

$$\boxed{\begin{array}{c} Re \to +\infty \\ \text{or} \\ \nu \to 0 \end{array}}$$

- KINETIC ENERGY BUDGET.

Consider the (local) kinetic energy $e_c = \frac{1}{2}|u|^2$

Make use of N.S. equations to obtain:

$$\boxed{\frac{1}{2}\frac{\partial |u|^2}{\partial t} + \operatorname{div} I = -\nu \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial x_j}\right)^2 + u \cdot f}$$

- It is called a conservation equation

- I : (spatial) vector current

$$I = \frac{1}{2}|u|^2 u + pu - \nu \nabla\left(\frac{1}{2}|u|^2\right)$$

- $-\nu \sum_{i,j=1}^{3} (\partial_j u_i)^2 \underset{?}{\leq} 0$ viscous dissipation

- $u \cdot f$: injected energy (power).

- Hints of the proof: take the scalar product of u with N.S.

  - remark → Using incompressibility, pressure contributions are of divergence-type: $u \cdot \nabla p = \nabla \cdot (pu)$

    → Using incompressibility, non linear contributions are of divergence-type: $u \cdot [(u \cdot \nabla) u] = \nabla \cdot \left(\frac{1}{2}|u|^2 u\right)$

    → Again with incomp.

    $$u \cdot \Delta u = \operatorname{div} \nabla \left(\frac{1}{2}|u|^2\right) - \sum_{i,j=1}^{3} (\partial_j u_i)^2$$

- Consequences on global and averaged kinetic energy.

Consider $E_c(t) = \int_{x \in V} \frac{1}{2} |u|^2(x,t)\, d^3x$

→ Thus $\frac{\partial E_c}{\partial t}$ governed by competition of (global) viscous dissipation and (global) injected power

+ flux of I through the boundaries.

divergence theorem

- In a statistically homogeneous (and isotropic) flow, the average current $\langle I(x) \rangle$ independent on x, therefore $\langle \operatorname{div} I \rangle = \operatorname{div} \langle I \rangle = 0$

Therefore $$\frac{\partial \langle e_c \rangle(t)}{\partial t} = \underbrace{- \langle \nu \sum_{i,j=1}^{3} (\partial_j u_i)^2 \rangle}_{-\varepsilon} + \langle f \cdot u \rangle$$

- (As observed) $\frac{\partial \langle e_c \rangle}{\partial t} = 0$: statistically stationary.

$\langle e_c \rangle = \frac{1}{2} \sigma^2 = \frac{3}{2} \langle u_1^2 \rangle$

↳ Axiomatics of Kolmogorov gives a consistent phenomenology of the kinetic energy budget at infinite Reynolds number, or equivalently, when $\nu \to 0$.

- Axiomatics of Kolmogorov, and the phenomenology of (3d) fully developed turbulence.

(1) When stirred at large scales, the velocity field reaches a statistically stationary regime such that

$$\lim_{\nu\to 0}\lim_{t\to\infty} \langle \tfrac{1}{3}|u|^2(t)\rangle = \sigma^2 < +\infty$$

rem: . Far from being obvious from the dynamics.
→ see illustration with the stochastic heat equation.

. The limit $\nu\to 0$ can be tested experimentally (using for instance Helium gaz close to the critical point; see work of Castaing et al.

(2) To ensure such an efficient way to dissipate energy, (maintaining in particular finite kinetic energy), the flow will develop small scales (non provided by the forcing) such that

$$\lim_{\nu\to 0}\lim_{t\to\infty} \varepsilon(t) = \langle \nu \sum_{i,j=1}^{3} (\partial_j u_i)^2(t) \rangle = \varepsilon > 0$$

- Incidentally velocity gradient variance diverges like $\frac{1}{\nu}$
- Incidentally, ε is governed by the flow at large scales, no other choices than $\boxed{\varepsilon = \mathcal{O}\left(\frac{\sigma^3}{L}\right)}$.

- Incidentally, $\varepsilon = \langle f\cdot u\rangle$, the flow "follows" the forcing, even at infinite Reynolds number, and keeps a finite correlation with it.

→ Experimental observations :

In a statistical homogeneous and isotropic framework, the two-point structure of turbulence is fully characterized by the correlation function of the streamwise component

$$\boxed{f(r) = \cancel{f(\ldots)} = \langle u_1(x_1, y, z, t)\, u_1(x_1+r, y, z, t)\rangle}$$

• In particular:

$$\boxed{C_{ij}(h) = \langle u_i(0)\, u_j(h)\rangle = \frac{f(|h|) - g(|h|)}{|h|^2} h_i h_j + g(|h|)\,\delta_{ij}}$$

where $g(|h|) = f(|h|) + \frac{1}{2}|h|\, f'(|h|)$

"transverse" correlation function

• Incidentally, the correlation of velocity gradients is determined the variance of the gradient of the streamwise component,

$$f''(0) = -\langle (\partial_{x_1} u_1)^2\rangle .$$

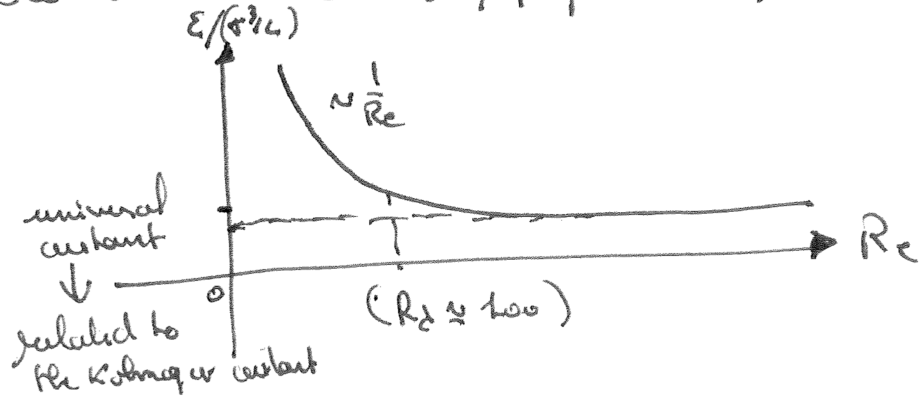
Making use of $\left\langle \frac{\partial u_i(x)}{\partial x_q} \frac{\partial u_j(x)}{\partial x_p}\right\rangle = -\frac{\partial^2}{\partial h_q \partial h_p} C_{ij}(h) \Big|_{h=0}$

we get

$$\left\langle \frac{\partial u_i(x)}{\partial x_q} \frac{\partial u_j(x)}{\partial x_p}\right\rangle = 2\langle (\partial_{x_1} u_1)^2\rangle \left[\delta_{ij}\delta_{qp} - \frac{1}{4}\delta_{iq}\delta_{jp} - \frac{1}{4}\delta_{ip}\delta_{jq}\right]$$

such that $\boxed{\varepsilon = \nu\langle \partial_q u_i\, \partial_q u_i\rangle = 15\nu\langle (\partial_{x_1} u_1)^2\rangle}$

See Batchelor (53), Pope (00)



See compilation of data by Sreenivasan, Kaneda etc.

③ Concerning the variance of velocity differences at infinite Reynolds number

→ Velocity gradients are infinite

→ (kinetic energy budget makes sense only in a distributional sense)

→ fluid velocity is rough, but bounded and continuous "weak"

➡ Consider the (longitudinal) velocity increment

$$\delta_\ell u_L(x) = u_L(x+\ell) - u_L(x)$$

➡ 2/3-law : Reach the statistically stationary regime, and observe ~~that~~ in good approximation, that

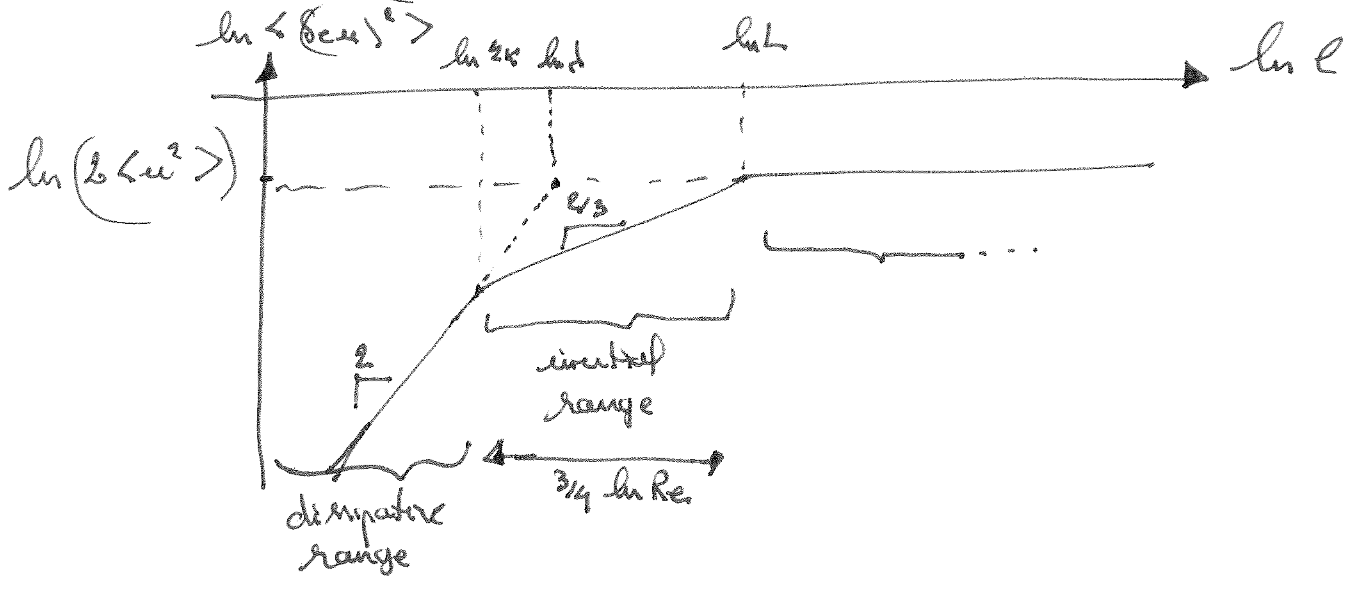
$$\lim_{\nu\to 0} \langle (\delta u)^2 \rangle \underset{\ell\to 0}{\sim} C_2 (\varepsilon \ell)^{2/3}$$

with $C_2 \sim \cancel{1}$, indept on large scale geometries and/or Reynolds number.

➡ At a finite Re, this implies a new lengthscale η_K defined as the typical scale at which velocity looses regularity:

$$\langle (\delta u)^2 \rangle \triangleq \begin{cases} C_2 (\varepsilon \ell)^{2/3} & \ell \gtrsim \eta_K \\ \ell^2 \langle (\partial_x u)^2 \rangle & \ell \lesssim \eta_K \end{cases}$$

➡ $\eta_K \sim \left(\frac{\nu^3}{\varepsilon}\right)^{1/4} \sim L\, Re^{-3/4}$



rem : Picture way too schematical to obtain the spectral interpretation of turbulence (boundary terms will dominate in the interesting range of scales) (9)

Nonetheless, keep in mind that this corresponds to

$$E(k) = \sigma^2 \int_0^{\infty} \cos(k\ell) \left[1 - \frac{(S_{\ell\ell})^2}{4\sigma^2} \right] d\ell$$

$$\underset{k\to\infty}{\sim} (cte)\, k^{-5/3}$$

↳ a.k. { BOS
Lohse - Muller Groeling 95

- First illustration of the standard phenomenology of turbulence.

↳ Studying solution of the stochastic heat equation.

Consider in dimension d the (randomly) forced heat equation

$$\frac{\partial u}{\partial t} = \nu \Delta u + f(x,t)$$

where $f(x,t)$ is a Gaussian field, delta correlated in time, ~~delta~~ correlated in space:

for instance : $\langle f(x,t_1) f(y,t_2) \rangle = A\, e^{-\frac{|x-y|^2}{2L^2}} \delta(t_1-t_2)$

Using fondamental solution of H.E. (Green function), starting from $u(x,0)=0$, we get

$$\boxed{u(x,t) = \int_0^t \int_{\mathbb{R}^d} \frac{1}{[4\pi\nu(t-s)]^{d/2}}\, e^{-\frac{|x-y|^2}{4\nu(t-s)}} f(y,s)\, dy\, ds}$$

→ Consequences on velocity variance :

by homogeneity → $\langle u^2(t) \rangle = \int_0^t ds \frac{1}{[4\pi\nu(t-s)]^d} \int_{\mathbb{R}^d\times\mathbb{R}^d} e^{-\frac{|y|^2+|z|^2}{4\nu(t-s)}} e^{-\frac{|y-z|^2}{2L^2}} dy\,dz$

Gaussian integrals : $\int_{\mathbb{R}^d\times\mathbb{R}^d} e^{-\frac{|y|^2+|z|^2}{4\nu(t-s)} - \frac{|y-z|^2}{2L^2}} dy\,dz$

$$= \left[\int_{\mathbb{R}\times\mathbb{R}} e^{-\frac{y^2+z^2}{4\nu(t-s)} - \frac{(y-z)^2}{2L^2}} dy\,dz \right]^d = \left[4\pi\nu(t-s) \sqrt{\frac{2L^2}{2L^2+8\pi\nu(t-s)}} \right]^d$$

such that

$$\langle u^2(t)\rangle = \int_0^t ds \left[\frac{2L^2}{2L^2+8\pi\nu(t-s)}\right]^{d/2}$$

$$= \int_0^t ds \frac{1}{\left[1+\frac{8\pi\nu(t-s)}{2L^2}\right]^{d/2}}$$

$$\tau = \frac{8\pi\nu(t-s)}{2L^2}$$

$$= \frac{L^2}{4\pi\nu} \times \int_0^{\frac{4\pi\nu t}{L^2}} \frac{1}{\left[1+\tau\right]^{d/2}} d\tau$$

so :

- $d=1 \Rightarrow \langle u^2(t)\rangle \xrightarrow[t\to\infty]{} \infty$ as $\sqrt{t}$
- $d=2 \Rightarrow \langle u^2(t)\rangle \xrightarrow[\infty]{} \infty$ as $\ln t$
- $d\geq 3 \Rightarrow \langle u^2(t)\rangle \xrightarrow[\infty]{} \frac{L^2}{4\pi\nu}\frac{2}{d-2}$

In all cases, the stochastic heat equation (diffusion) is not efficient enough to dissipate the injected energy.
- or it does not reach a stationary regime ($d\leq 2$)
- or variance behaves as $1/\nu$ ($d\geq 3$)

→ Consequences on viscous dissipation:

Similarly,

$$\partial_i u(x,t) = -\int_0^t \int_{\mathbb{R}^d} \frac{1}{[4\pi\nu(t-s)]^{d/2}} \frac{x_i - y_i}{8\nu(t-s)} e^{-\frac{|x-y|^2}{4\nu(t-s)}} f(y,s)\, d^d y\, ds$$

such that (by homogeneity)

$$\langle |\nabla u|^2 \rangle = \int_0^t \frac{ds}{[4\pi\nu(t-s)]^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{y\cdot z}{[8\nu(t-s)]^2} e^{-\frac{|y|^2+|z|^2}{4\nu(t-s)} - \frac{|y-z|^2}{2L^2}} dy\,dz$$

↓ isotropy

$$= d \int_0^t \frac{ds}{[4\pi\nu(t-s)]^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{y_1 z_1}{[8\nu(t-s)]^2} e^{-\frac{|y|^2+|z|^2}{4\nu(t-s)} - \frac{|y-z|^2}{2L^2}} dy\,dz$$

$$= d \int_0^t \frac{ds}{[4\pi\nu(t-s)]^d} \left[\int_{\mathbb{R}\times\mathbb{R}} \frac{yz}{[8\nu(t-s)]^2} e^{-\frac{y^2+z^2}{4\nu(t-s)} - \frac{(y-z)^2}{2L^2}} dy\,dz \right] \left[\int_{\mathbb{R}\times\mathbb{R}} e^{-\frac{y^2+z^2}{4\nu(t-s)} - \frac{(y-z)^2}{2L^2}} dy\,dz \right]^{d-1}$$

$$\propto \frac{d}{2L^2} \int_0^t \frac{1}{\left[1 + 8\pi\nu \frac{t-s}{2L^2}\right]^{\frac{d+1}{2}}} ds = \frac{d}{2\cancel{L^2}} \frac{\cancel{L^2}}{4\pi\nu} \int_0^{\frac{4\pi\nu t}{L^2}} \frac{1}{[1+\tau]^{\frac{d+1}{2}}} d\tau$$

$$= \frac{d}{8\pi\nu} \int_0^{\frac{4\pi\nu t}{L^2}} \frac{1}{(1+\tau)^{\frac{d+1}{2}}} d\tau$$

- $d=1 \Rightarrow \langle |\nabla u|^2 \rangle \xrightarrow[\infty]{} \infty$ as $\ln t$
- $d \geq 2 \Rightarrow \boxed{\nu \langle |\nabla u|^2 \rangle \xrightarrow[\infty]{} \frac{1}{4\pi} \frac{d}{d-1}}$

ε is finite, ~~in~~dependantly on viscosity !

(but variance diverges)

- Second illustration : Fractional Gaussian field.

Can we give a (probabilistic) meaning to the velocity field depicted in the phenomenology of Kolmogorov ?

- a random field such that
  1. Variance is finite (when $\nu \to 0$)
  2. Viscous dissipation finite (when $\nu \to 0$)
  3. Hölder continuity $\lim_{\nu\to 0} \langle (u(x+\ell) - u(x))^2 \rangle \underset{\ell \to 0}{\sim} \ell^{2/3}$

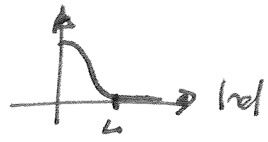
?

- Stay simple (i.e. schematical), adopt a Gaussian approximation
- Consider
$$\boxed{u_\eta(x) = \int_{\mathbb{R}^d} \mathcal{L}_L(x-y)\, |x-y|_\eta^{H-\frac{d}{2}}\, W(dy)}$$

- $W(dy)$: "White Gaussian noise"
  (On a discrete set, made up of independent Gaussian random variable, zero average, variance $(dx)^d$)
  obeying the following rule of calculation:
$$\begin{cases} E\left[\int f(y)\, W(dy)\right] = 0 \\ E\left[\int f(y)\, W(dy) \int g(y)\, W(dy)\right] = \int_{\mathbb{R}^d} f(y) g(y)\, d^d y \end{cases}$$

- A "regularized" norm $|x|_\eta = \begin{cases} |x| & \text{for } |x| \gtrsim \eta \\ \eta & \text{for } |x| \lesssim \eta \end{cases}$

over a "small" length scale η (that will eventually depend on ν)

- $\mathcal{L}_L$ a large scale cut-off, non universal 

As mentioned, the field is regularized over $\eta(\nu)$, s.t. it is differentiable.
what happens when $\eta(\nu) \longrightarrow 0$?

- Consider the variance. (by homogeneity)

$$\langle u_\eta^2(0) \rangle = \int_{\mathbb{R}^d} \mathcal{L}_L^2(y) \, |y|_\eta^{2H-d} \, d^d y$$

↳ for the sake of presentation, consider $d=1$

$$\Rightarrow \langle u_\eta^2(0) \rangle = \int_{\mathbb{R}} \mathcal{L}_L^2(y) \, |y|_\eta^{2H-1} \, dy$$

the limit exists iif the singularity at the origin is integrable, which is the case for $\boxed{H>0}$

$$\Rightarrow \langle u_\eta^2(0) \rangle \xrightarrow[\eta \to 0]{} \int_{\mathbb{R}} \mathcal{L}_L^2(y) \, |y|^{2H-1} \, dy < +\infty$$

It is finite and non universal:

Now, consider first $\eta \to 0$ (variance remains bounded), and consider the increment of the field

$$\delta_\ell u(x) = u(x+\ell) - u(x)$$

$$\langle \delta_\ell u^2(0) \rangle = \int_{\mathbb{R}} \Phi_\ell^2(y) \, dy$$

$$\Rightarrow \Phi_\ell(y) = f_L(\ell-y) \, |\ell-y|^{H-\frac{1}{2}} - f_L(y) \, |y|^{H-\frac{1}{2}}$$

Now, the question is : How behaves this asymptotic velocity field when $\ell \to 0^+$?

$$\langle \delta u_\ell^2 \rangle = \ell^{2H} \int_{\mathbb{R}} \left[\mathcal{L}_L(\ell(1-y))\, |1-y|^{H-\frac{1}{2}} - \mathcal{L}_L(\ell y)\, |y|^{H-\frac{1}{2}} \right]^2 dy$$

$$\underset{\ell \to 0}{\overset{?}{\sim}} \mathcal{L}_L^2(0)\, \ell^{2H} \int \left[|1-y|^{H-\frac{1}{2}} - |y|^{H-\frac{1}{2}} \right]^2 dy$$

$\longrightarrow$ true if the integral is finite.

We only have to check integrability at infinity.

remark that $|1-y|^{H-\frac{1}{2}} - |y|^{H-\frac{1}{2}} \underset{y \to \infty}{\sim} y^{H-\frac{1}{2}} \left[(1-\frac{1}{y})^{H-\frac{1}{2}} - 1 \right]$

$$y^{H-\frac{1}{2}} \left[-\frac{1}{y}(H-\tfrac{1}{2}) \right]$$

$$\underset{y \to \infty}{\sim} (H-\tfrac{1}{2})\, y^{H-3/2}$$

$\Rightarrow$ the integrand behaves $(H-\frac{1}{2})^2\, y^{2H-3}$ for $y \to +\infty$

$\Rightarrow$ integral is finite iif $\boxed{H < 1}$

- Conclusions, for $0 < H < 1$, the field is, asymptotically, of finite variance (depends on the cut-off), and the structure function behaves as

$$\lim_{\eta \to 0} \langle \delta u_{\ell,\eta}^2 \rangle \underset{\ell \to 0}{\sim} \mathcal{L}_L^2(0)\, \ell^{2H} \int \left[|1-y|^{H-\frac{1}{2}} - |y|^{H-\frac{1}{2}} \right]^2 dy$$

So the field is not differentiable, and universal (depends only on $\mathcal{L}_L(0)$) !

- Further calculations shows that

$$\langle |\nabla u_\eta|^2 \rangle \underset{\eta \to 0}{\sim} \eta^{2H-2} \; cte(H,d)$$

(it is indeed non differentiable)

- $\Rightarrow$ Thus, it is enough to take

$$\eta(\nu) = \nu^{-\frac{1}{2H-2}} = \nu^{\frac{1}{2-2H}} \xrightarrow[\nu \to 0]{} 0$$

to ensure that

$$\nu \langle |\nabla u_{\eta(\nu)}|^2 \rangle \xrightarrow[\nu \to 0]{} cte.$$

- In particular, $\boxed{H = \frac{1}{3}}$

we have $\langle (\delta_\ell u)^2 \rangle \sim \ell^{2/3}$

and $\eta(\frac{1}{3}) = \nu^{3/4} \propto \eta_K$

Second part: Energy transfer through scales, and Intermittency.

(17)

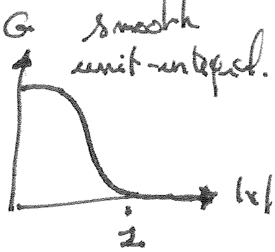
- Former kinetic Energy budget evidenced the role played by viscous dissipation. Further interpretation lead us to expect non differentiable velocity fields at infinite Reynolds numbers.

➡ ~~Consider~~
➡ Budget exhibits anomalous (i.e. distributional) quantities
➡ It is thus tempting to study the energy budget of a coarse-grained version of velocity, such as:

$$u^\ell(x,t) = \int G_\ell(x-y)\, u(y,t)\, d^3y \xrightarrow[\ell\to 0]{\text{pointwise}} u(x,t)$$

where G_ℓ is an isotropic mollifier = approximation of the Dirac δ-function at scale ℓ:

$$G_\ell(x) = G_\ell(|x|) = \frac{1}{\ell^3}\, G\left(\frac{|x|}{\ell}\right)$$



➡ Kinetic budget exhibits a new term (see literature on Large Eddy Simulations) involving the subgrid stress $\tau_{ij}(x) = (u_i u_j)^\ell(x) - u_i^\ell(x) u_j^\ell(x)$

➡ Instead, remark that

$$|u^\ell(x,t)|^2 = \int G_\ell(x-y)\, G_\ell(x-z)\, \underbrace{u(y,t)\cdot u(z,t)}_{\text{two-points}}\, d^3y\, d^3z$$

➡ Budget of the point-split kinetic energy: ?

$$\boxed{e_{r}(x) = \frac{1}{2}\, u(x,t)\cdot u(x+r,t)}$$

- Here, we follow Greg Eyink's derivation (his notes on turbulence theory are a must).

Define : $u = u(x,t)$, $p = p(x,t)$

$u' = u(x+r,t)$; $p' = p(x+r,t)$

$\delta u = u' - u = u(x+r;t) - u(x,t)$

From N.S., we get:

$$\boxed{\frac{1}{2}\frac{\partial(u.u')}{\partial t} + \operatorname{div} J = -\nu \sum_{i,j=1}^{3} \partial_j u_i \, \partial_j u'_i + \frac{1}{2}(u.f' + u'.f) + \frac{1}{4}\nabla_r \cdot \left[\delta u \, |\delta u|^2\right]}$$

- Similar interpretations as the former budget, but action of a new term.
- $J = \left(\frac{1}{2}u.u'\right)u + \frac{1}{2}(pu' + p'u) + \frac{1}{4}|u'|^2 \delta u - \nu \operatorname{grad}\left(\frac{1}{2}u.u'\right)$

<u>Hints on the proof</u>:

- the viscous contribution is written as

$$u.\Delta u' + u'.\Delta u = \operatorname{div}\left[\operatorname{grad}(u.u')\right] - 2\sum_{i,j=1}^{3} \partial_j u_i \, \partial_j u'_i$$

- Similarly, pressure only participates in redistribution of energy in space.

- Crucial contribution of the non linear term :
  (again, take a look at the notes of Greg Eyink)

$$u\cdot\left[(u'\cdot\nabla)u'\right] + u'\cdot\left[(u\cdot\nabla)u\right] = u_i\,\partial_j(u'_i u'_j) + u'_i\,\partial_j(u_i u_j)$$

$$= \underbrace{u_i\,\partial_j(u'_i u'_j) - u_i u_j\,\partial_j u'_i}_{} + \underbrace{\partial_j(u'_i u_i u_j)}_{\mathrm{div}\left[u\,(u\cdot u')\right]}$$

$$\hookrightarrow \;= u_i u'_j\,\partial_j u'_i - u_i u_j\,\partial_j u'_i$$

$$= u_i\,\delta u_j\,\partial_j u'_i \qquad (*)$$

Remark that :

$$\nabla_r\cdot\left[\delta u\,|\delta u|^2\right] = |\delta u|^2\,\underbrace{\partial_{r_i}\delta u_i}_{0 \text{ by incompressibility}} + (\delta u\cdot\nabla_r)\,|\delta u|^2$$

$$= \delta u_j\,\partial_{r_j}\delta u_i\,\delta u_i = 2\,\delta u_i\,\delta u_j\,\partial_{r_j}\delta u_i$$

$$= 2\,\delta u_i\,\delta u_j\,\partial_j u'_i$$

$$= 2\,u'_i\,\delta u_j\,\partial_j u'_i - 2\,u_i\,\delta u_j\,\partial_j u'_i$$

$$= \delta u_j\,\partial_j |u'|^2 - 2\,(*)$$

$$= \mathrm{div}\left[|u'|^2\,\delta u\right] - 2\,(\text{nonlinear contribution})$$

- This *local* budget of the point-split energy allows to discuss the local conservation, or non-conservation, of kinetic energy in a more general framework:

  - without assuming differentiability (but assuming Hölder-continuity) (with appropriate functional spaces)

  - gives a proof of Onsager's conjecture

    See devoted works of $\begin{cases} \text{Eyink} \\ \text{Constantin - E - Titi} \\ \text{Duchon - Robert} \end{cases}$

    and more recents ones of Buckmaster, De Lellis, Isett, Székelyhidi, etc...

    $\to$ ask Dubrulle.

- Adopt a simpler statistical approach, and assume statistical stationarity, homogeneity, isotropy.

We get:

$$\frac{1}{4}\nabla_r\cdot\langle \delta u\,|\delta u|^2\rangle - \nu\sum_{i,j=1}^{3}\langle \partial_j u_i\,\partial_j u_i'\rangle + \frac{1}{2}\langle u\cdot f' + u'\cdot f\rangle = 0$$

- Viscous contribution $\nu\sum_{i,j=1}^{3}\langle \partial_j u_i\,\partial_j u_i'\rangle = -\nu\,\underbrace{\Delta\,\mathrm{trace}\,(C(r))}_{\xrightarrow[\nu\to 0]{} \text{ bounded function of } r}$

where $C_{ij}(r) = \langle u_i(x)\,u_j(x+r)\rangle$ $\quad \xrightarrow[\nu\to 0]{} 0$!

- forcing contribution: $\lim_{|r|\to 0}\lim_{\nu\to 0}\frac{1}{2}\langle u\cdot f' + u'\cdot f\rangle = \langle u\cdot f\rangle = \varepsilon$

We are left with

$$\boxed{\lim_{|r|\to 0} \lim_{\nu\to 0} \nabla_r \cdot \langle \delta u \, |\delta u|^2 \rangle = -4\,\varepsilon}$$

which says (heuristically) when $\nu \to 0$ and then $|r| \to 0$ that $\boxed{\langle \delta u \, |\delta u|^2 \rangle = -\frac{4}{3}\,\varepsilon\, r}$

- Consequences for the coarse-grained energy : (Switch off forcing)

$$\underbrace{\frac{d\langle |u^\ell|^2\rangle}{dt}}_{\substack{\text{evolution of kinetic}\\ \text{energy of "scales"} \geq \ell}} = \int G_\ell(x-y)\, G_\ell(x-z)\, \frac{d}{dt} \langle u(y) \cdot u(z) \rangle \, d^3y\, d^3z$$

$$\xrightarrow[\nu \to 0]{} \int G_\ell(x-y)\, G_\ell(x-z)\, \frac{1}{4} \nabla_{|y-z|} \langle \delta u \, |\delta u|^2 \rangle \, d^3y\, d^3z$$

$$\underset{\ell \ll L}{\simeq} \; -\varepsilon \, .$$

That says that kinetic energy "cascade" through scales at the rate ε, independently of the scale.

• Experimental check and derivation of the 4/5 - law

Only the longitudinal velocity increment is accessible in wind tunnels (along the mean flow)

→ Make use of statistical homogeneity and isotropy to relate
$\langle \delta u \, |\delta u|^2 \rangle$ to $\langle \delta u_{\parallel} |\delta u_{\parallel}|^2 \rangle$

where $\delta u_{\parallel}(r) = \left[u(x+r) - u(x) \right] \cdot \frac{r}{|r|}$

For full derivation, see Frisch C95).

Hints of the proof:

• remark that

$$\langle \delta u_j |\delta u|^2 \rangle = \underbrace{\langle u'_i u'_i u'_j \rangle - \langle u_i u_i u_j \rangle}_{0 \text{ by homogeneity}} + \underbrace{\langle u_i u_i u'_j \rangle - \langle u'_i u'_i u_j \rangle}_{\text{will not contribute cause } \nabla_{r_j} \cdot \to 0}$$

$$+ 2 \langle u_i u'_i u_j \rangle - 2 \langle u_i u'_i u'_j \rangle$$

→ require knowledge of $b_{ij,m}(r) = \langle u_i(x) u_j(x) u_m(x+r) \rangle$

→ Batchelor, Monin-Yaglom, Frisch

$$\boxed{b_{ij,m}(r) = C(|r|) \delta_{ij} \frac{r_m}{|r|} + D(|r|) \left[\delta_{im} \frac{r_j}{|r|} + \delta_{jm} \frac{r_i}{|r|} \right] + F(|r|) \frac{r_i r_j r_m}{|r|^3}}$$

Incompressibility $\partial_{r_m} b_{ij,m}(r) = 0$ gives D and F as functions of C and its derivative

- Obtain $B_{ijm} = \langle \delta u_i \delta u_j \delta u_m \rangle$ from $b_{ij,m}$

- Notice that $\langle (\delta_{\parallel} u)^3 \rangle = B_{ijm} \frac{r_i r_j r_m}{|r|^3} = -12\, C$

showing that $\langle \delta u |\delta u|^2 \rangle$ can be expressed only in terms of the longitudinal velocity increment.

- Go on with the calculation, and get

$$\boxed{\lim_{\nu \to 0} \left\langle \left[\left(u(x+r) - u(x) \right) \cdot \frac{r}{|r|} \right]^3 \right\rangle \underset{|r| \to 0}{\sim} -\frac{4}{5}\, \varepsilon\, |r|}$$

- Experimental check $\to$ Gagne et al. PoF (2004)

• Higher order statistics, Intermittency, and their stochastic modeling

We have seen that, at "small" scales (i.e. in the inertial range):

• $\langle (\delta_\ell u_{\parallel})^2 \rangle \simeq C_K (\varepsilon \ell)^{2/3}$

• We were able to draw a Gaussian representation using fractional Gaussian fields.

$$\delta_\ell u_{\parallel} \overset{\text{"law"}}{=} \sigma \left(\frac{\ell}{L}\right)^{1/3} \underbrace{\mathcal{N}(0,1)}_{\substack{\text{Gaussian random variable} \\ \text{0-average, unit variance}}}$$

as far as 1-point statistics of increments are concerned.

$\boxed{\text{BUT}}$ $\langle (\delta_\ell u_{\parallel})^3 \rangle = -\frac{4}{5} \varepsilon \ell \neq 0$

• The Gaussian interpretation is not realistic

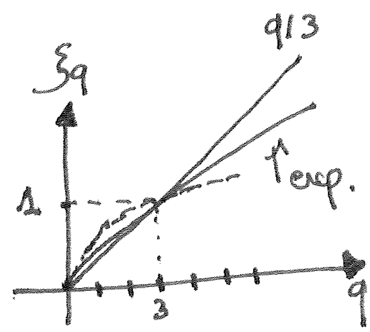
• Furthermore, the probability density functions (PDFs) of increments are strongly non-Gaussian

• See Frisch (95)

• In a (mostly) equivalent way,

$$\langle (\delta_\ell u_{\parallel})^q \rangle = C_q (\varepsilon \ell)^{\xi_q}$$

with ξ_q a non linear function of q



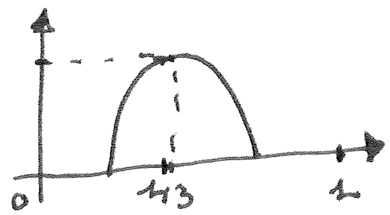
- Instead, a more realistic picture would be given by this equality in "law"

$$\boxed{\delta_\ell u_{\parallel} \overset{\text{"law"}}{=} \sigma \left(\frac{\ell}{L}\right)^{h} \mathcal{N}(0,1)}$$

where h is no more unique, and fluctuates around $\frac{1}{3}$ according to a given universal (in a certain sense) probability density:

At a scale ℓ, this density is given by $\boxed{P_\ell(h) = \frac{1}{Z(\ell)} \left(\frac{\ell}{L}\right)^{1-D(h)}}$

$D(h)$: called the singularity spectrum, universal (independent of scales, Reynolds number, large scale anisotropies, etc.)



In essence, this is the multifractal formalism.

(See textbook of Frisch).

- If furthermore, h and the $\mathcal{N}(0,1)$ random variable are assumed independent, we can show that

$$\langle (\delta_\ell u_{\parallel})^q \rangle = c_q \left(\frac{\ell}{L}\right)^{\zeta_q}$$

with $\boxed{\zeta_q = \min_h \left[qh + 1 - D(h) \right]}$

- Former picture still too schematical, since

$$C_q = \sigma^q \langle (\mathcal{N}(0,1))^q \rangle$$

$$= 0 \quad \text{for } q \text{ odd.}$$

In particular $\boxed{C_3 = 0}$, not consistent with 4/5-law

- Vast litterature on the precise choice for $D(h)$ able to reproduce experimental data. ("lognormal", She-Lévêque etc.)

- From the theoretical side, it is very convenient to assume Gaussian statistics for h

  - corresponding to $\boxed{D(h) = 1 - \frac{(h-c_1)^2}{2c_2}}$

    with $c_1 = \frac{1}{3} + \frac{3}{2} c_2$

    $c_2 = 0{,}025$ (obtained from Experiments - DNS)

  - corresponding to $\boxed{\zeta_q = c_1 q + c_2 \frac{q^2}{2}}$

  - $\zeta_3 = 1$

  - ζ_q realistic of data for $q \lesssim 6$

- Beyond Fractional Gaussian fields: towards Multifractality

Main probabilistic tools borrowed from Yaglom, Mandelbrot, Kahane and many others over the last 60 years.

I follow here, mostly, some propositions made in a concise way by Robert and Vargas (CMP, 2008).

- Consider $$\boxed{u_\eta(x) = \int_{\mathbb{R}} L_L(x-y)\, |x-y|_\eta^{H-\frac{1}{2}}\, e^{\gamma X_\eta(y) - \gamma^2 \langle X_\eta^2 \rangle}\, W(dy)}$$

- $X_\eta(x)$ a zero-average Gaussian field, with appropriate (defined later) covariance structure

- γ: a free parameter, that will eventually play the role of the intermittency coefficient $c_2 = \gamma^2 = 0.025$.

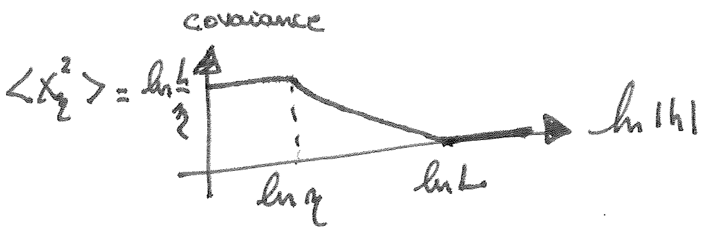
- We assume X_η and W independent.

- rem: Since X_η is Gaussian, we have $\boxed{\langle e^{\gamma X_\eta} \rangle = e^{\frac{\gamma^2}{2}\langle X_\eta^2 \rangle}}$ (0-average)

- rem: For theses conditions, the covariance of the velocity field $\langle u_\eta(0)\, u_\eta(x) \rangle$ is the same as the corresponding Fractional Gaussian Field (take $\gamma = 0$)

- To reproduce the formerly depicted phenomenology of intermittency, we have to take for the covariance of X_η a logarithmic structure, of the form:

$$\boxed{\langle X_\eta(0)\, X_\eta(h) \rangle = \ln \frac{L}{|h|_\eta}\, \mathbb{1}_{|h| \leq L}}$$



- In a good approximation, that warrants statistical homogeneity, X_η is given by a Fractional Gaussian field of vanishing $H = 0$.

$$\boxed{X_\eta(x) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} \mathbb{1}_{|x-y| \leq L}\, |x-y|_\eta^{-1/2}\, W'(dy)}$$

- W' and W are independent.
- Remark $\langle X_\eta^2 \rangle$ diverges when $\eta \to 0$

- Hints :

$$\langle X_\eta^2 \rangle = \frac{1}{2} \int_{|y| \leq L} |y|_\eta^{-1}\, dy = \int_0^L |y|_\eta^{-1}\, dy$$

$$\simeq \int_0^\eta \eta^{-1}\, dy + \int_\eta^L \frac{dy}{y} = \ln \frac{L}{\eta} + O(1)$$

- Nomenclature : $\lim_{\eta \to 0} e^{\gamma X_\eta(x) - \frac{\gamma^2}{2} \langle X_\eta^2 \rangle}$ is called a $\boxed{\text{Multiplicative Chaos}}$ by Kahane.

- As mentioned, $\langle (\delta_\ell u_\eta)^2 \rangle$ is not affected by intermittency. in particular $\lim_{\eta \to 0} \langle (\delta_\ell u_\eta)^2 \rangle \underset{\ell \to 0}{\sim} \text{cte}\, \ell^{2H}$
- Proposition: For $\gamma^2 < H$, we have

$$\lim_{\eta \to 0} \langle (\delta_\ell u_\eta)^4 \rangle \text{ finite, and} \underset{\ell \to 0}{\sim} \text{cte}\, \ell^{4H - 4\gamma^2}$$

Hints:

Similarly, consider $\delta_\ell u_\eta(0) = \int \phi_\ell(y)\, e^{\gamma X_\eta(y) - \gamma^2 \langle X_\eta^2 \rangle} W(dy)$

We get, using $\phi_\ell(y) = \mathbb{1}_L(\ell - y)|\ell - y|^{H - 1/2} - \mathbb{1}_L(y)|y|^{H - \frac{1}{2}}$,

$$\langle (\delta_\ell u_\eta)^4 \rangle = 3 \int \phi_\ell^2(y)\, \phi_\ell^2(z) \langle e^{2\gamma X_\eta(y) + 2\gamma X_\eta(z) - 4\gamma^2 \langle X_\eta^2 \rangle} \rangle \, dy\, dz$$

$$= 3 \int \phi_\ell^2(y)\, \phi_\ell^2(z)\, e^{4\gamma^2 \langle X_\eta(y) X_\eta(z) \rangle} dy\, dz$$

$$= 3 \int \phi_\ell^2(y)\, \phi_\ell^2(z) \left(\frac{L}{|y - z|_\eta^+} \right)^{4\gamma^2} dy\, dz$$

Rescale the dummy variables y and z by ℓ to exhibit scalings.

$$\Rightarrow \langle (\delta_\ell u)^4 \rangle \underset{\ell \to 0}{\sim} 3\, \mathbb{1}_L^4(0) \times \text{factor}(H, \gamma^2) \times \ell^{4H - 4\gamma^2}$$

If the factor (H, γ^2) does not blow up, which is the case for $\gamma^2 < H$ (Cf. Jason Reneuve)

Conclusions :

We have built an intermittent velocity field u_η such that, for not too big values of γ,

$$\lim_{\eta \to 0} \langle (\delta_\ell u_\eta)^q \rangle \underset{\ell \to 0}{\sim} A_q \, \ell^{\xi_q}$$

with $$\boxed{\xi_q = qH - q(q-2)\frac{\gamma^2}{2}}$$

<u>But</u> too simplistic to reproduce 4/5-law ($A_3 = 0$)

For developments on this subject, see :

PEREIRA, GARBAN, CHEVILLARD, JFM (2016)

CHEVILLARD, GARBAN, RHODES, VARGAS, Annales Henri Poincaré (2019)

RENEUVE, PhD (2019).