Rotational dynamics of planetary cores: from geostrophic to inertial wave turbulence

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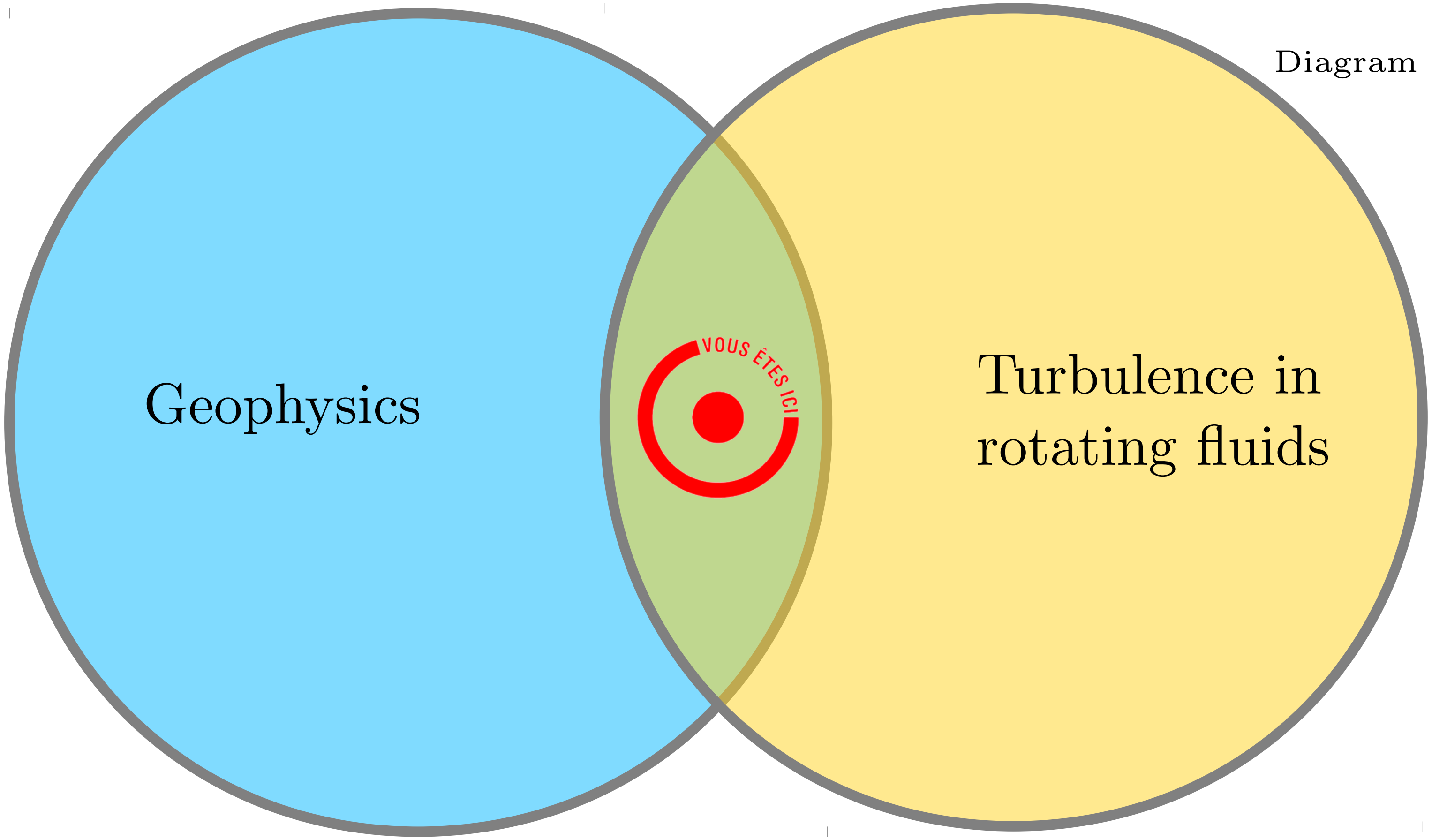
NCTR V — École de Physique des Houches — 7-12 Avril 2019

Motivations

Diagram not to scale



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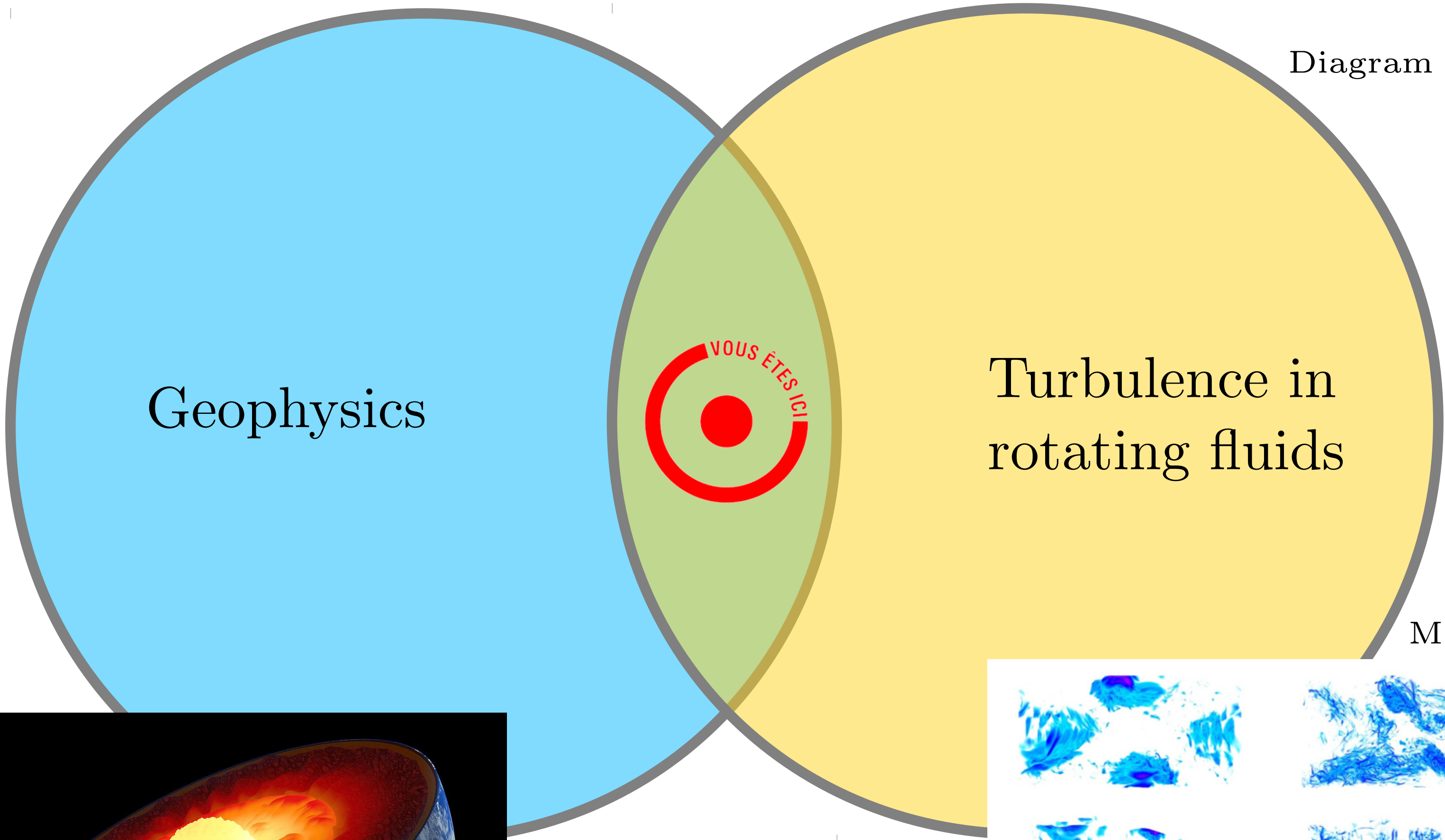
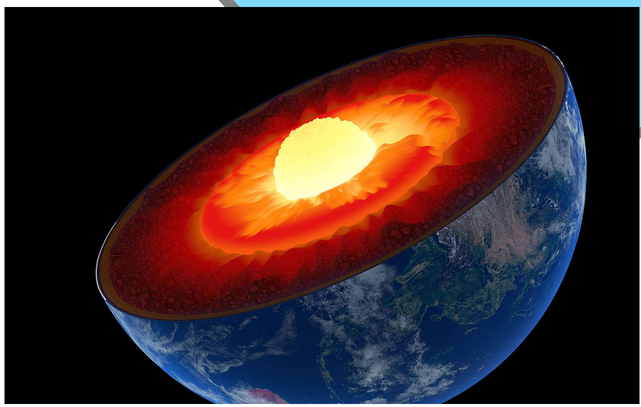
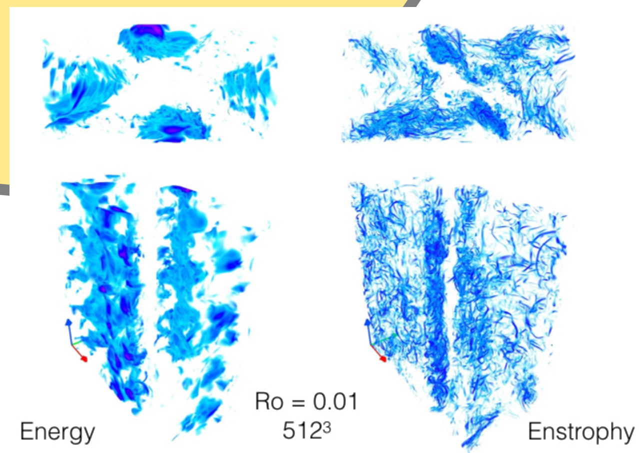




Flows in planetary cores

Mininni (2013)



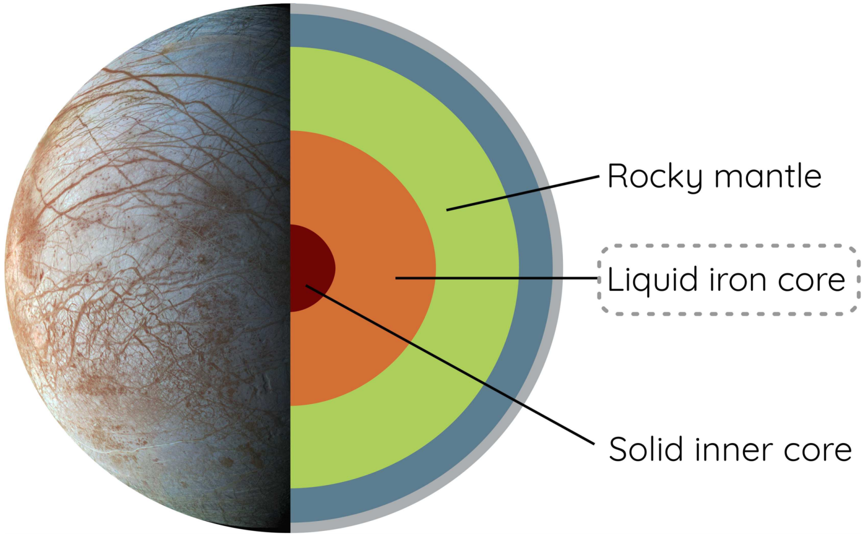
Waves and vortices in rotating fluids

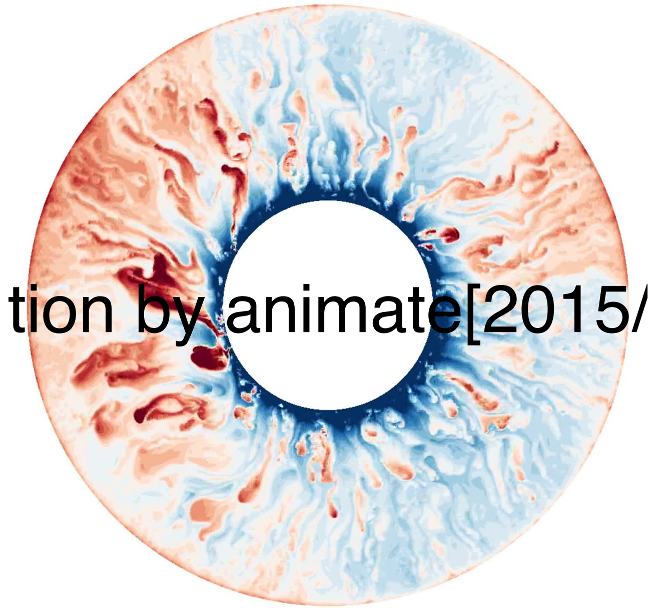
Outline

Outline

Sources of motion in planetary fluid layers

- A classical mechanism: thermo-solutal convection





Schaeffer *et al.* (2017), Code XSHELLS

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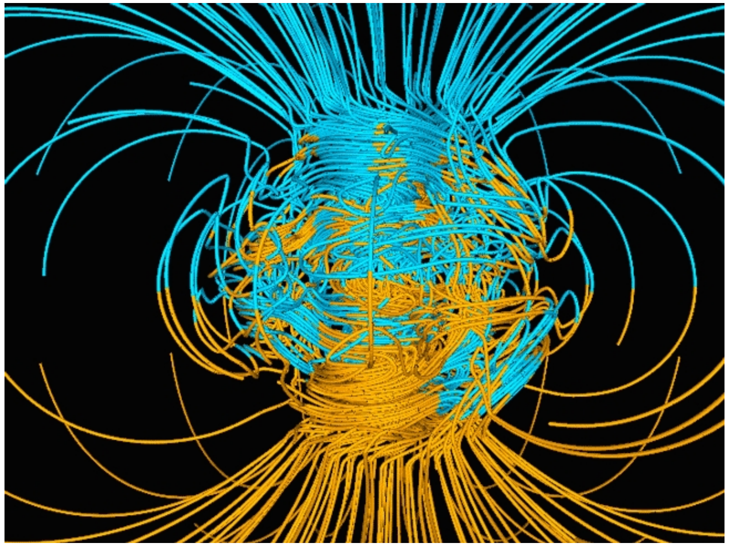
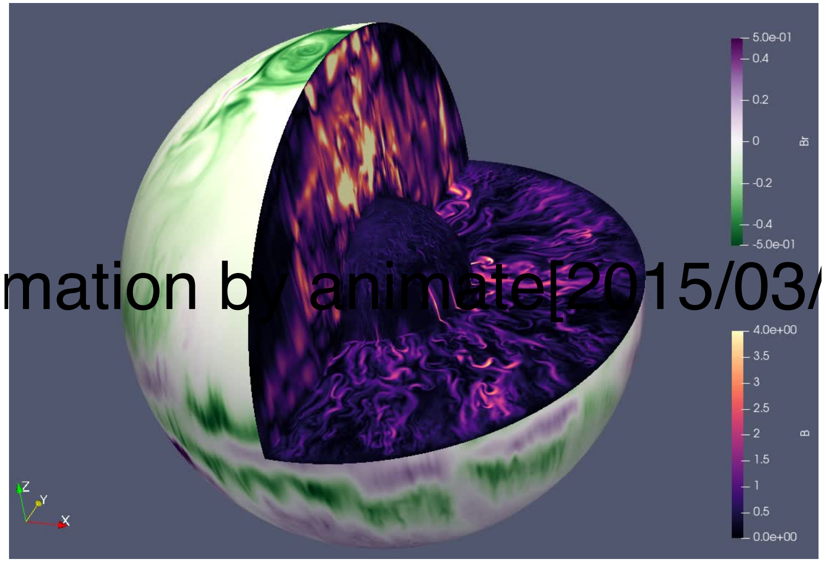
- Very efficient at sustaining magnetic fields...



Glatzmaiers and Roberts (1995)



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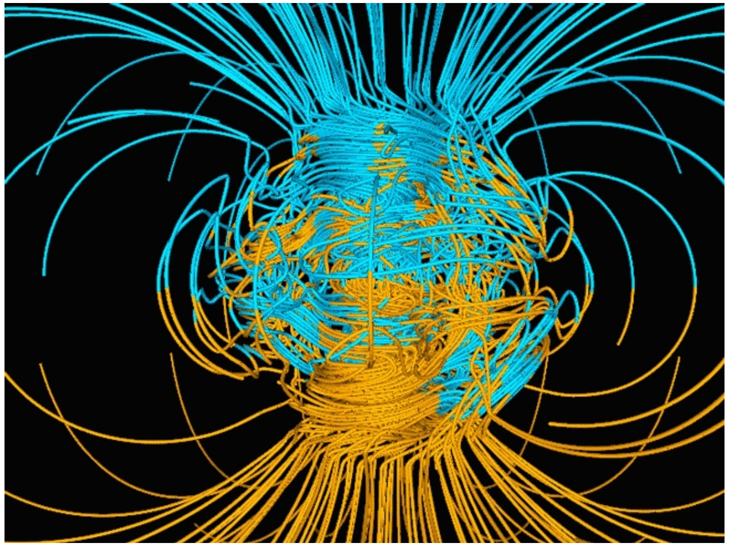

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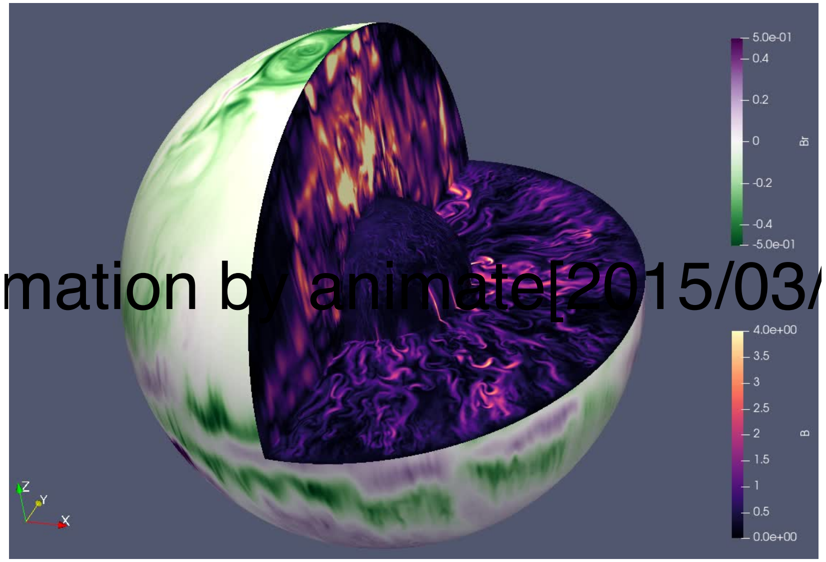

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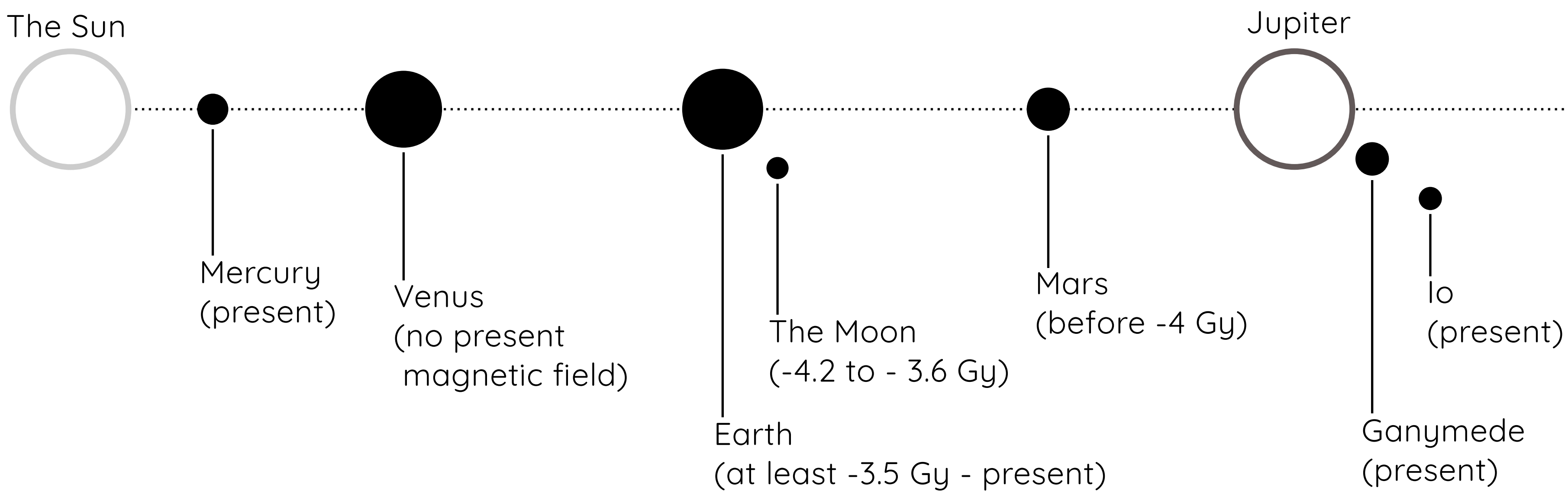
- ... but with tight energy budget and strongly dependent on poorly-constrained thermal history and properties

Magnetic fields in the Solar system



Le Reun (2019)

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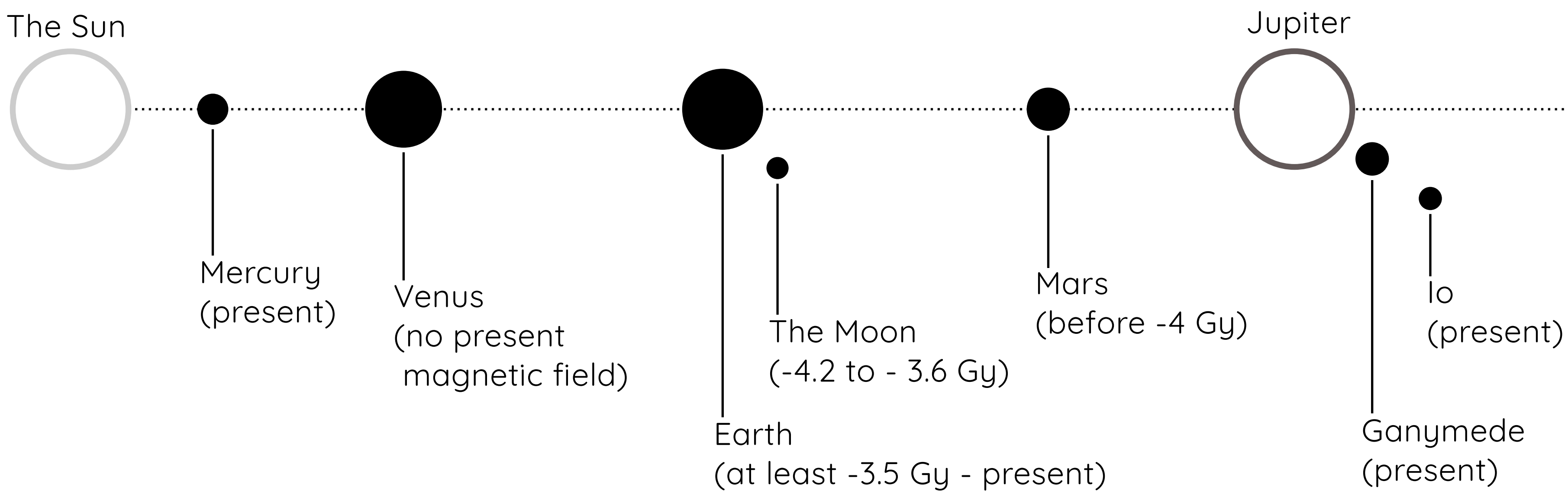
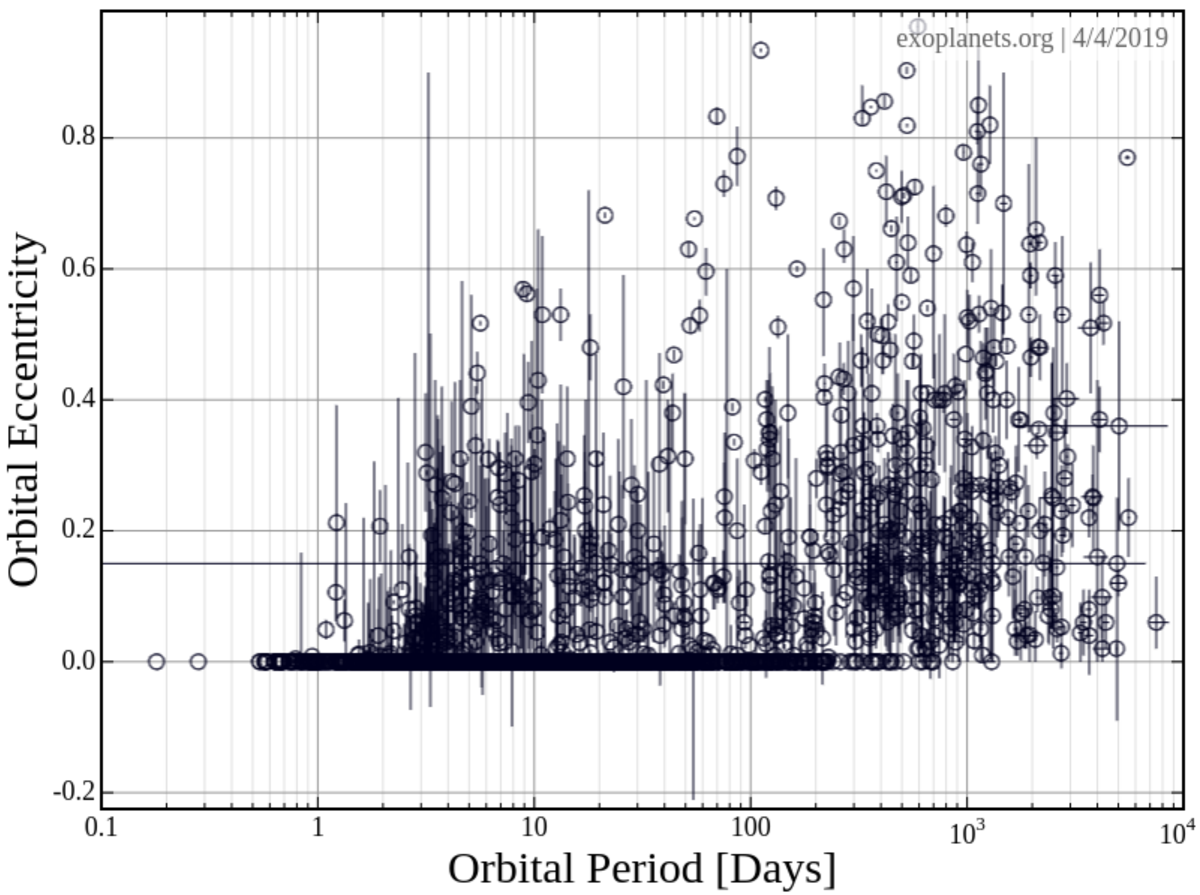


Le Reun (2019)

Could there be another source of dynamo action?

Celestial mechanics



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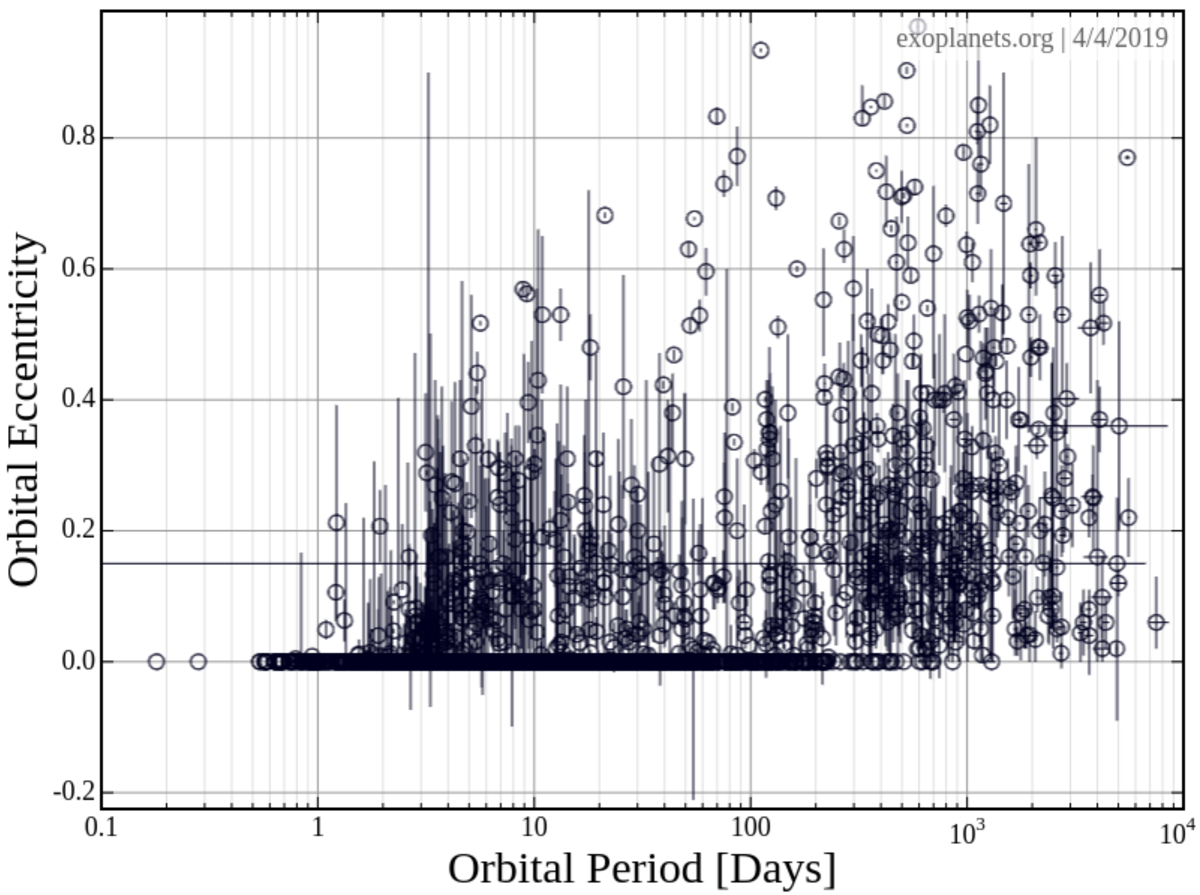
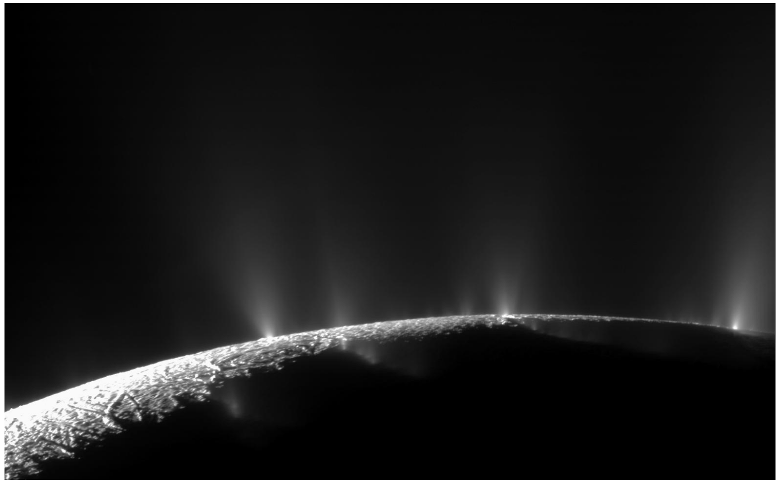
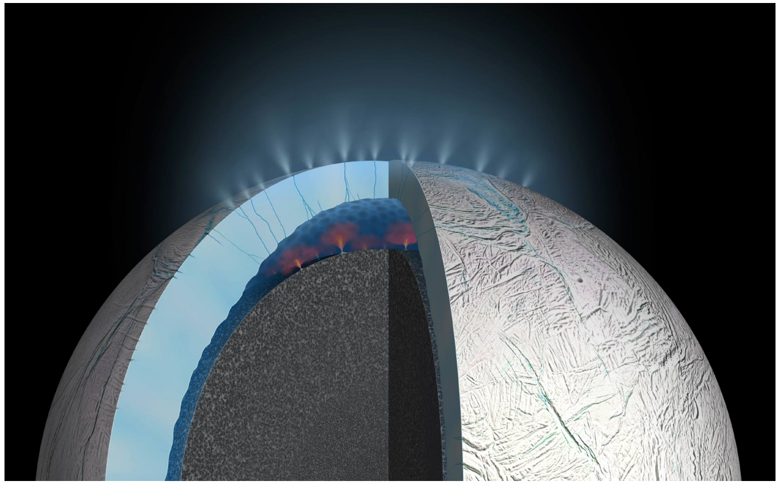


Origin of the dissipation required to circularize close-in orbits?

Hydro-thermal activity on Enceladus





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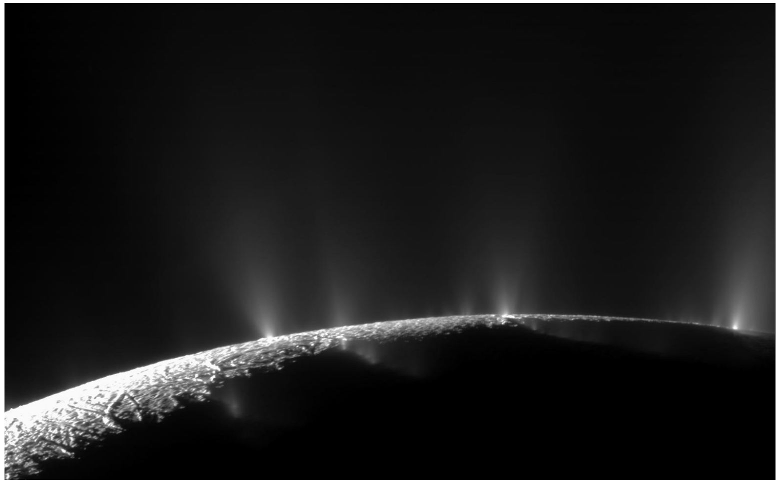

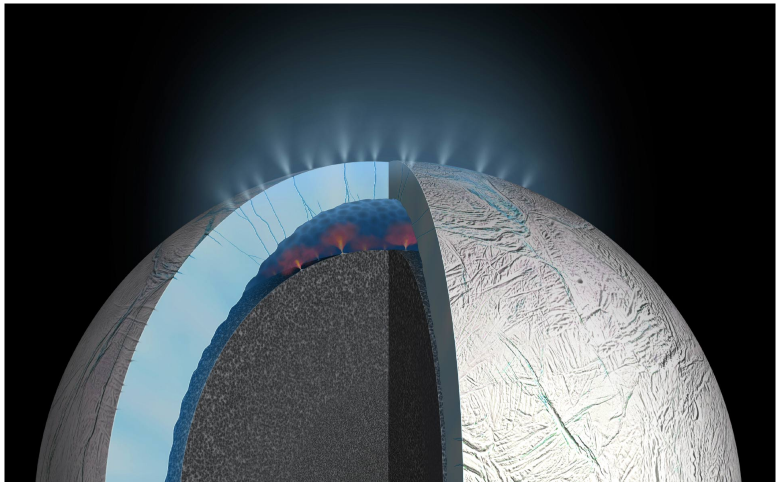

What is the heat source sustaining the liquid ocean?

Sources of motion in planetary fluid layers

- Gravitational interactions lead to mechanical forcing:

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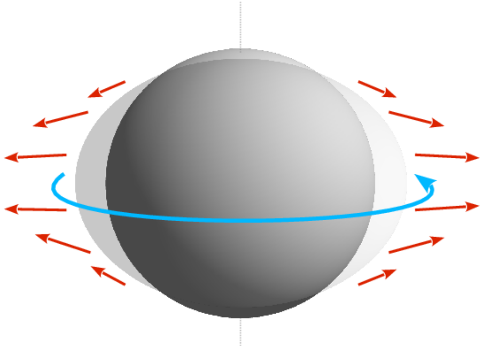
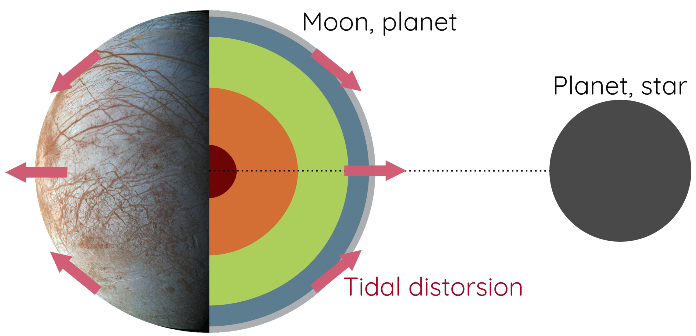


Tides



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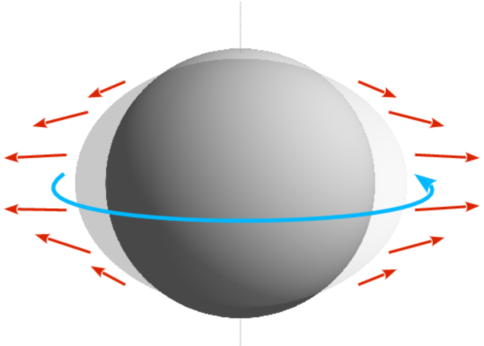

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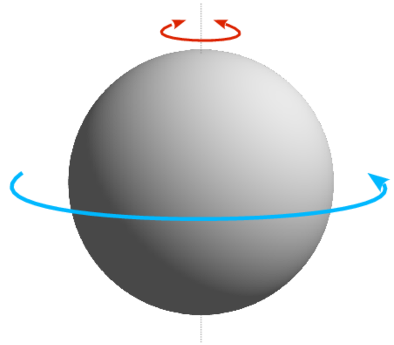
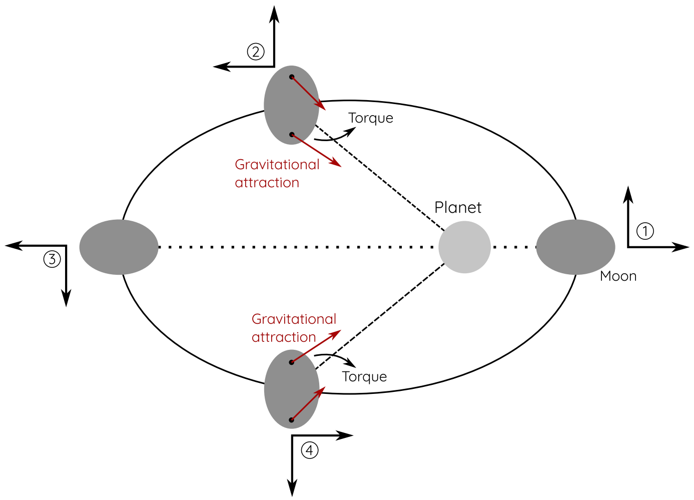
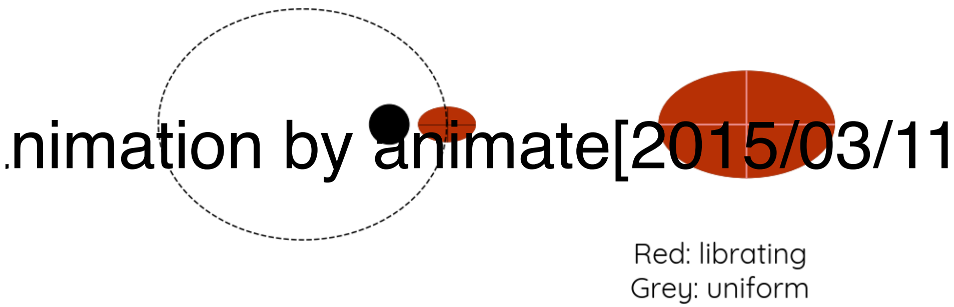


Libration





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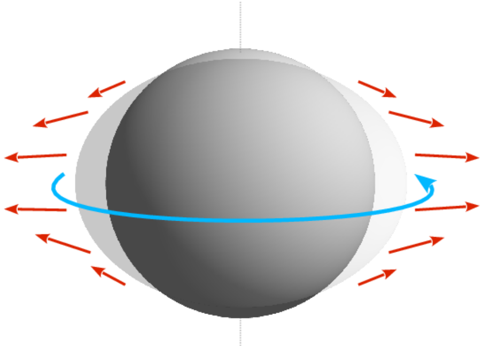

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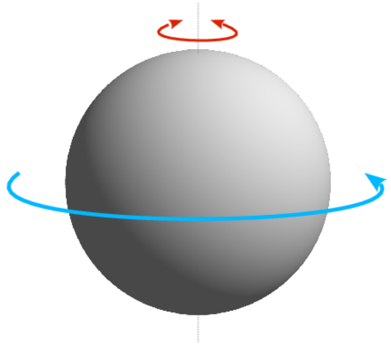

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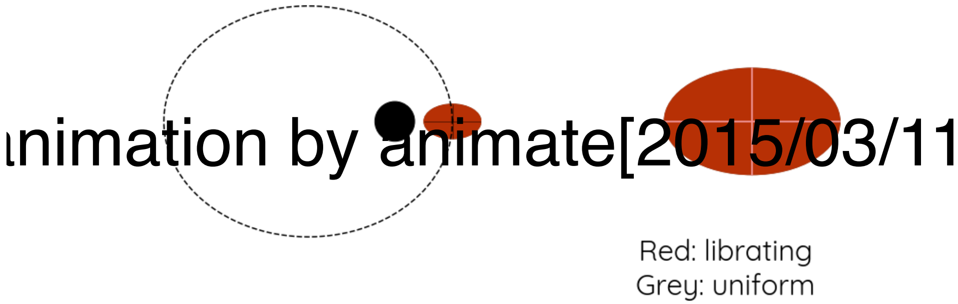

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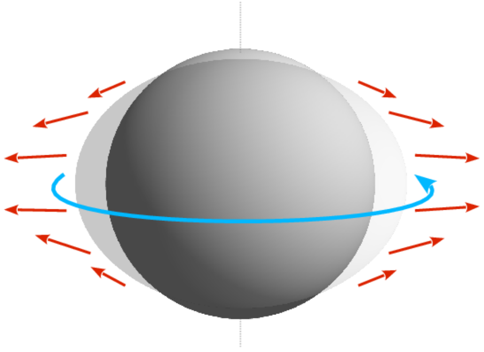

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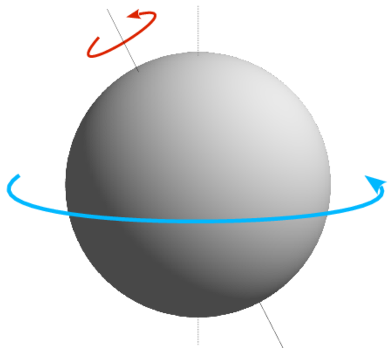
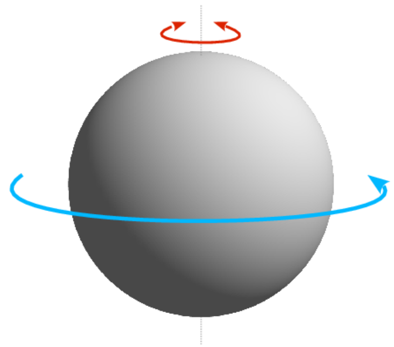


Precession/Nutation



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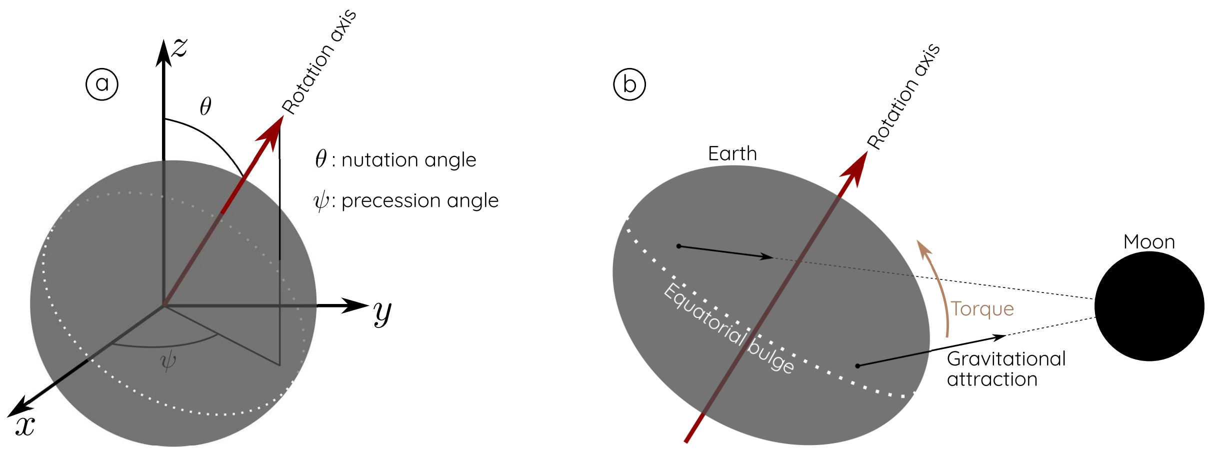

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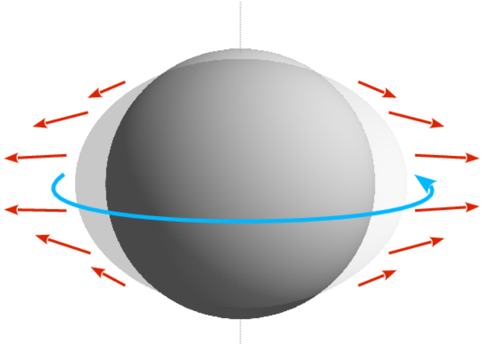

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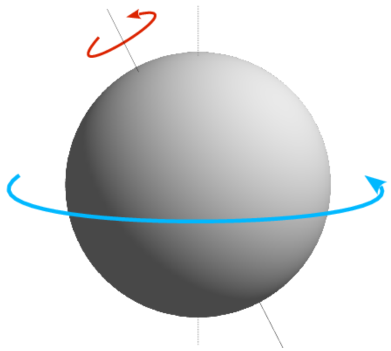

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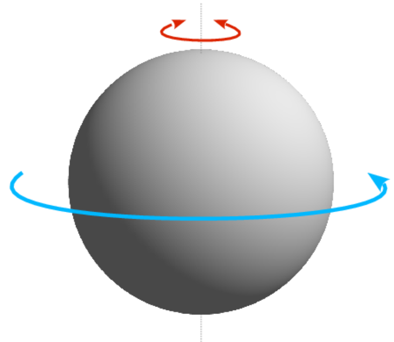
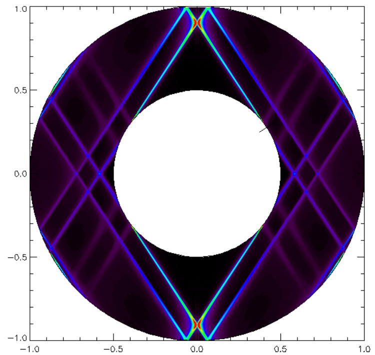
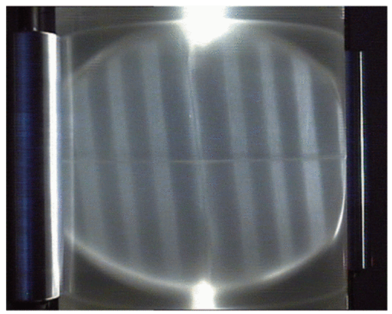


Libration

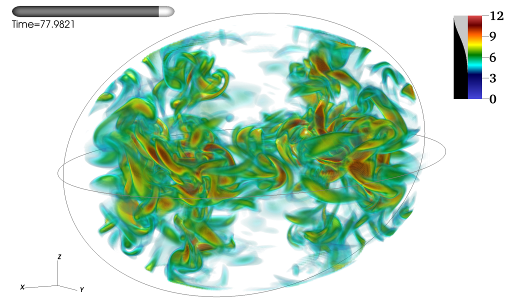
- These forcing can generate intense fluid motions:



Waves (direct forcing)
Ogilvie *et al.* 2007
Favier *et al.* 2014



Zonal flows
Morize *et al.* 2010
Calkins *et al.* 2012



Turbulence
Noir *et al.* 2012
Grannan *et al.* 2014
Favier *et al.* 2015

Rotating turbulence: waves versus vortices

Navier-Stokes in the rotating frame:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \underbrace{2\boldsymbol{\Omega} \times \boldsymbol{u}}_{\text{Coriolis}} = -\nabla P + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}$$

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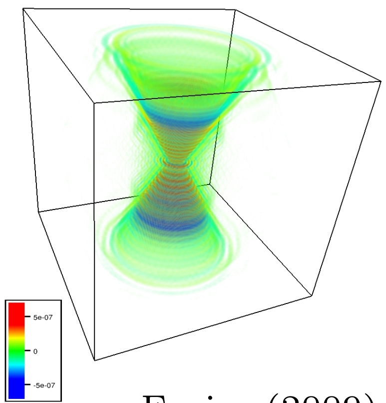
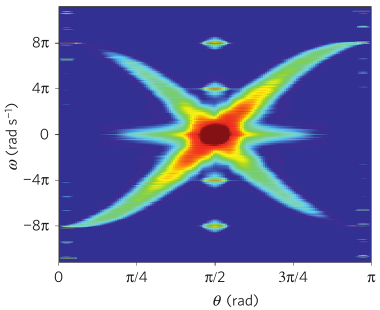
Inertial waves

$$\frac{\partial^2 \nabla^2 \boldsymbol{u}}{\partial t^2} + 4\Omega^2 \frac{\partial^2 \boldsymbol{u}}{\partial z^2} = 0$$



Favier (2009)



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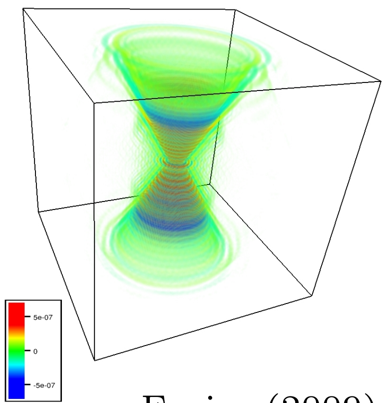

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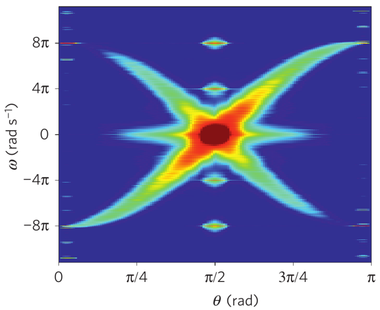

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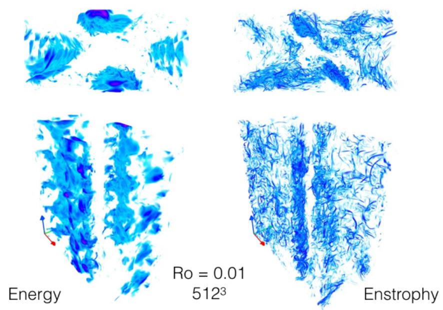

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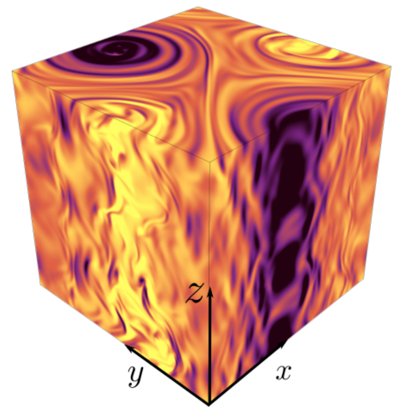

Le Reun (2017)

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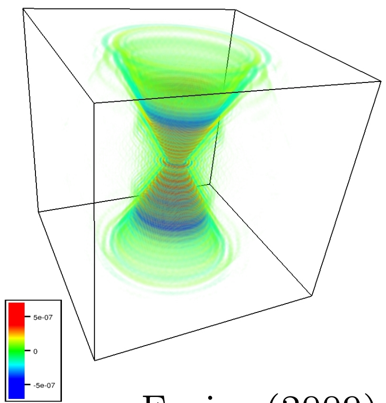

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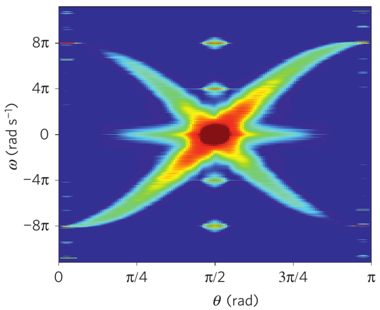

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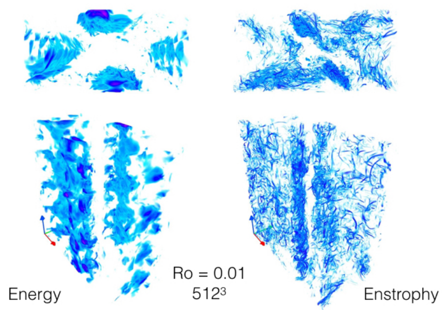

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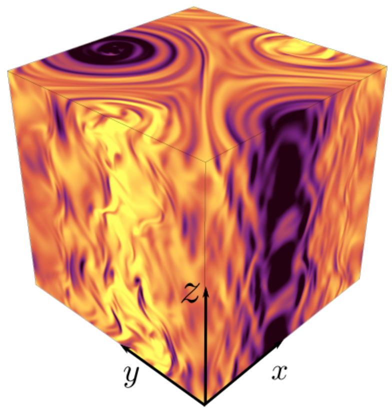
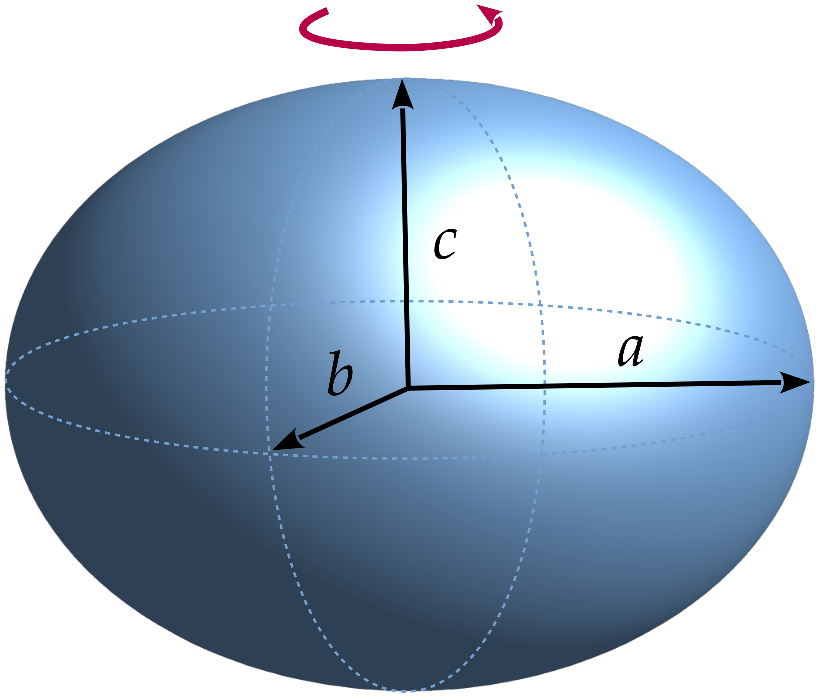


Le Reun (2017)

$$\boxed{\text{Role of the forcing } \boldsymbol{f}?}$$

Outline

Rotating tri-axial ellipsoids



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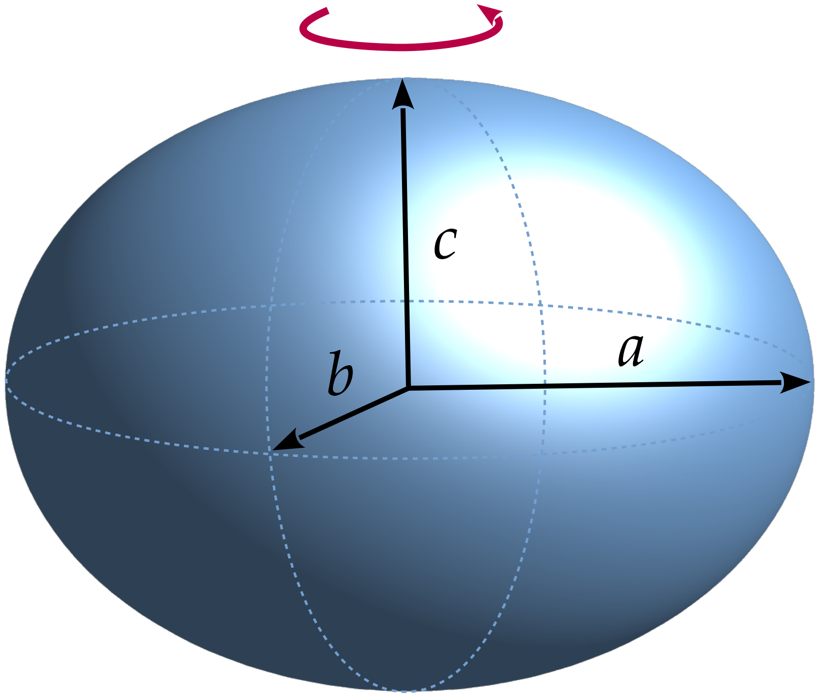

$$\beta = \frac{a^2 - b^2}{a^2 + b^2}$$

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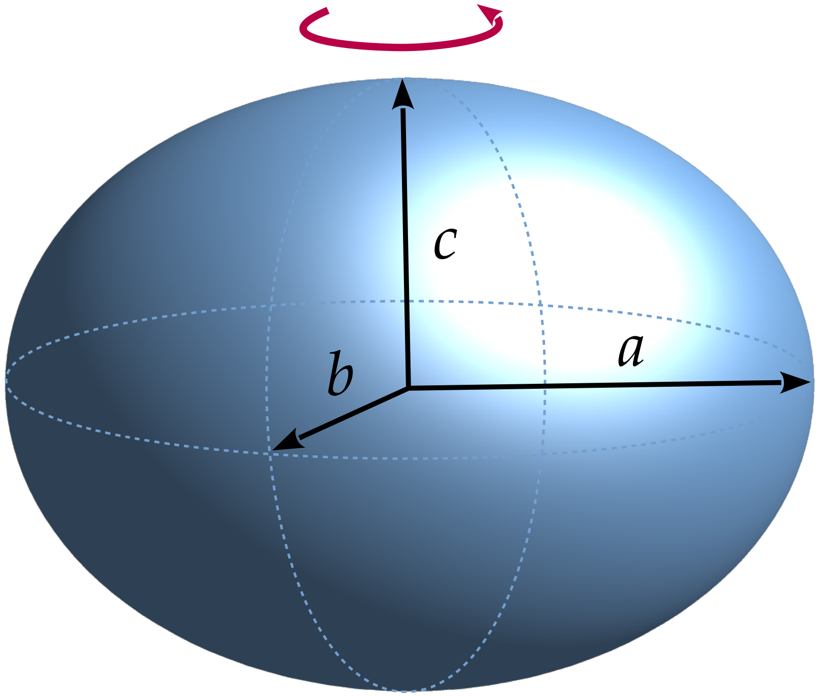

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Libration: modulation of the rotation rate

$$\boldsymbol{\Omega}(t) = \Omega_0 \left[1 + \epsilon \cos(f \Omega_0 t)\right] \boldsymbol{e}_z$$

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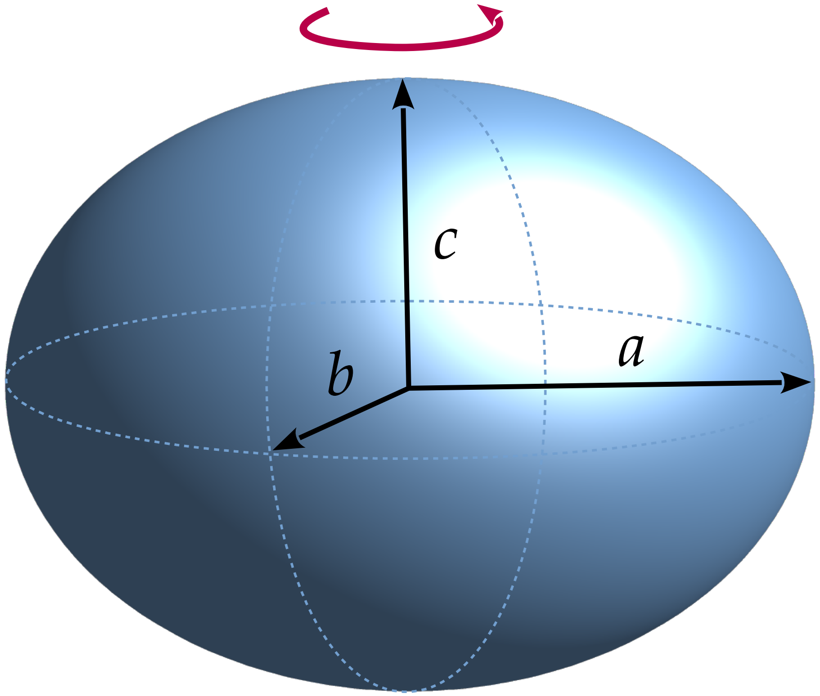

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Tides: modulation of the shape

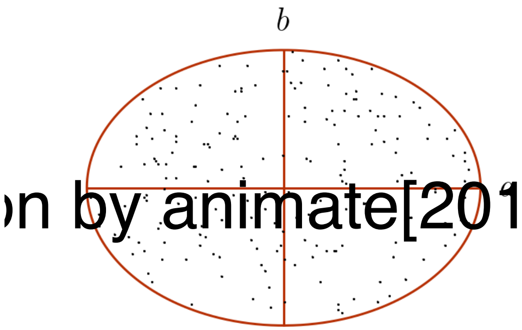
$$a(t), b(t), c(t)$$

Base flow driven by libration (Hough 1885)

$$\boldsymbol{\Omega}(t) = \Omega_0 \left[1 + \epsilon \cos(f\Omega_0 t)\right] \boldsymbol{e}_z$$

Mantle frame rotating at $\Omega(t)$

$$\boldsymbol{U}_b = \Omega_0 \epsilon \sin(\Omega_0 f t) \begin{bmatrix} -(1+\beta)Y \\ (1-\beta)X \\ 0 \end{bmatrix}$$



Frame rotating at Ω_0

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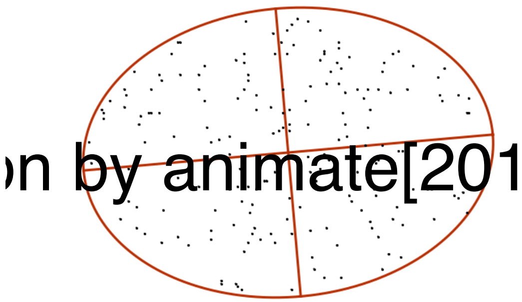

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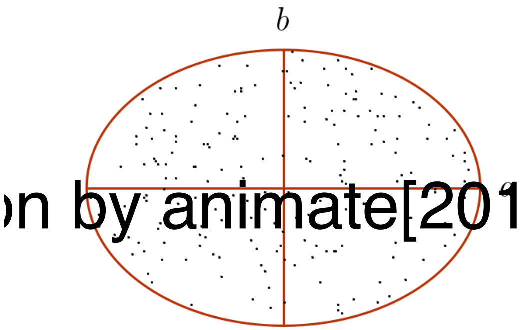
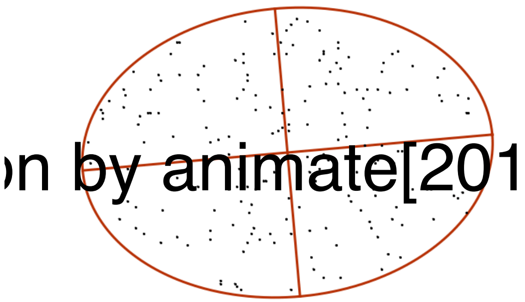




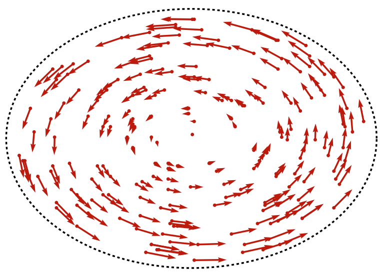
Dimensionless parameters:

- Input Rossby number $Ro = \epsilon\beta$
- Libration frequency f
- Ekman number $E = \nu/(\Omega_0 a^2)$

Base flow driven by tides

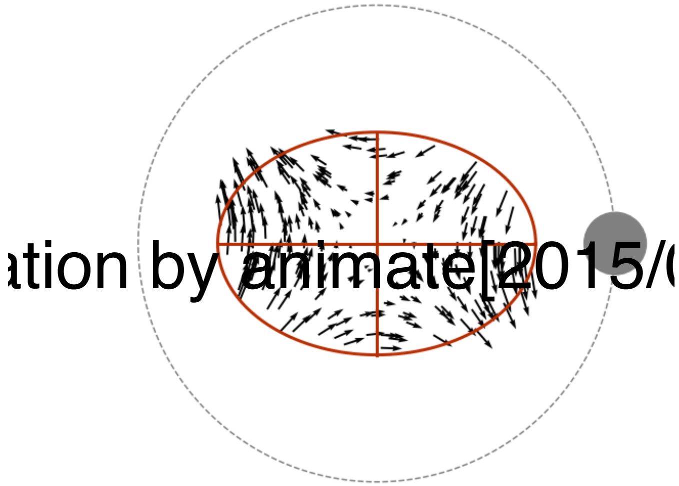
Orbital frame rotating at n

$$\boldsymbol{U}_b = (\Omega_0 - n)\begin{bmatrix} -(1+\beta)Y \\ (1-\beta)X \\ 0 \end{bmatrix}$$



Planetary frame rotating at Ω_0

$$\boldsymbol{U}_b = -\beta\gamma \begin{bmatrix} \sin 2\gamma t & \cos 2\gamma t & 0 \\ \cos 2\gamma t & -\sin 2\gamma t & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

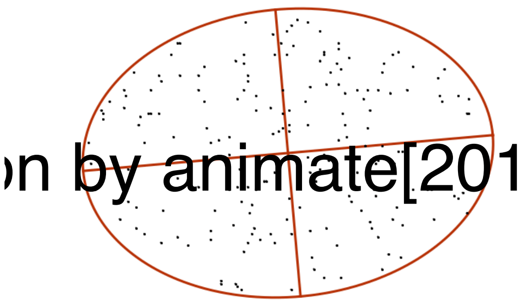


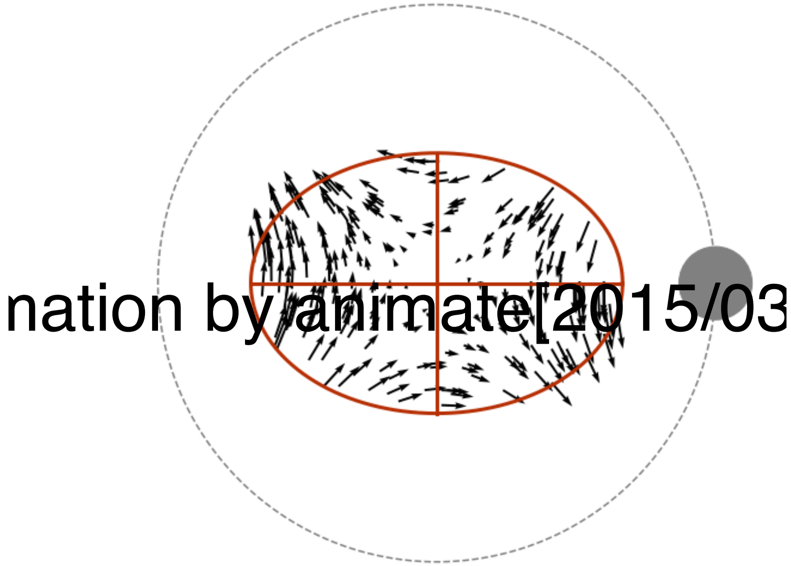
Dimensionless parameters:

- Input Rossby number $Ro = \beta$
- Differential rotation frequency $\gamma = \Omega_0 - n$
- Ekman number $E = \nu/(\Omega_0 a^2)$

Tides

Libration

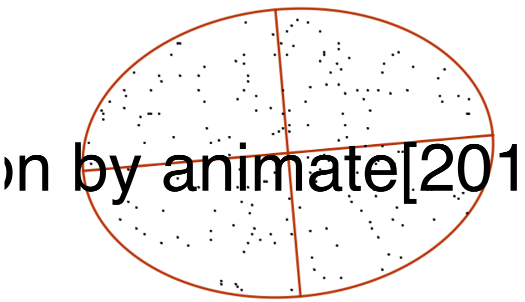


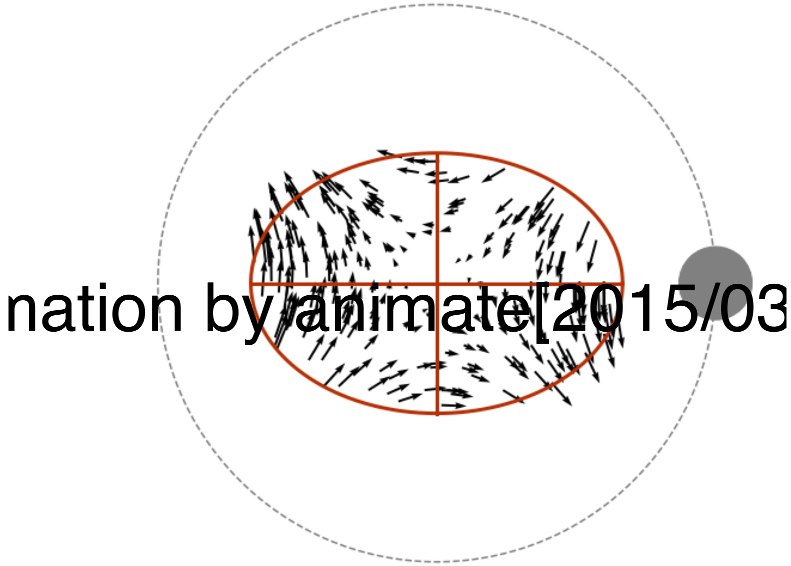


Are these flows stable?

Tides

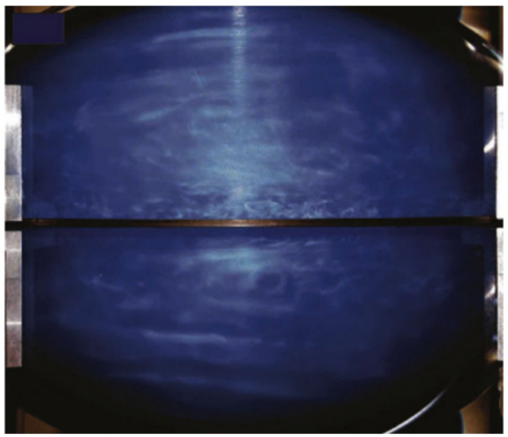
Libration



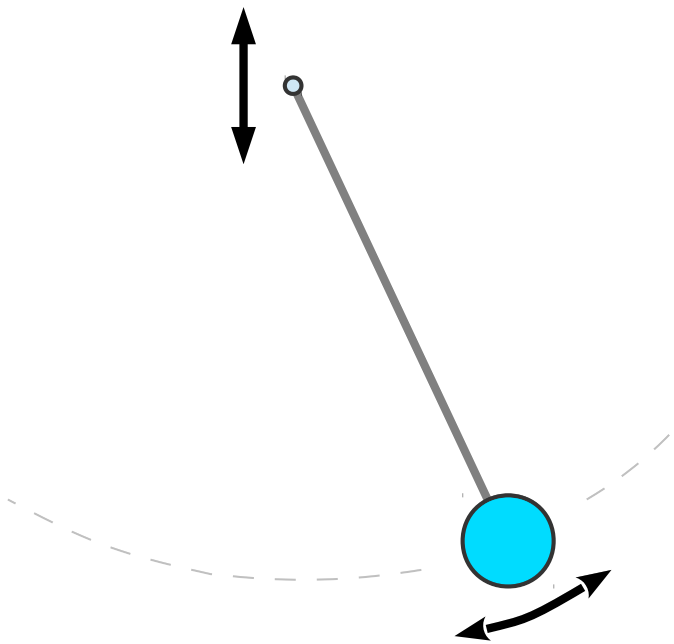


Are these flows stable?

- Boundary instabilities (centrifugal, Noir *et al.* (2009))
- Bulk instabilities (elliptical)



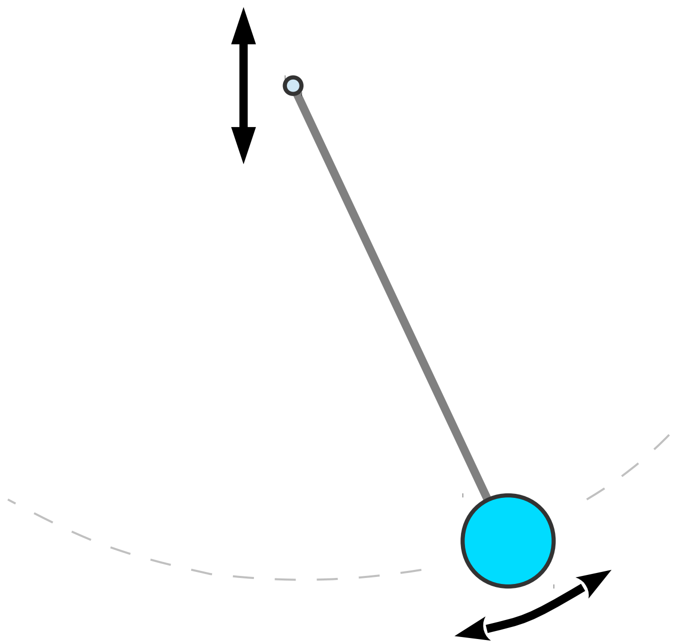
Parametric instabilities



Forced pendulum

Faraday waves (Youtube)

Parametric instabilities



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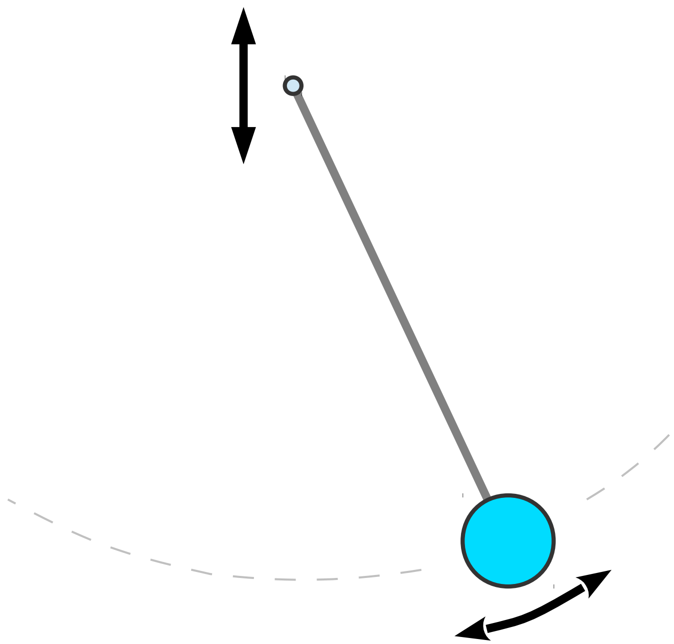
Faraday waves (Youtube)

Harmonic oscillator + Harmonic forcing

Parametric instabilities



Forced pendulum

Faraday waves (Youtube)

Harmonic oscillator + Harmonic forcing

Elliptical instability: Inertial waves + Flow with elliptic streamlines

The harmonic oscillator: inertial waves

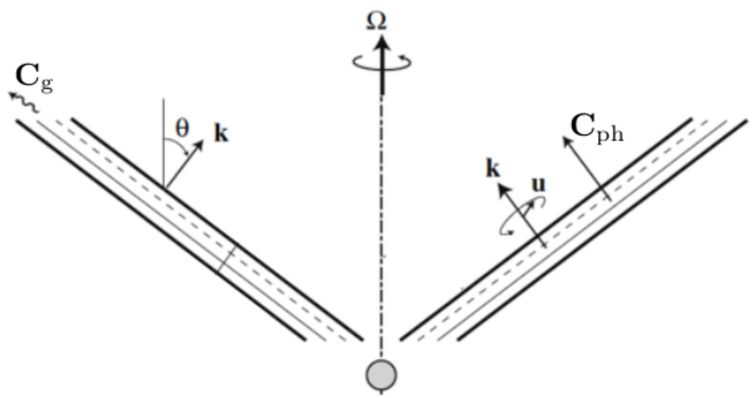
Navier-Stokes in the rotating frame:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\nabla P + \nu \nabla^2 \boldsymbol{u}$$

Poincaré equation (linear inviscid limit):

$$\frac{\partial^2 \nabla^2 \boldsymbol{u}}{\partial t^2} + 4\Omega^2 \frac{\partial^2 \boldsymbol{u}}{\partial z^2} = 0$$

a. Inertial wave



$$\boxed{\omega = \pm 2\Omega \cos\theta}$$

The harmonic oscillator: inertial waves

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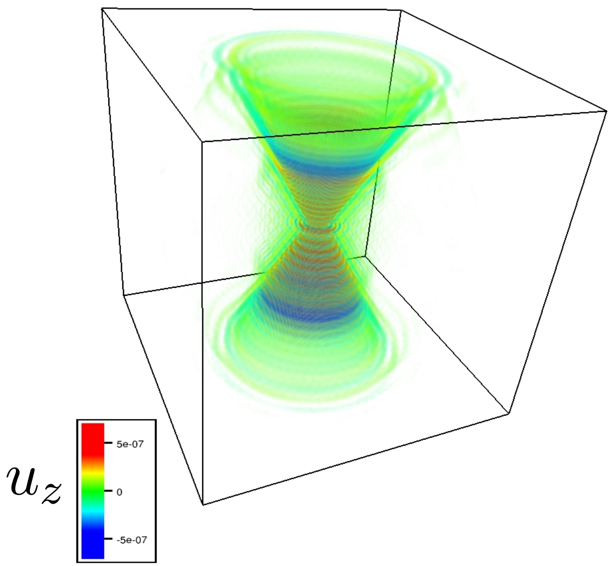

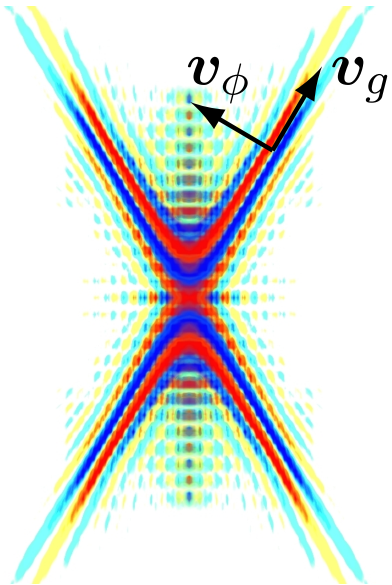

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The harmonic oscillator: inertial waves

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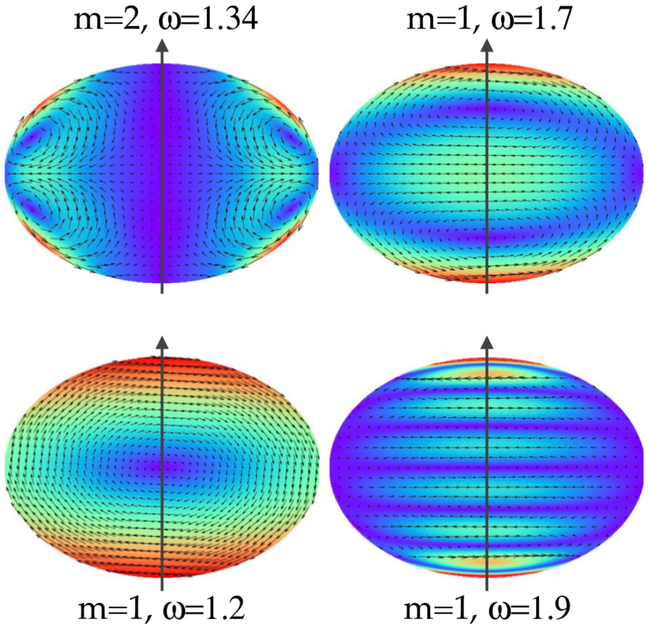
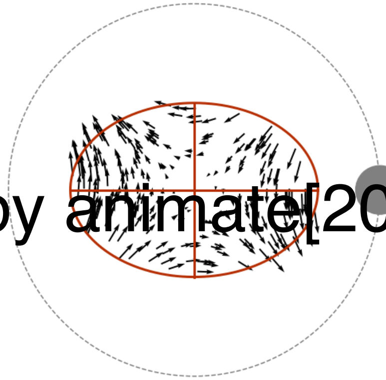


Kerswell (1994)

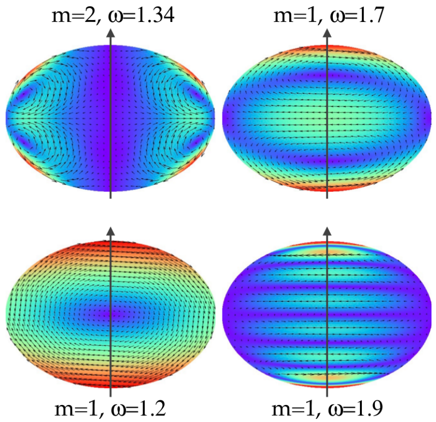
$$\boxed{\omega = \pm 2\Omega \cos\theta}$$

Elliptical instability

on by animate[201



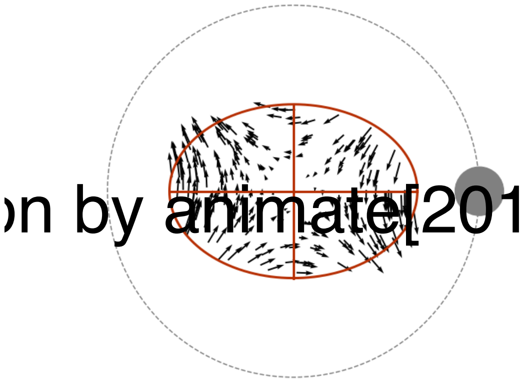
$+$



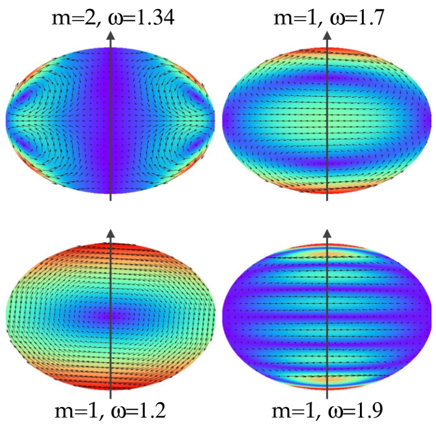
$=$

$\boxed{\text{Elliptical instability}}$

Elliptical instability



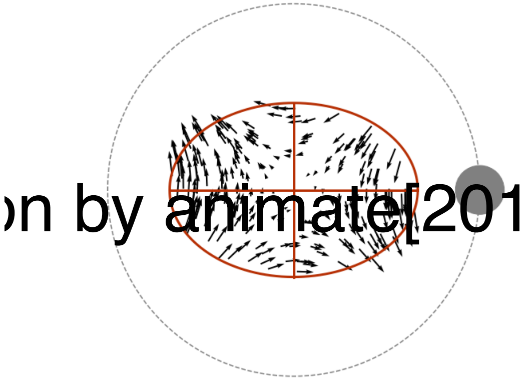
$+$



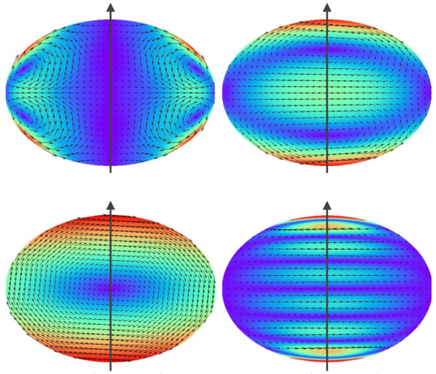
$=$ $\boxed{\text{Elliptical instability}}$

- Parametric sub-harmonic instability
- Extensively studied in the context of strained vortices (Bailly 1986, Pierrehumbert 1986)
- Suggested for geophysical applications by Malkus (1963,1989)
- Reintroduced by Kerswell (1996,2002)
- Observed experimentally in ellipsoidal containers (Le Bars 2015)

Elliptical instability



+



=

Elliptical instability

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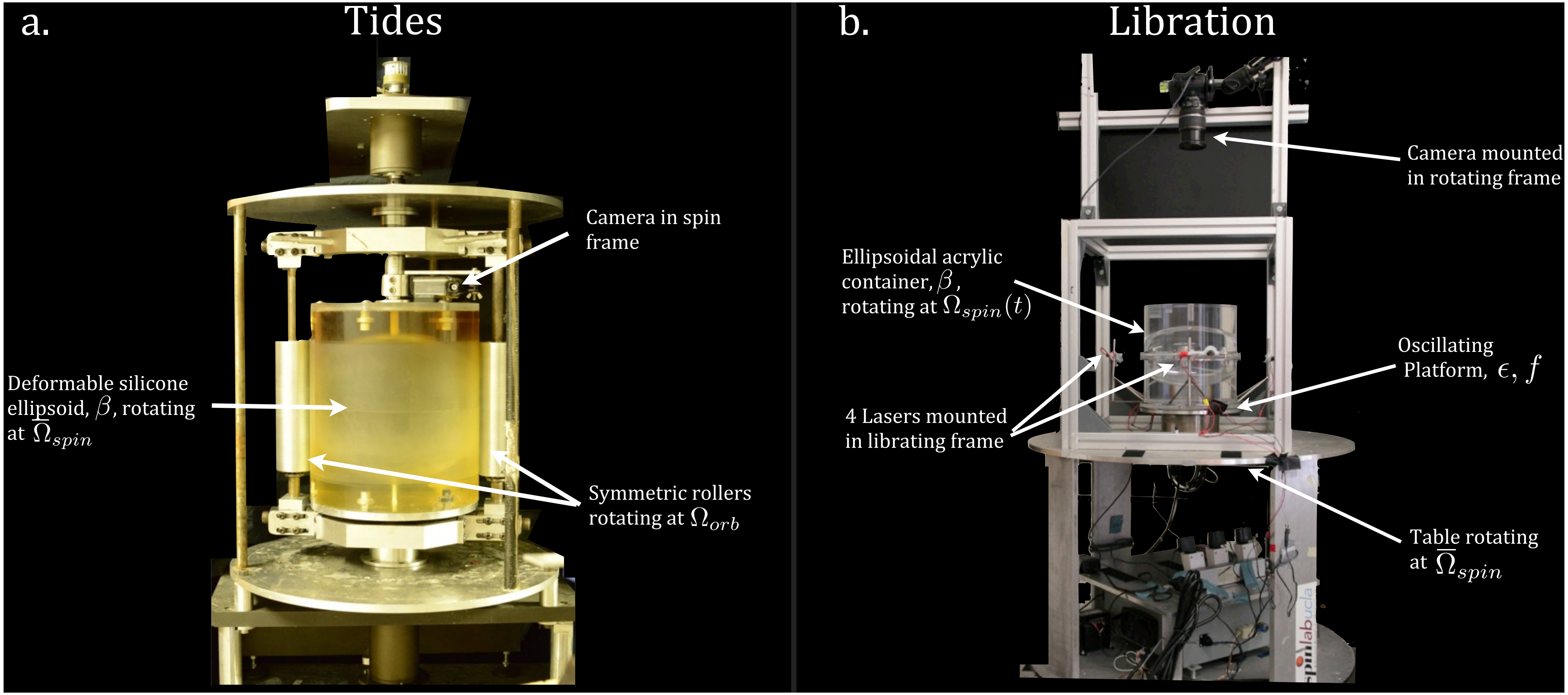
$\Rightarrow$ Energy is injected in resonant inertial waves only

Experimental setups



IRPHE, Marseille

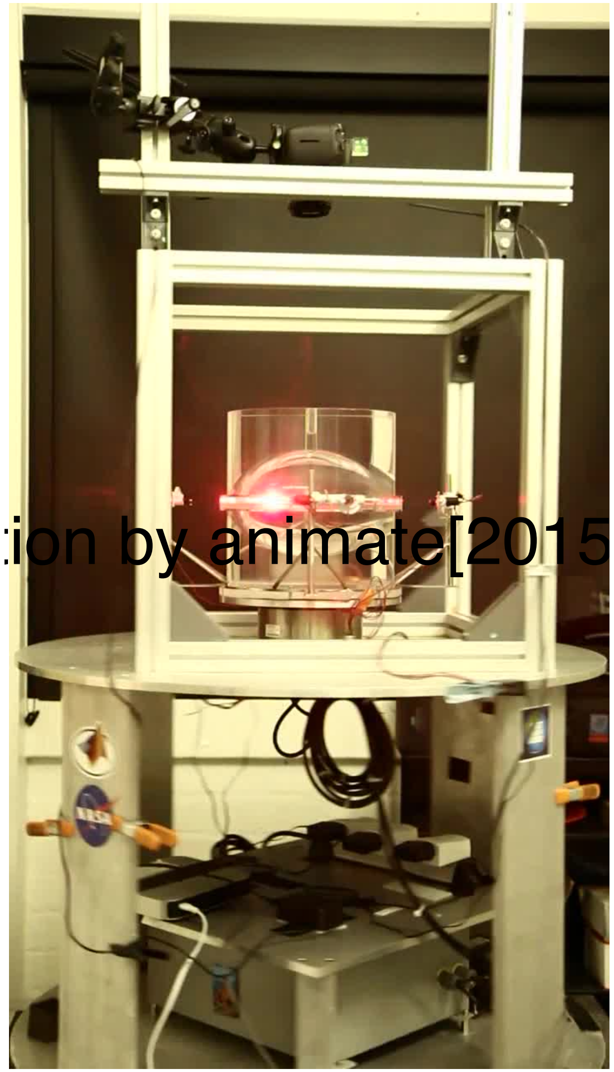
Morize *et al.* 2010
Grannan *et al.* 2017

SpinLab, Los Angeles

Noir *et al.* 2012
Grannan *et al.* 2014

Experimental setups

Experimental setups



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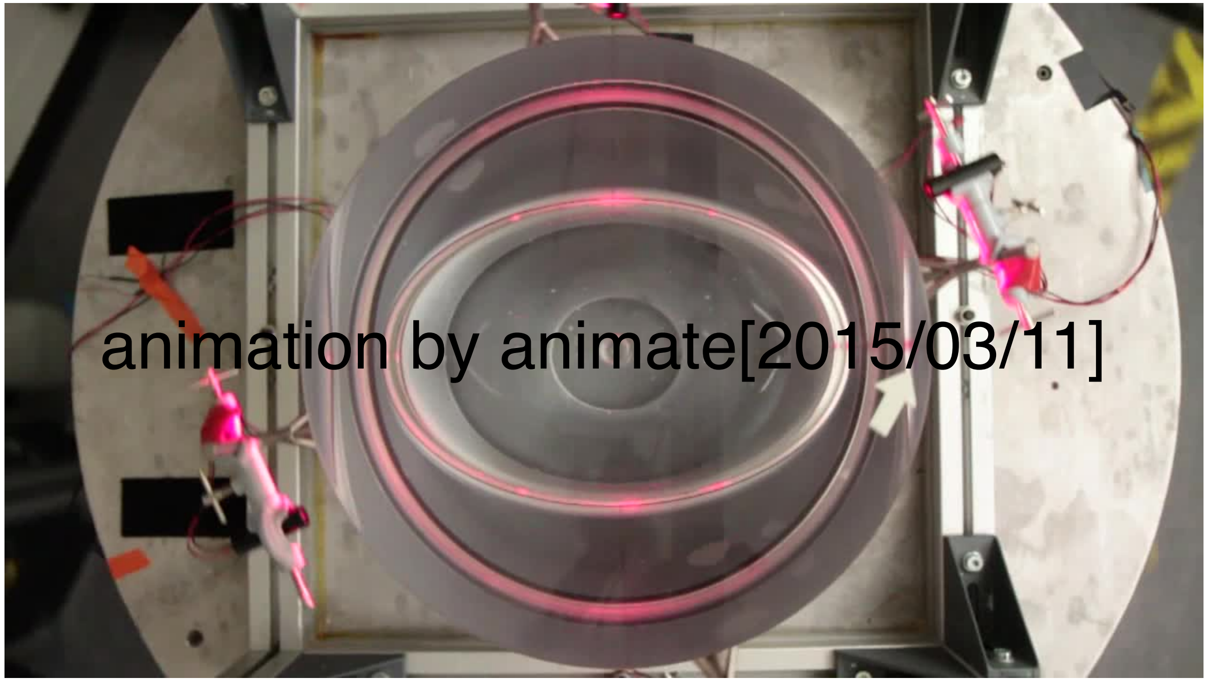
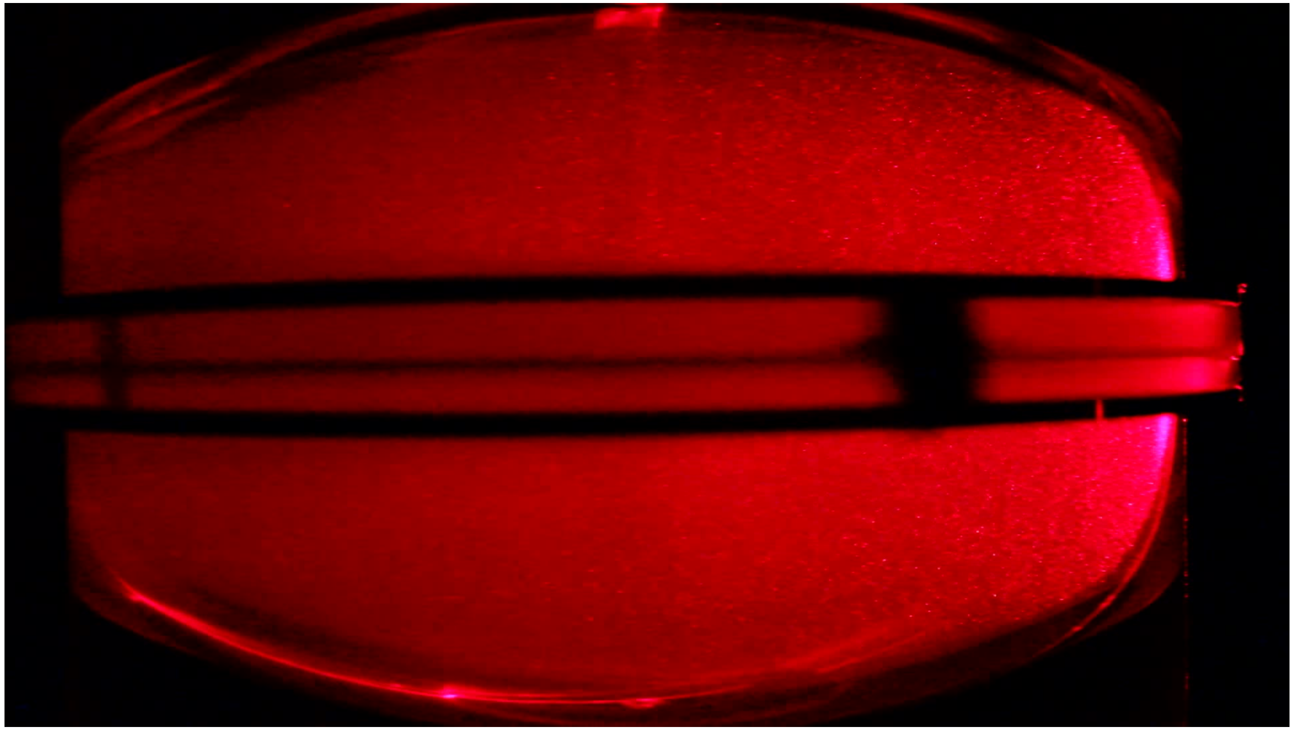
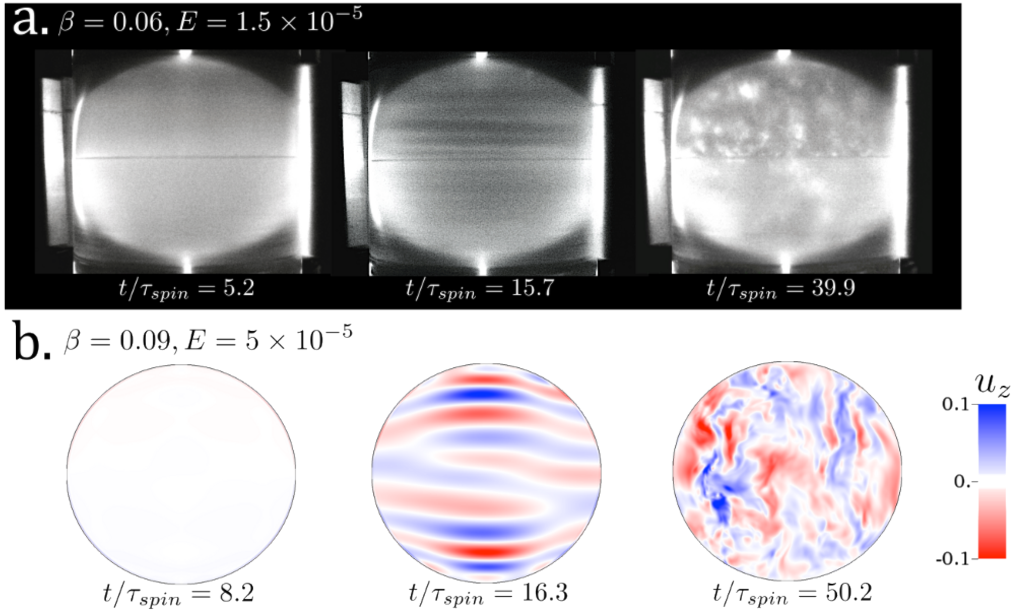


An example of libration-driven instability

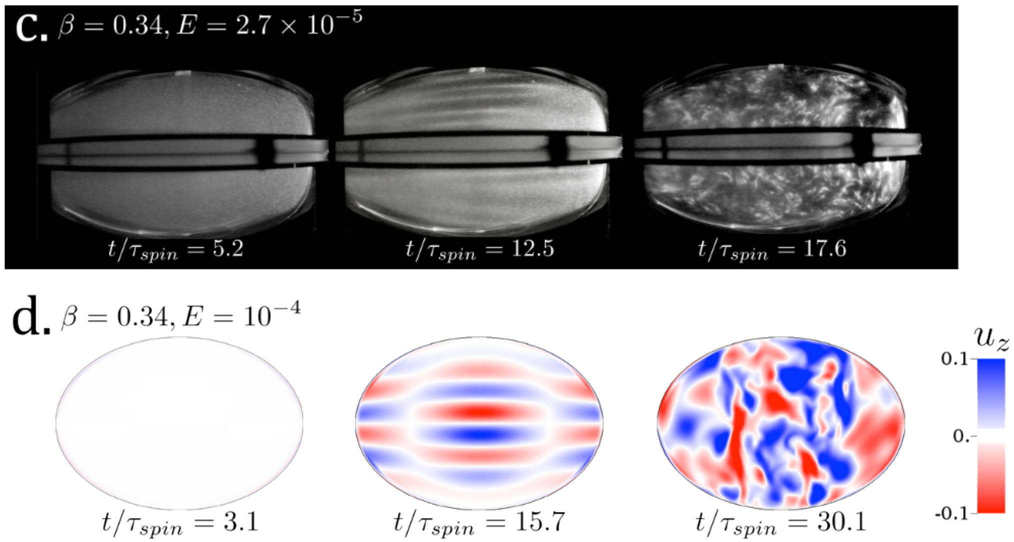


Results for $f/\Omega = 4$

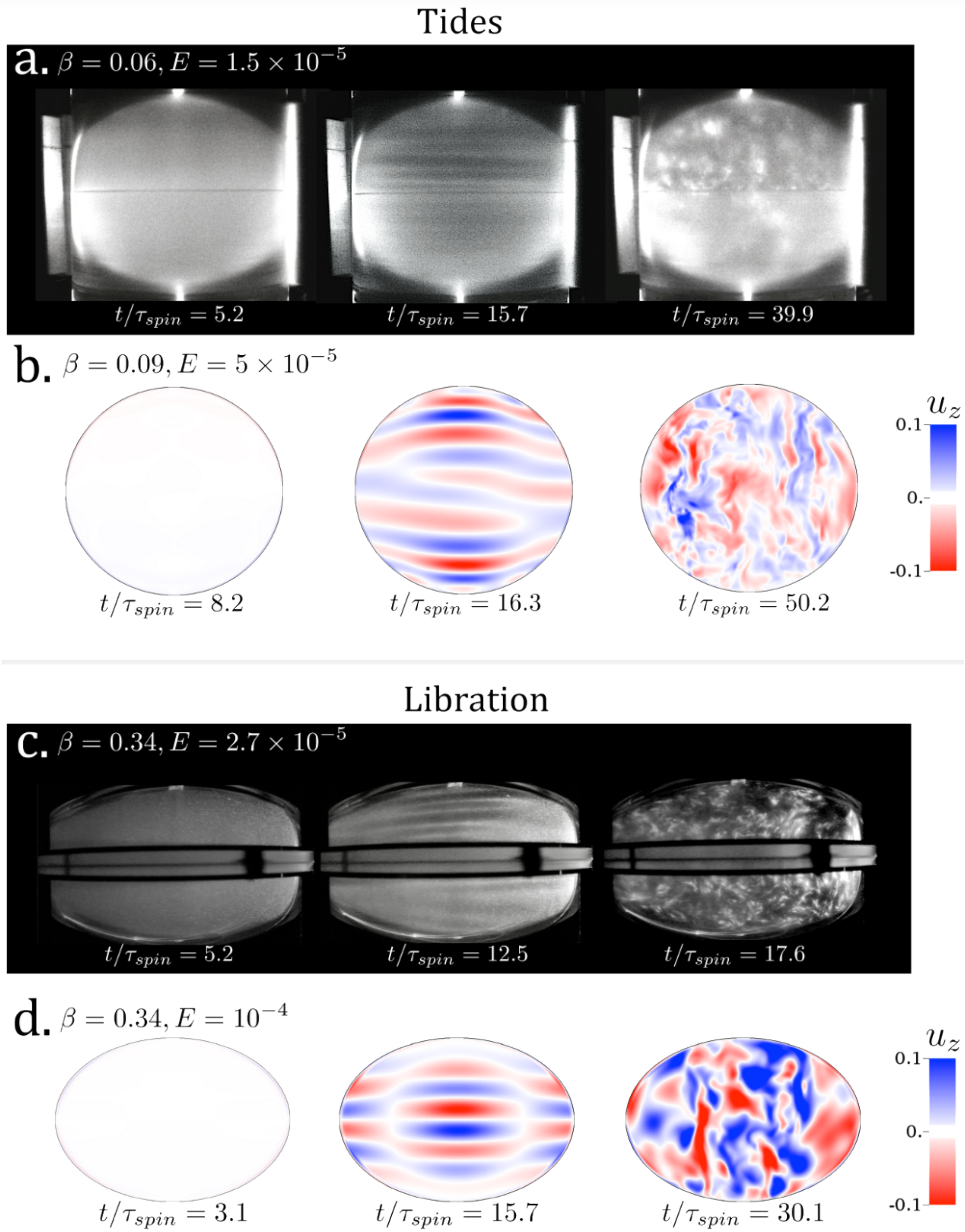
Tides



Libration



Results for $f/\Omega = 4$



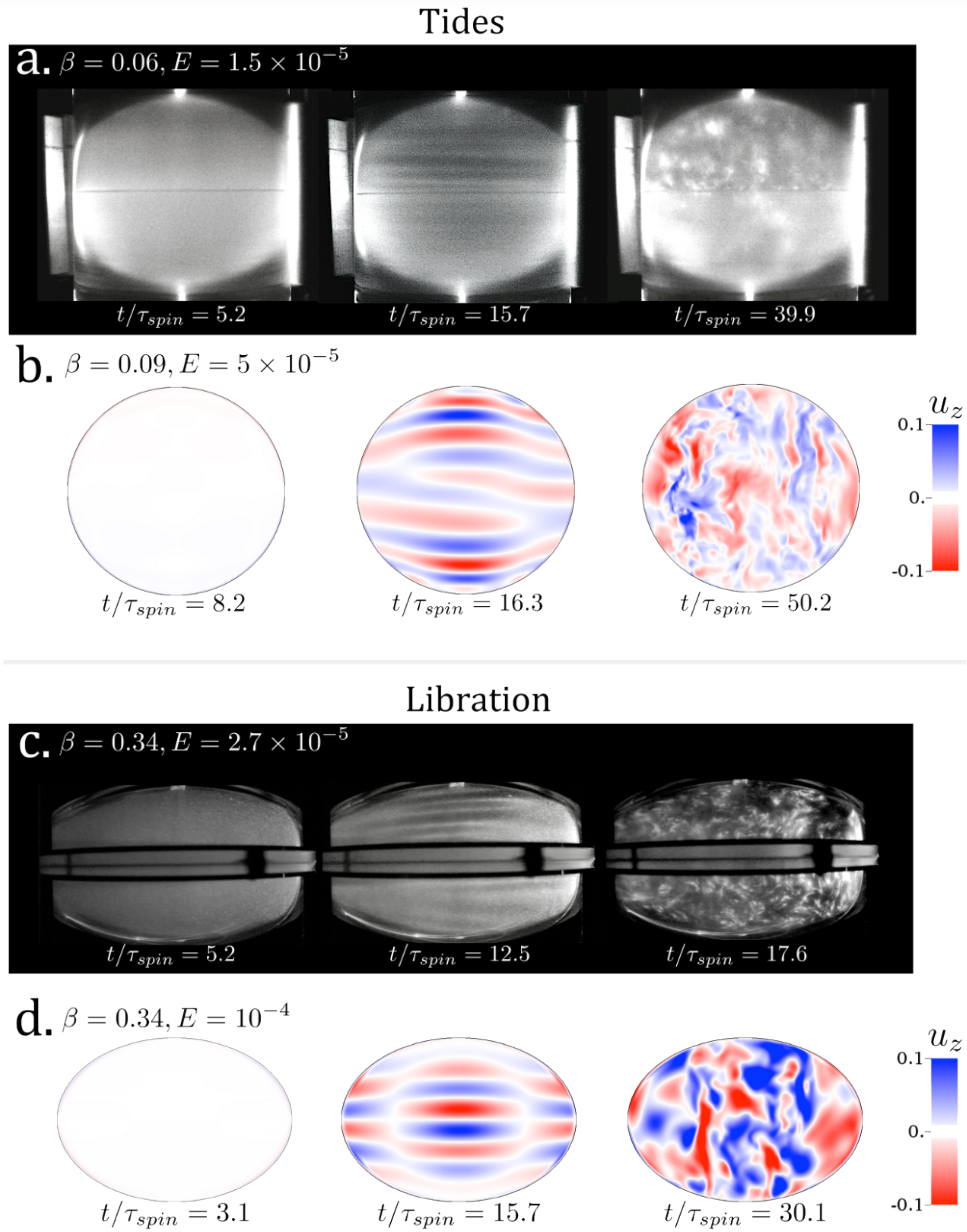
Instability threshold

Secondary resonances

Small-scale turbulence

...

Results for $f/\Omega = 4$



Instability threshold

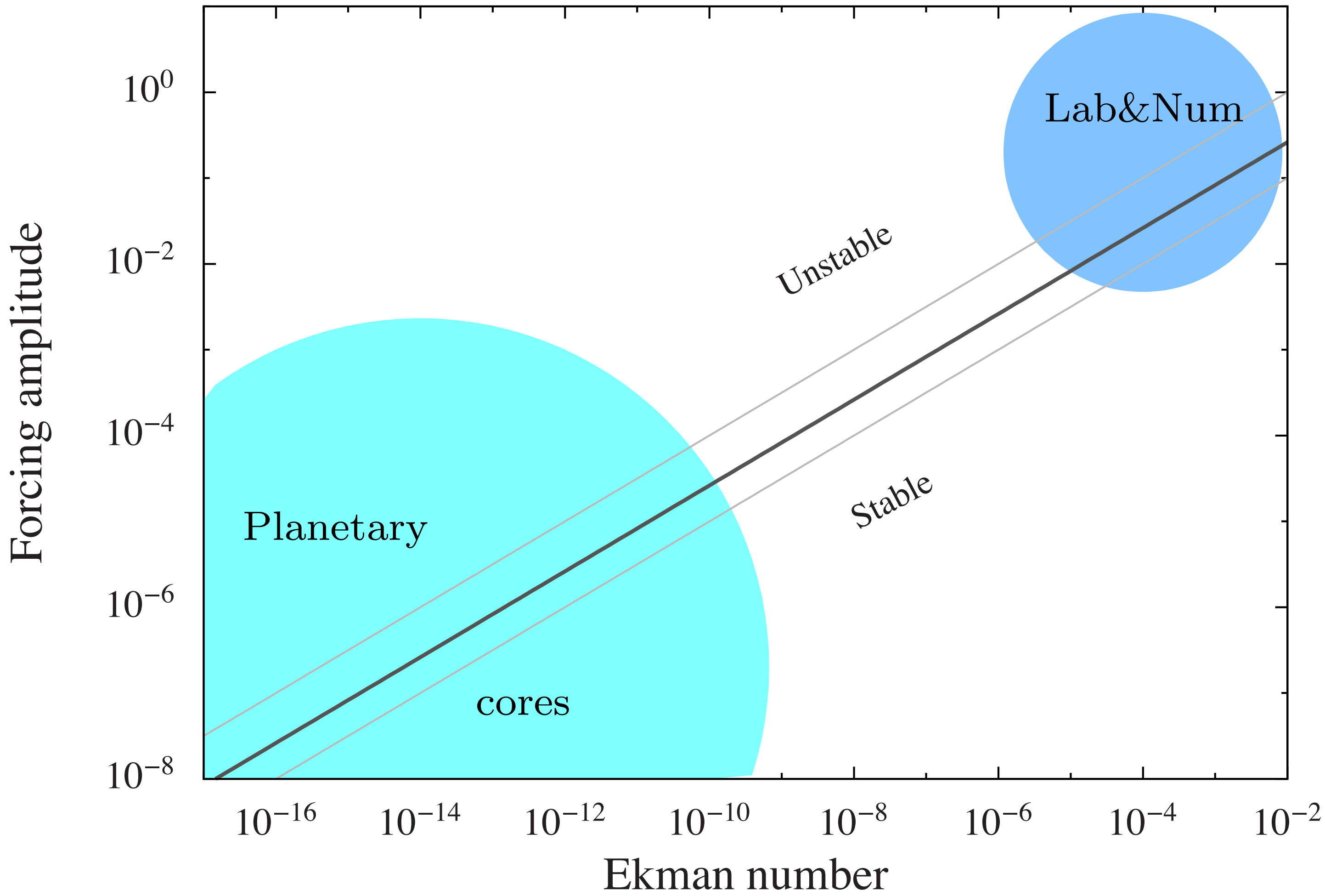
Secondary resonances

Small-scale turbulence

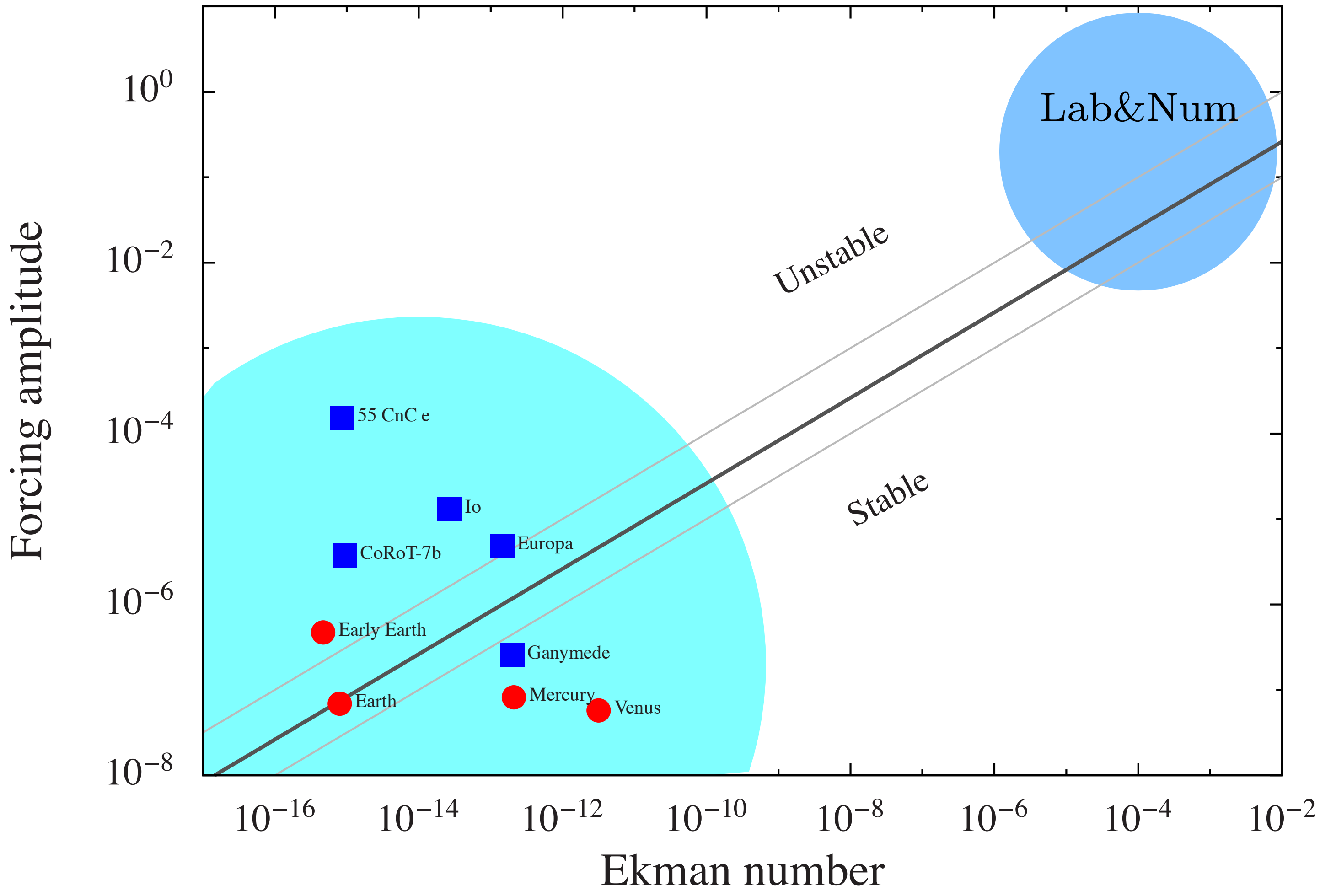
...

but...

From the lab to planets...



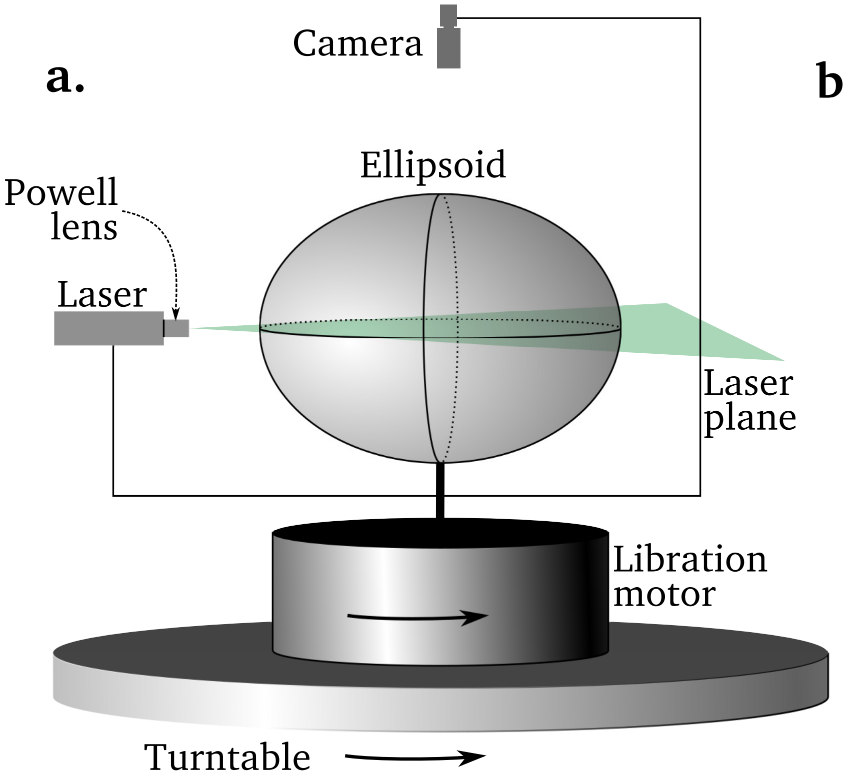
From the lab to planets...

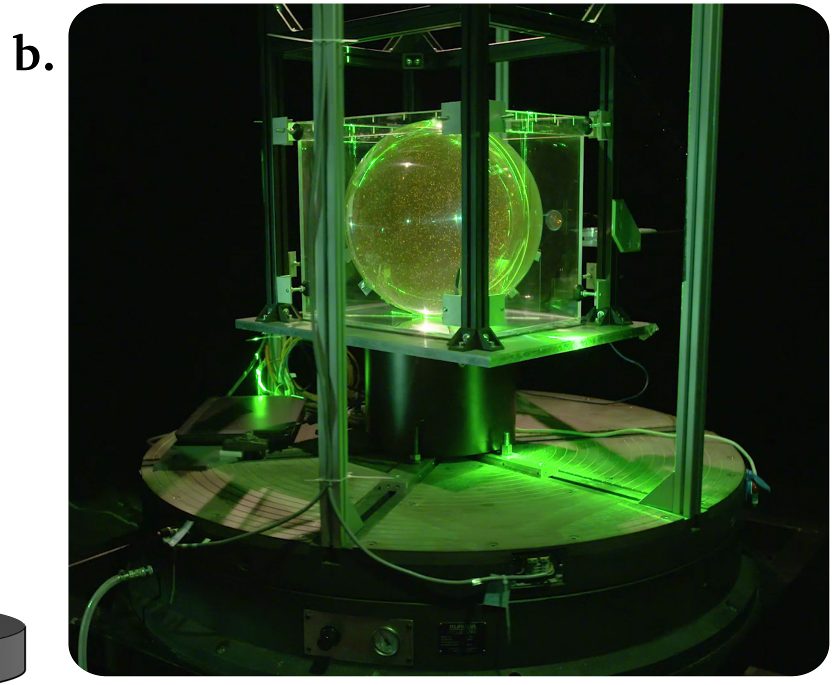


Cébron *et al.* 2012, 2013

Outline

A new (bigger) experiment





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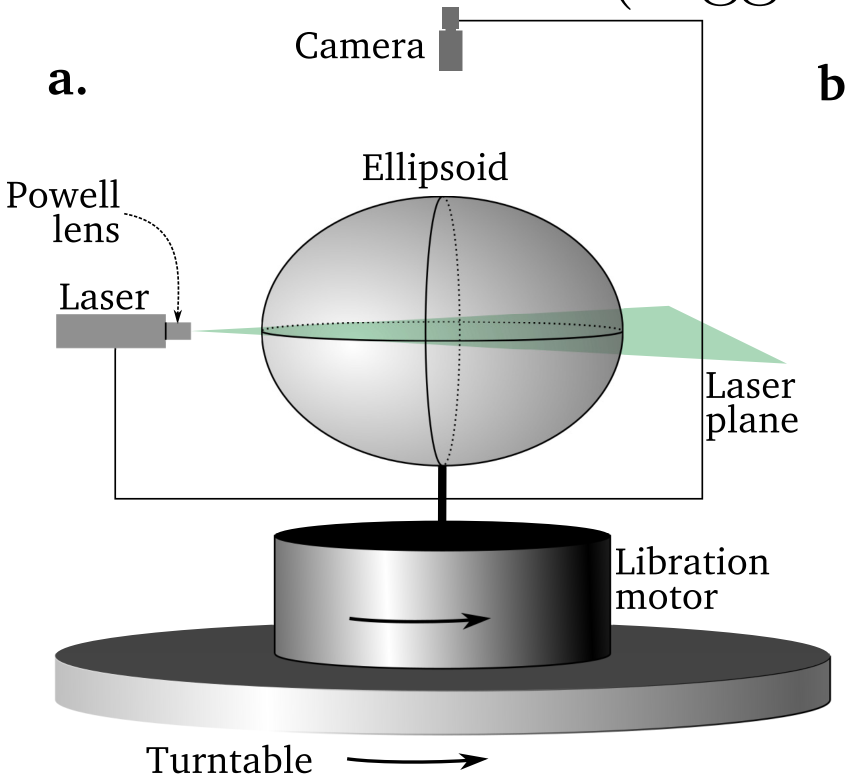
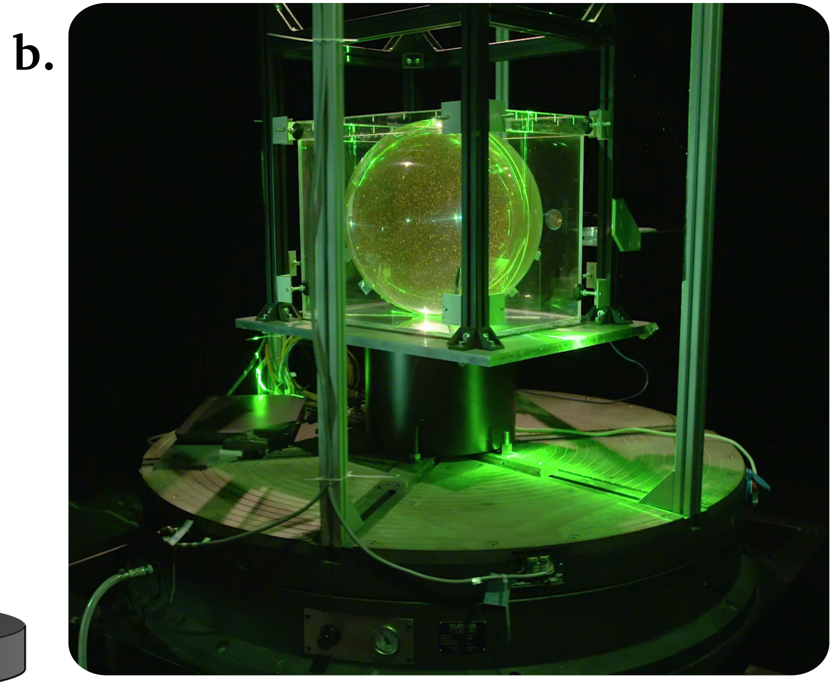
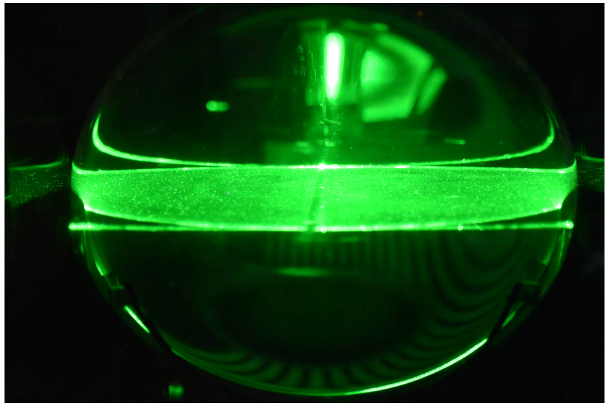
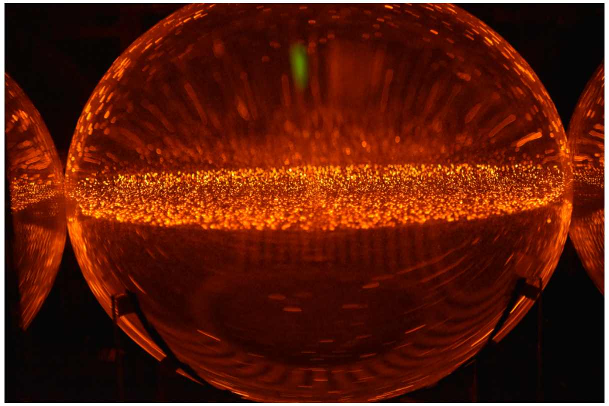


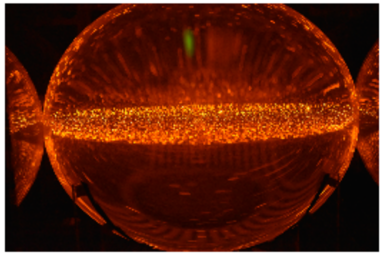


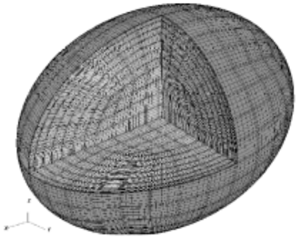




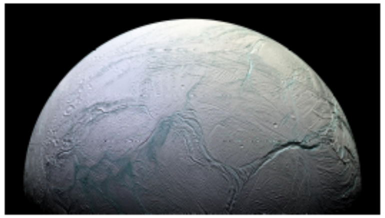
A new (bigger) experiment

		The experiment	Simulations	Planetary cores
Ellipticity	$\beta = \frac{a^2 - b^2}{a^2 + b^2}$	0.33	0.33	$\leq 10^{-3}$
Libration amplitude	$\varepsilon = \Delta\Omega/\Omega$	$0.05 - 0.30$	0.8	$\leq 10^{-3}$
Libration frequency	f	4	4	Any
Ekman number	$E = \nu/(a^2\Omega)$	$5 \times 10^{-6} - 10^{-5}$	$\geq 5 \times 10^{-5}$	$\leq 10^{-10}$



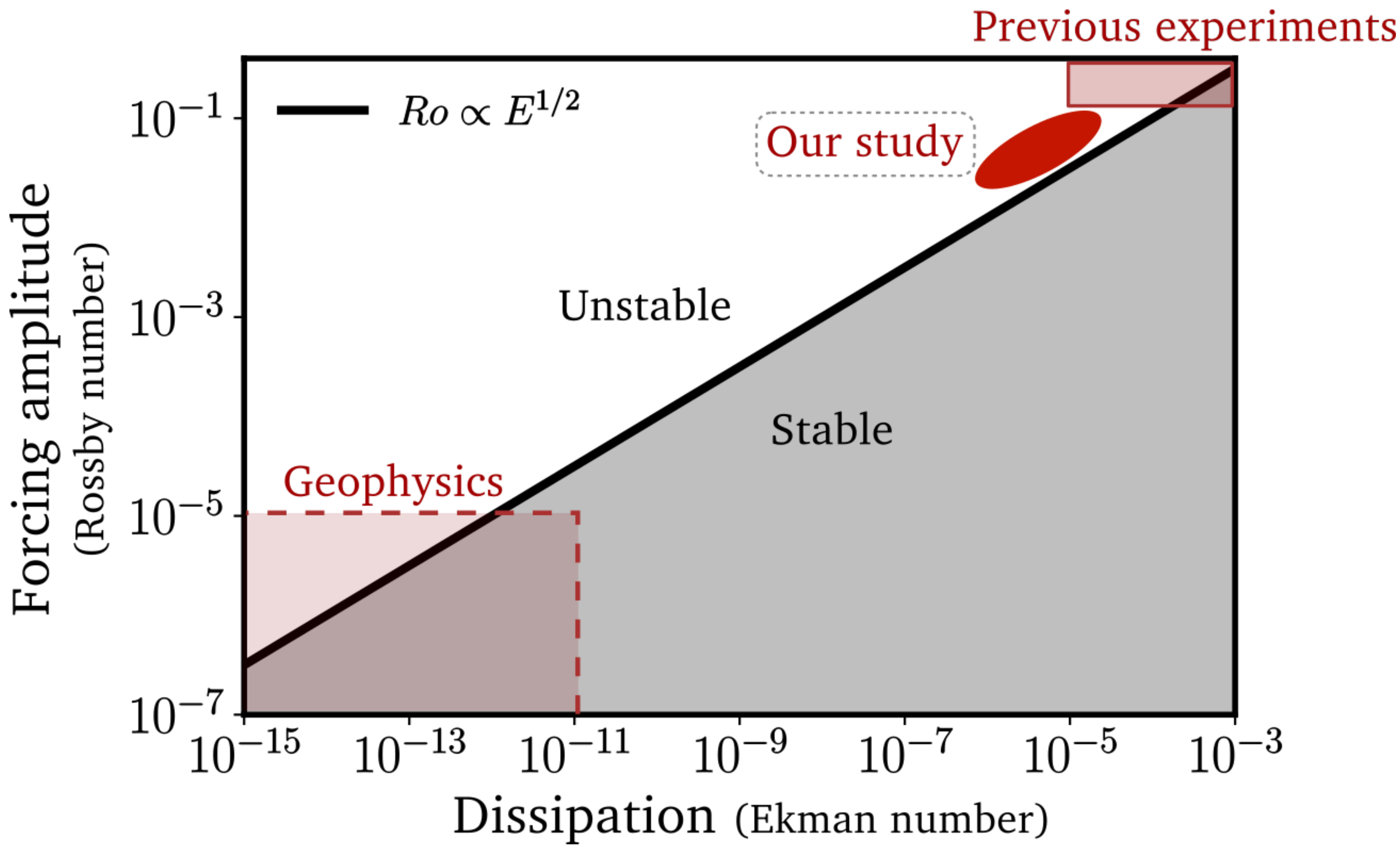


Favier et al. 2015



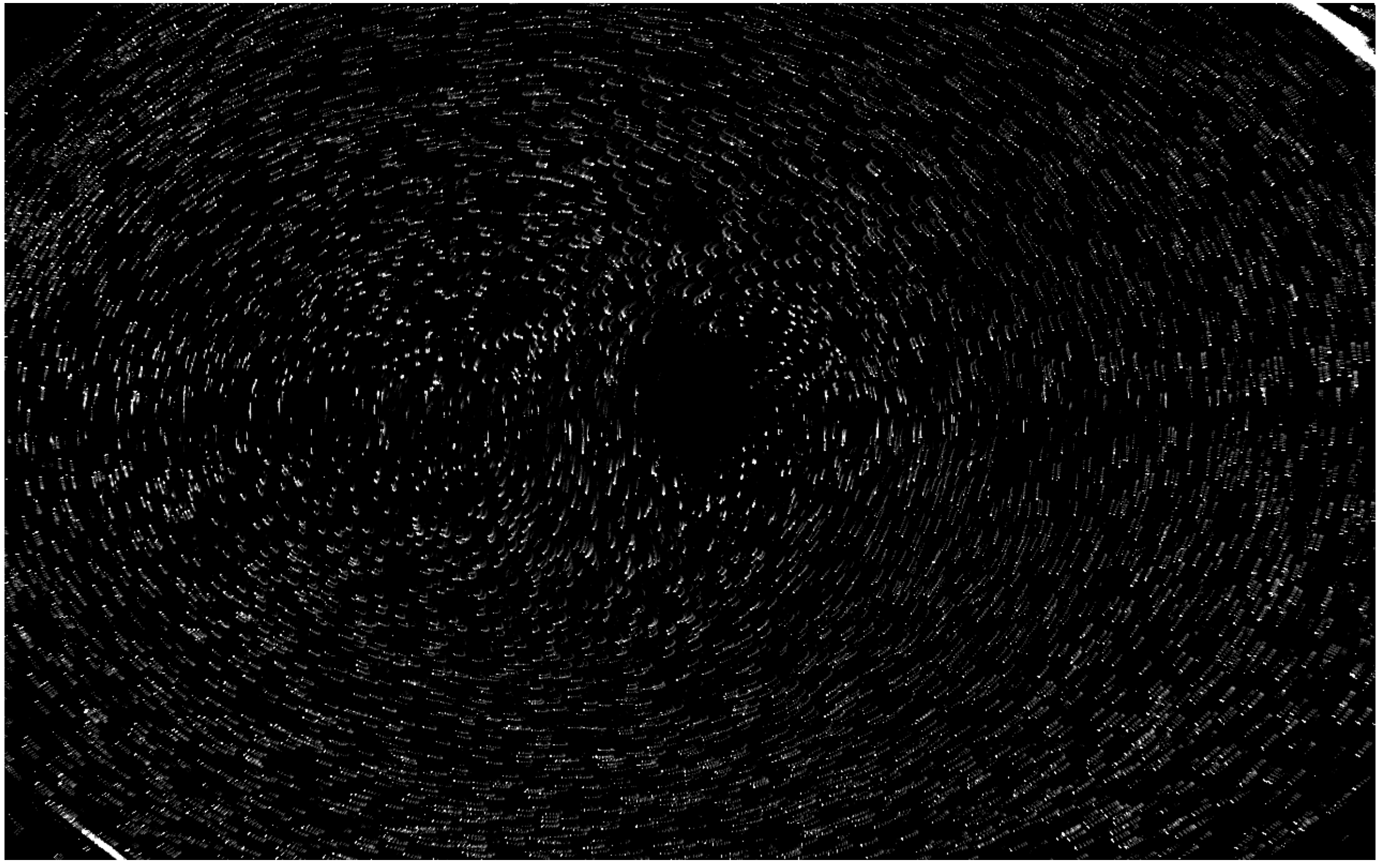
Cébron et al. 2012

A new (bigger) experiment



Low forcing amplitudes

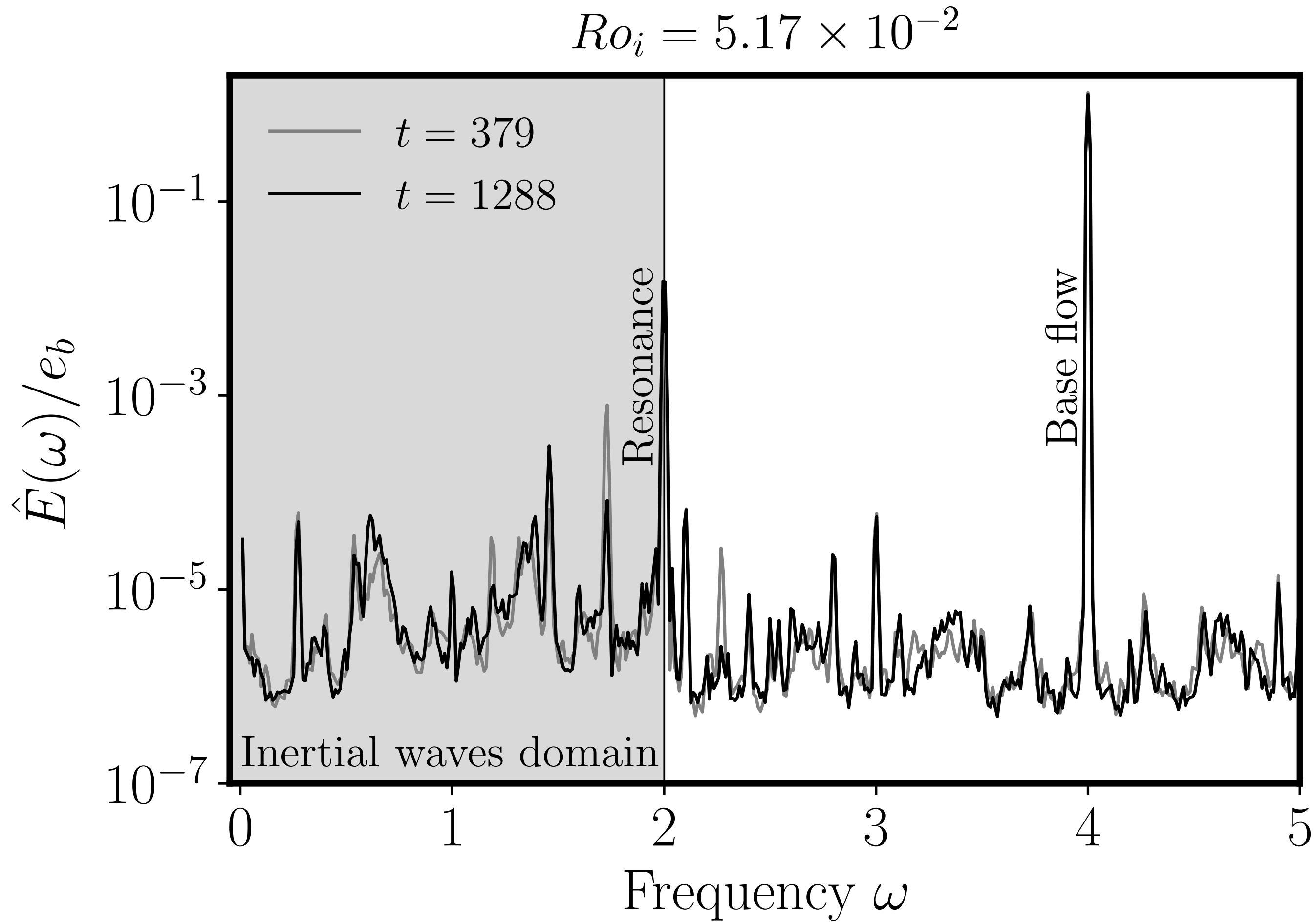
Equatorial flow in the quasi-steady state



$$E = 5 \times 10^{-6} \quad \text{and} \quad Ro = 5 \times 10^{-2} \quad (Ro_c = 2 \times 10^{-2})$$

Low forcing amplitudes

Temporal power spectra



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Low forcing amplitudes

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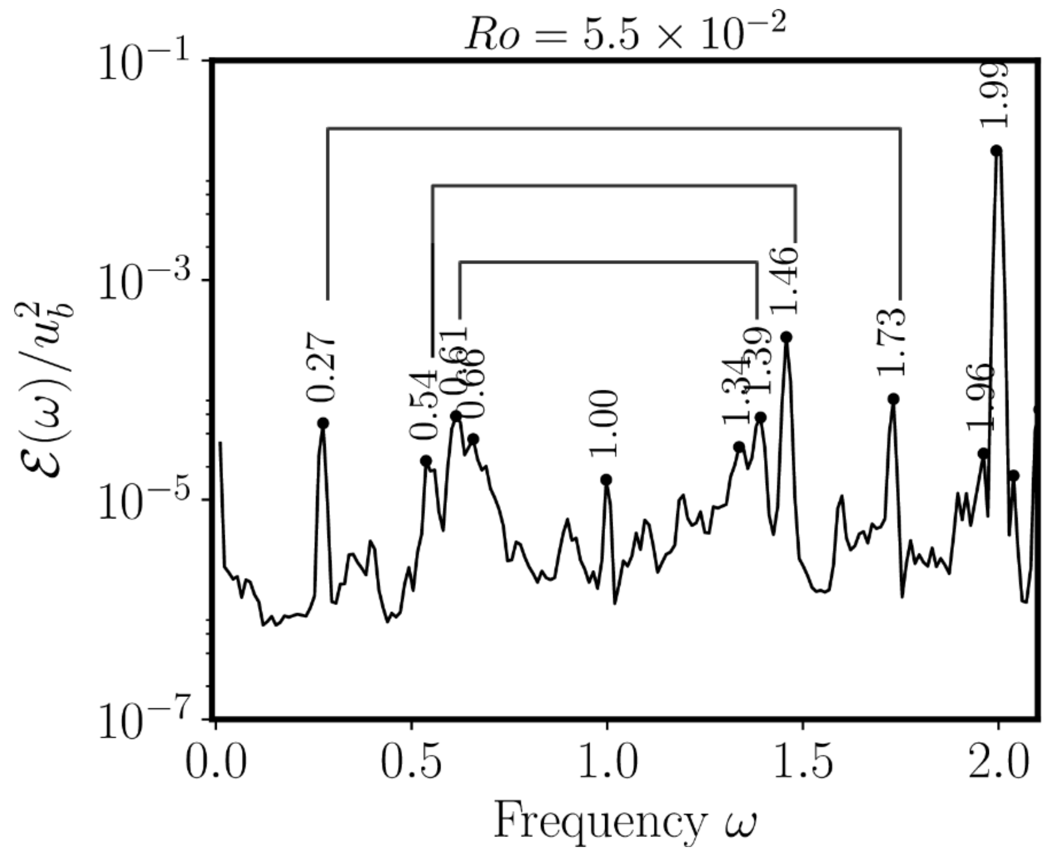
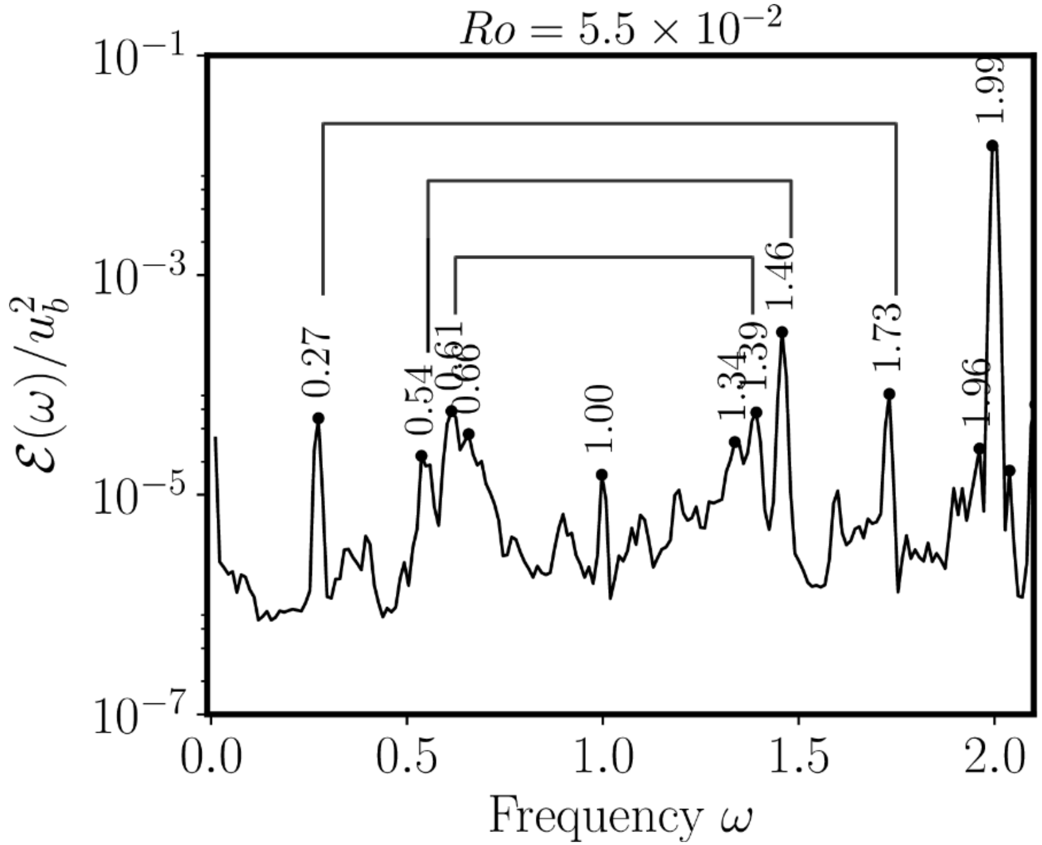


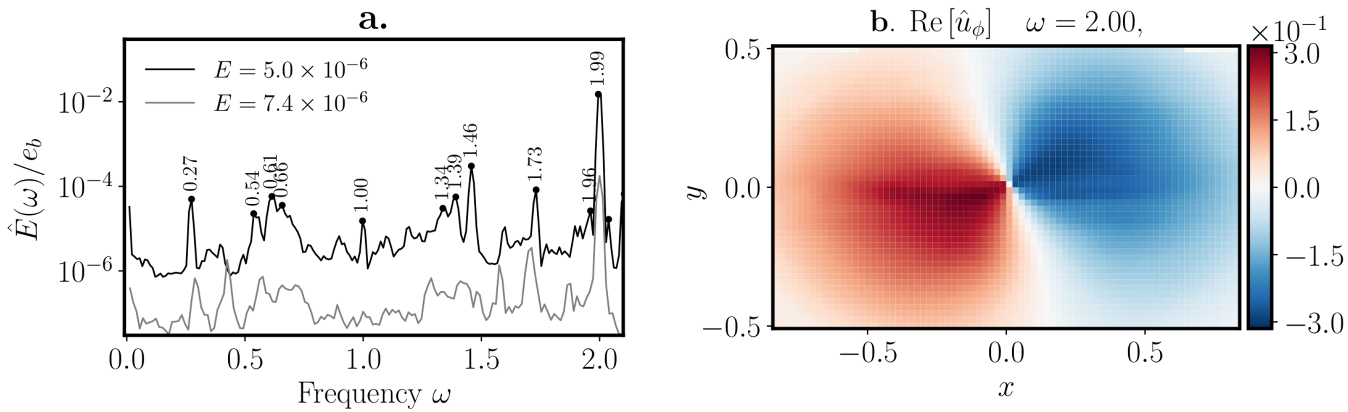
Low forcing amplitudes

Temporal power spectra



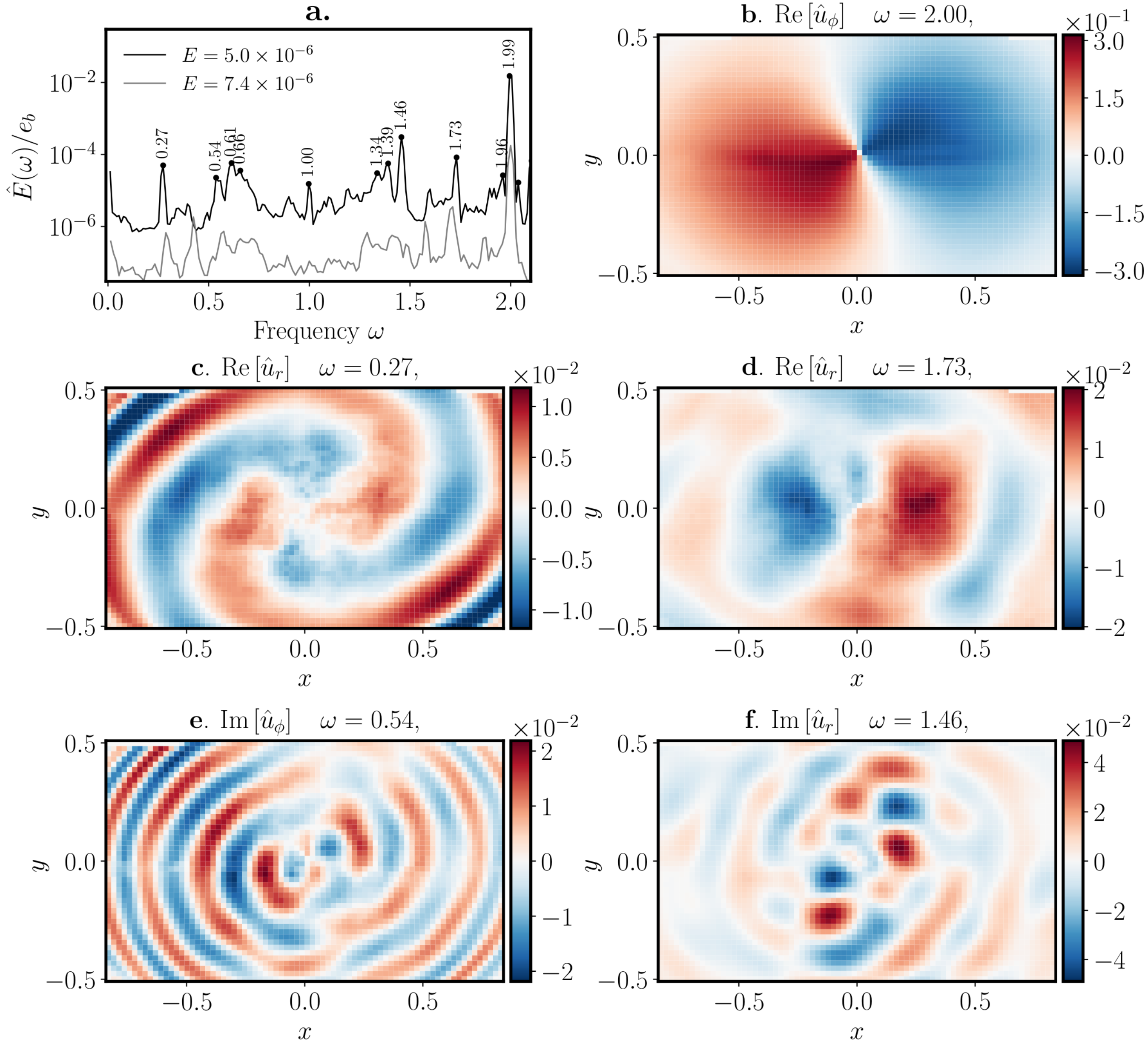
$$\omega_1 + \omega_2 = \omega_{\text{res}} \approx 2$$

Low forcing amplitudes



$m = \pm 1$
$(\Delta m = 2)$
$\omega = \pm 2$
$(\Delta \omega = 4)$

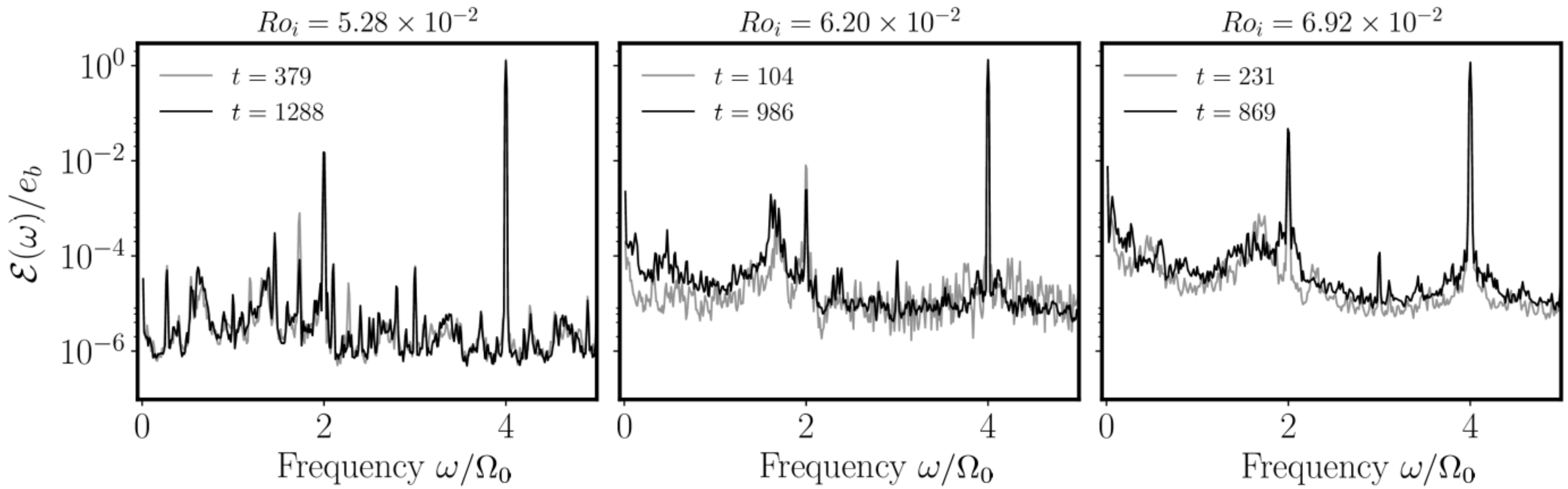
Low forcing amplitudes



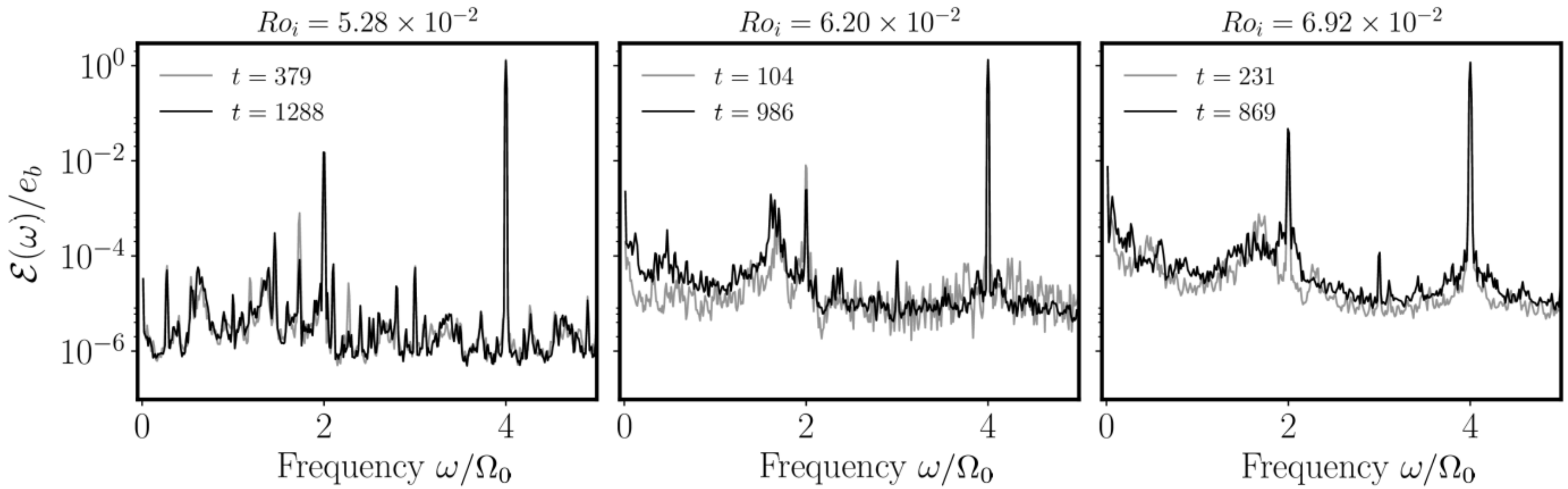
$m = \pm 1$
$(\Delta m = 2)$
$\omega = \pm 2$
$(\Delta \omega = 4)$

$\Delta m = 1$
$\Delta \omega = 2$

Increasing the forcing amplitude



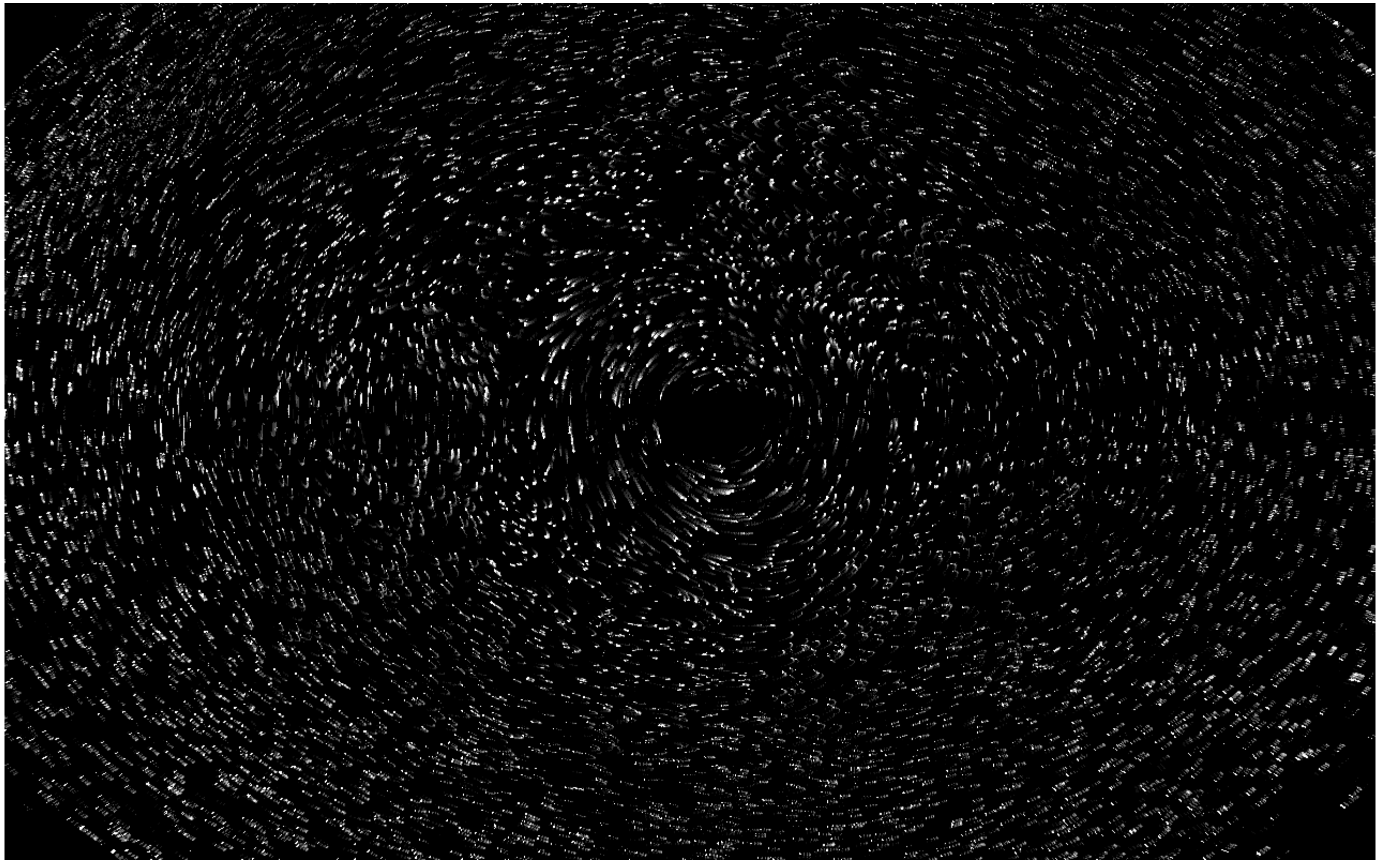
Increasing the forcing amplitude



- Disappearance of the sharp peaks
- Rise of the background level
- Increase of the mean flow amplitude

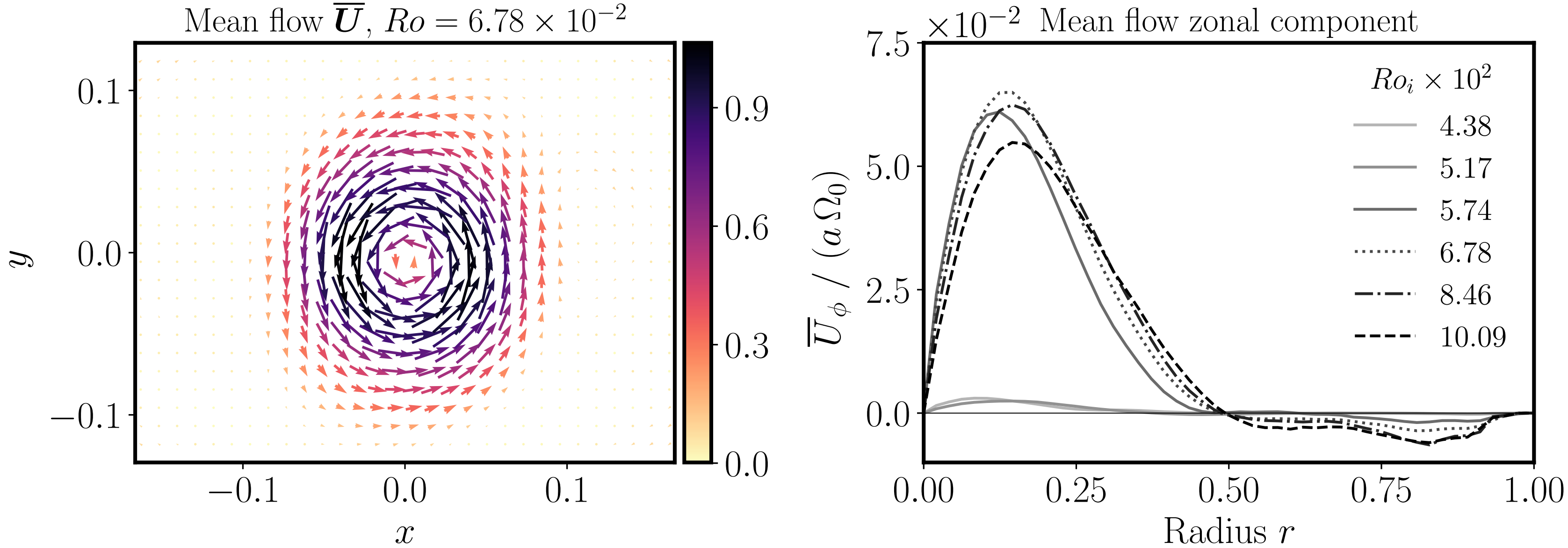
Rise of the geostrophic flow

Equatorial flow in the quasi-steady state

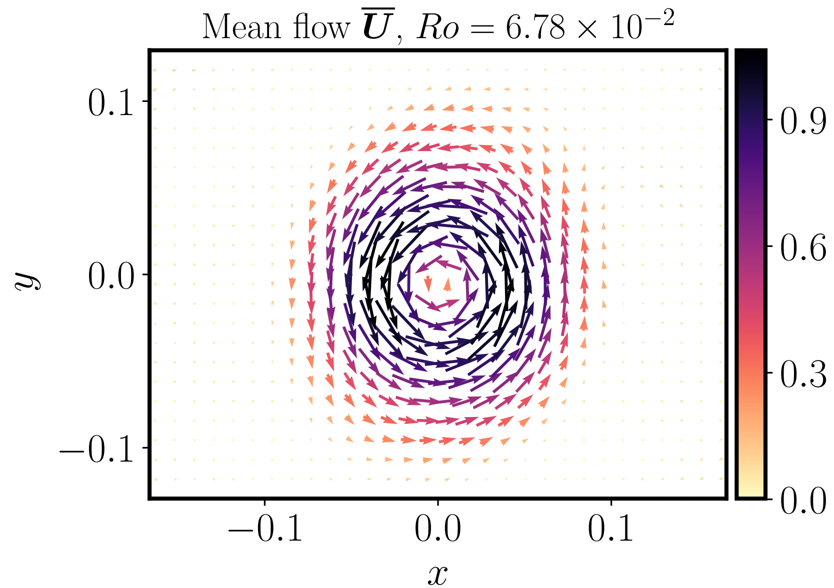


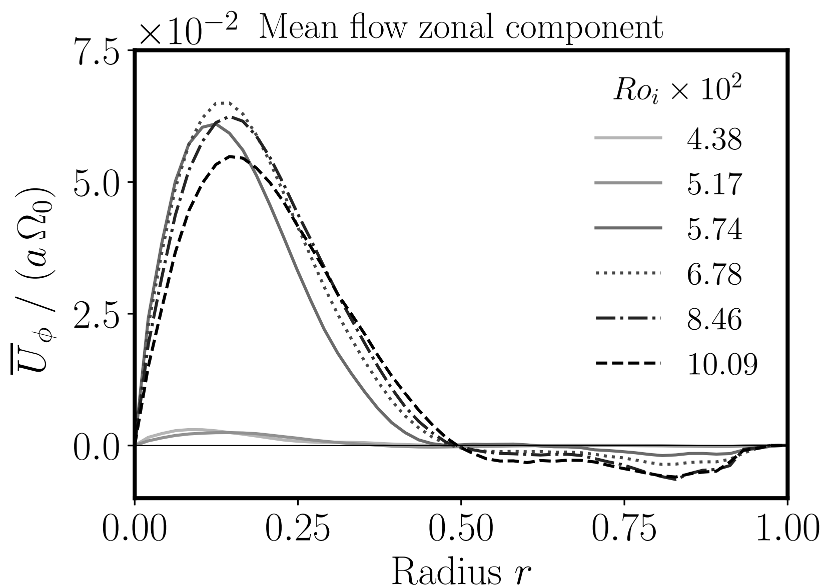
$$E = 5 \times 10^{-6} \quad \text{and} \quad Ro = 7 \times 10^{-2} \quad (Ro_c = 2 \times 10^{-2})$$

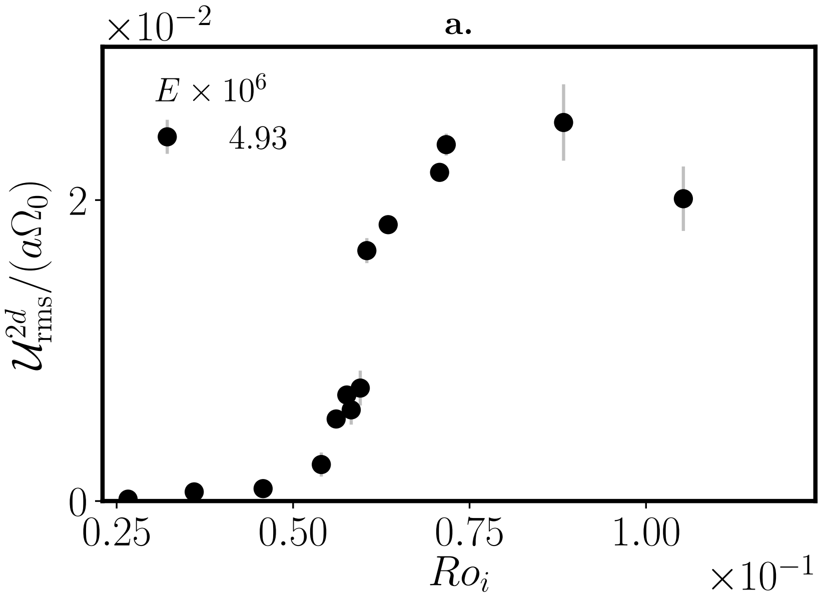
Rise of the geostrophic flow

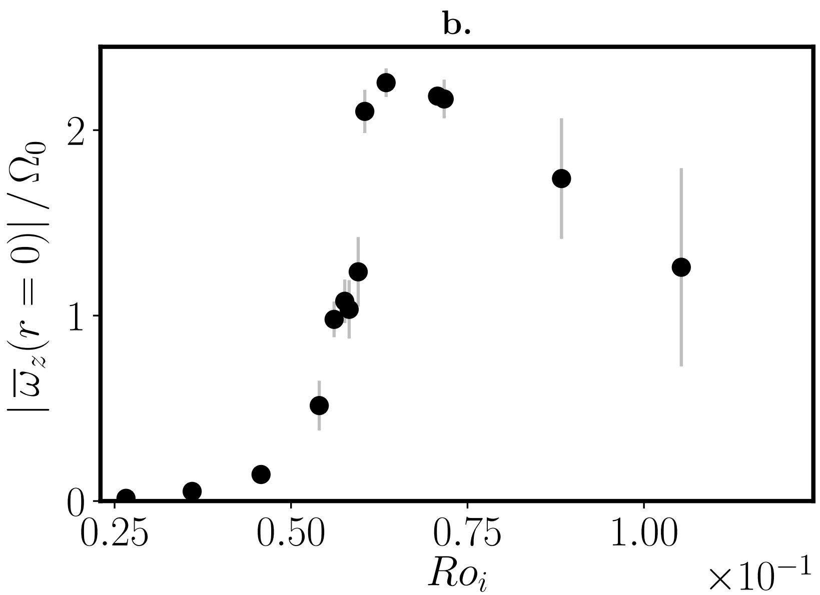


Rise of the geostrophic flow

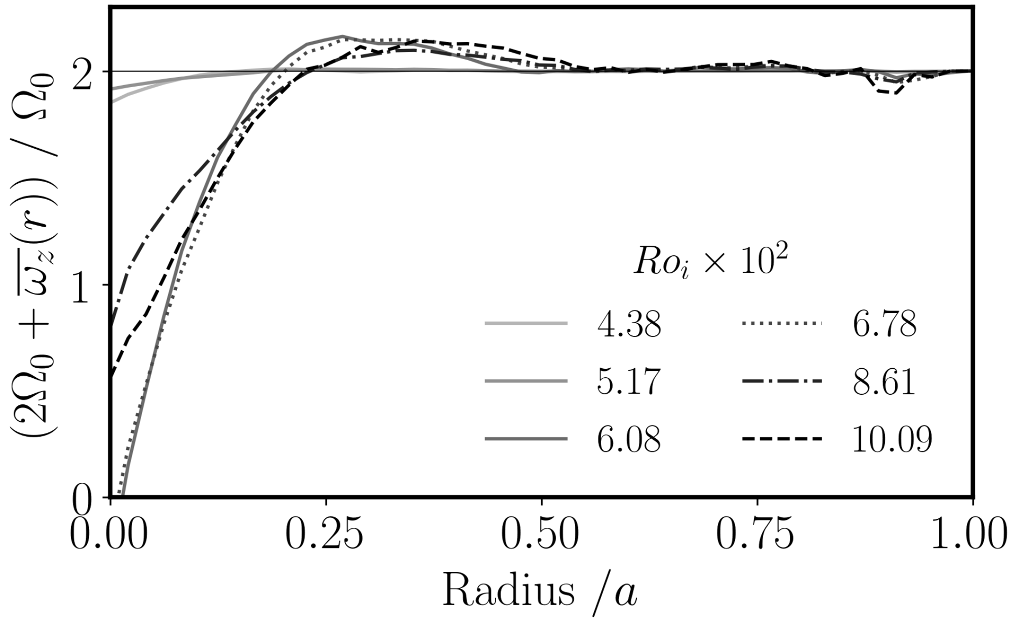




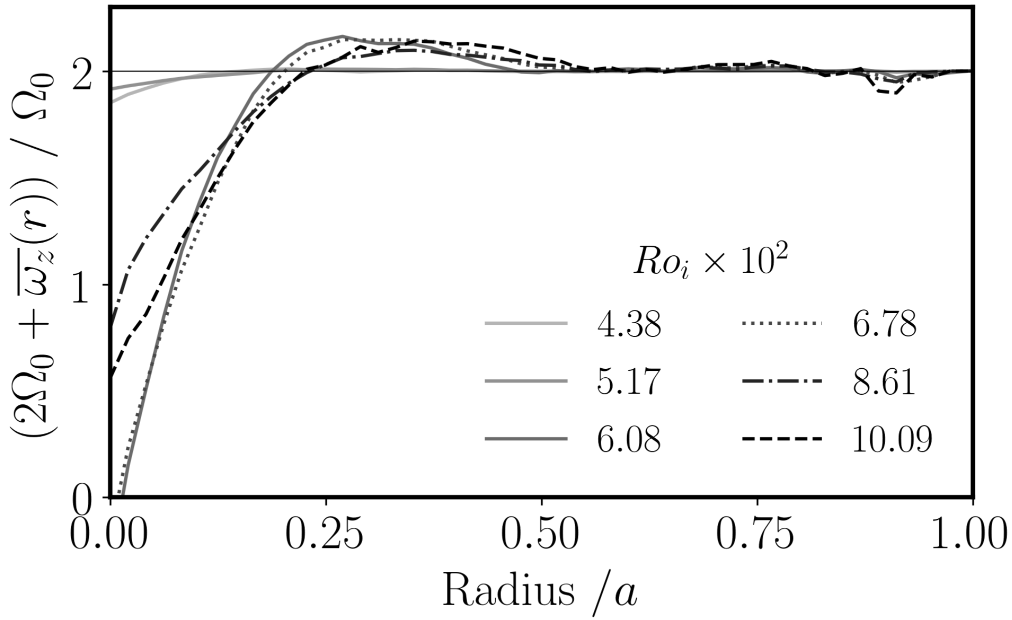


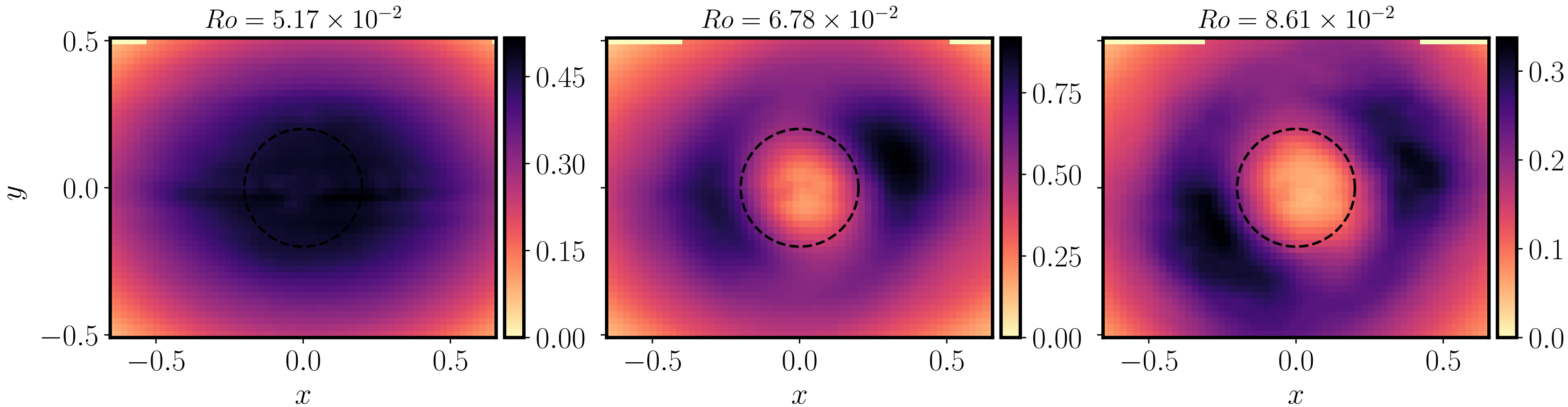


Feedback on the resonant waves

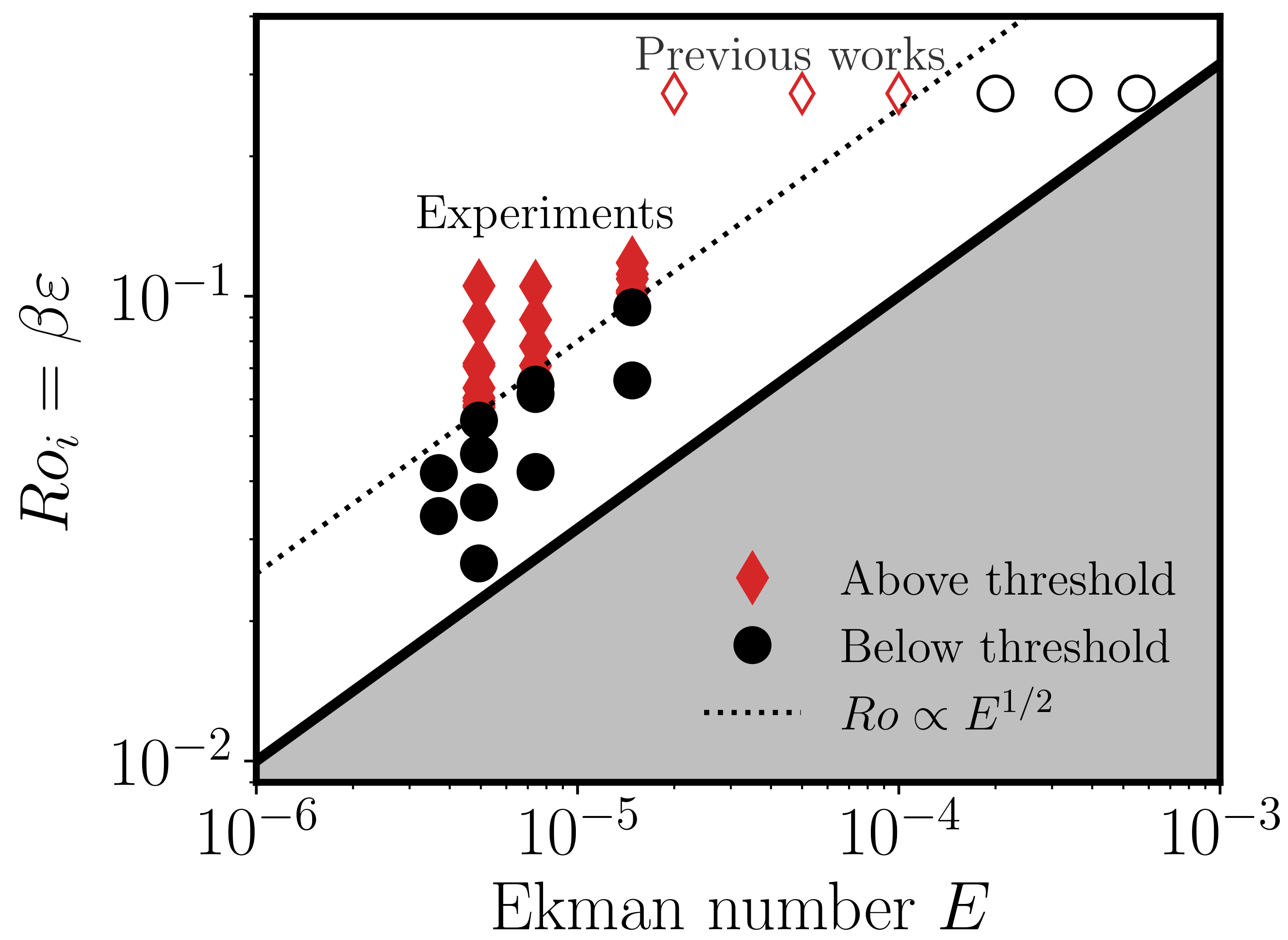


Feedback on the resonant waves





Summary of all experiments



Outline

Introduction

Base flows, instabilities and parameter regimes

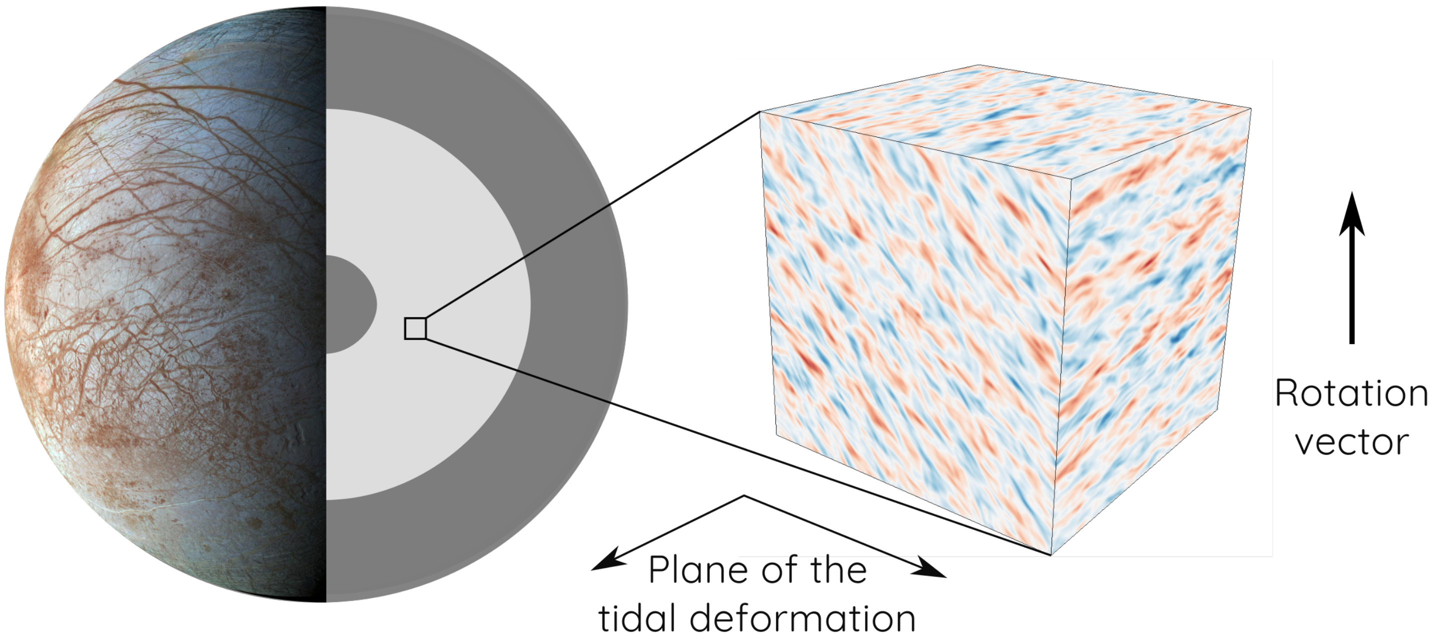
Towards the asymptotic regime in the lab?

Towards asymptotic regimes: the local approach

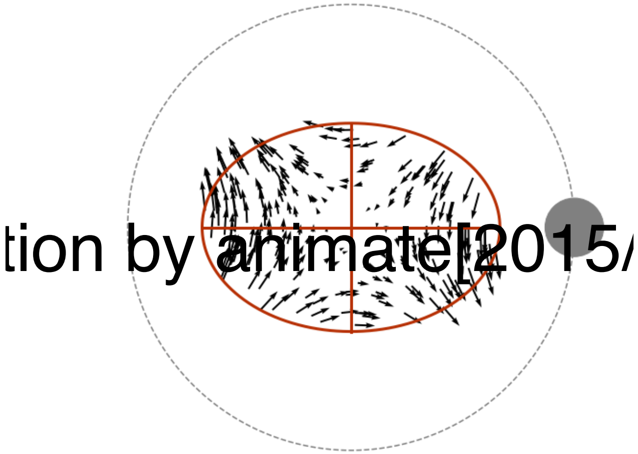
- Rotating case
- Stratified case

Conclusions and Perspectives

Decreasing forcing and dissipation: local approach



Tidal base flow



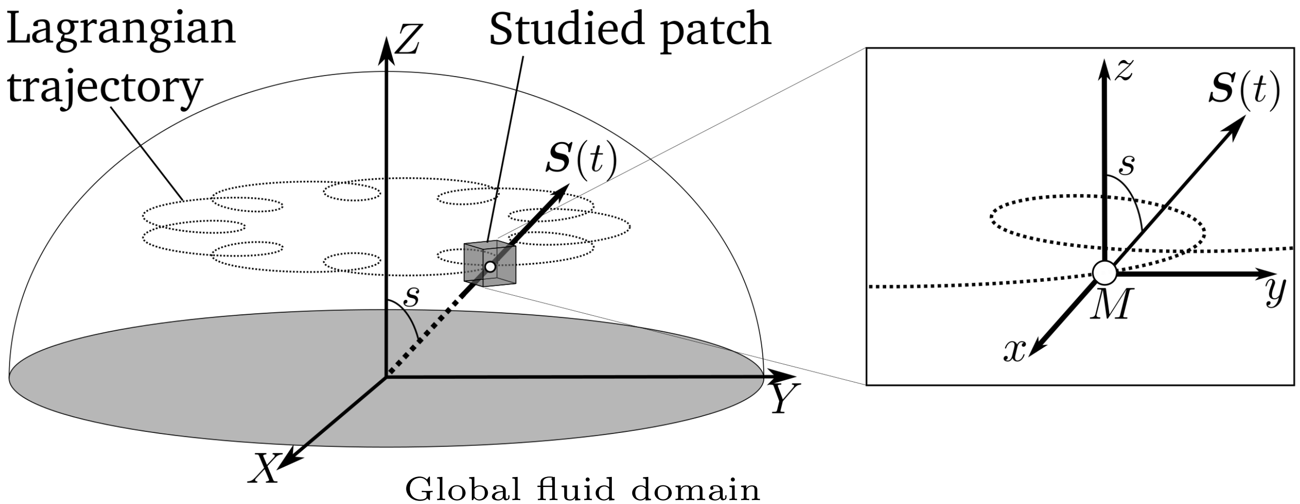
$\beta = (a^2 - b^2)/(a^2 + b^2)$ is the ellipticity
$\gamma = \Omega - n$ is the relative rotation rate

Solution of the Navier-Stokes eqs in the frame rotating at Ω:

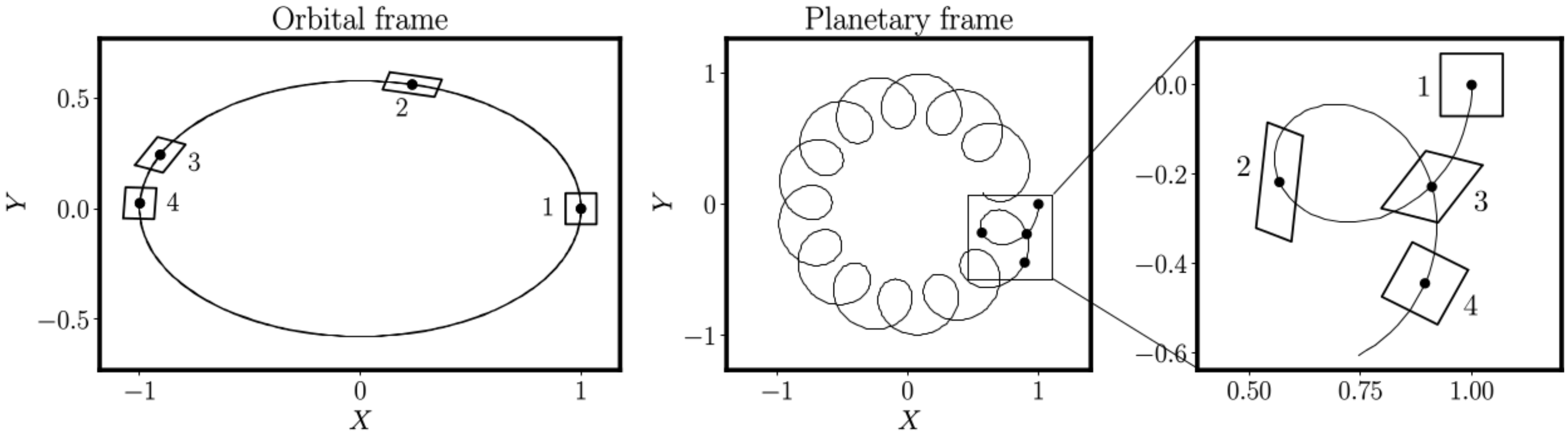
$$\boldsymbol{U}_B = -\gamma\beta \begin{pmatrix} \sin 2\gamma t & \cos 2\gamma t & 0 \\ \cos 2\gamma t & -\sin 2\gamma t & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{A}(t)\boldsymbol{X}$$

Sridhar & Tremaine 1992, Kerswell 2002, Barker 2016

Local model of tidally-driven elliptic instabilities

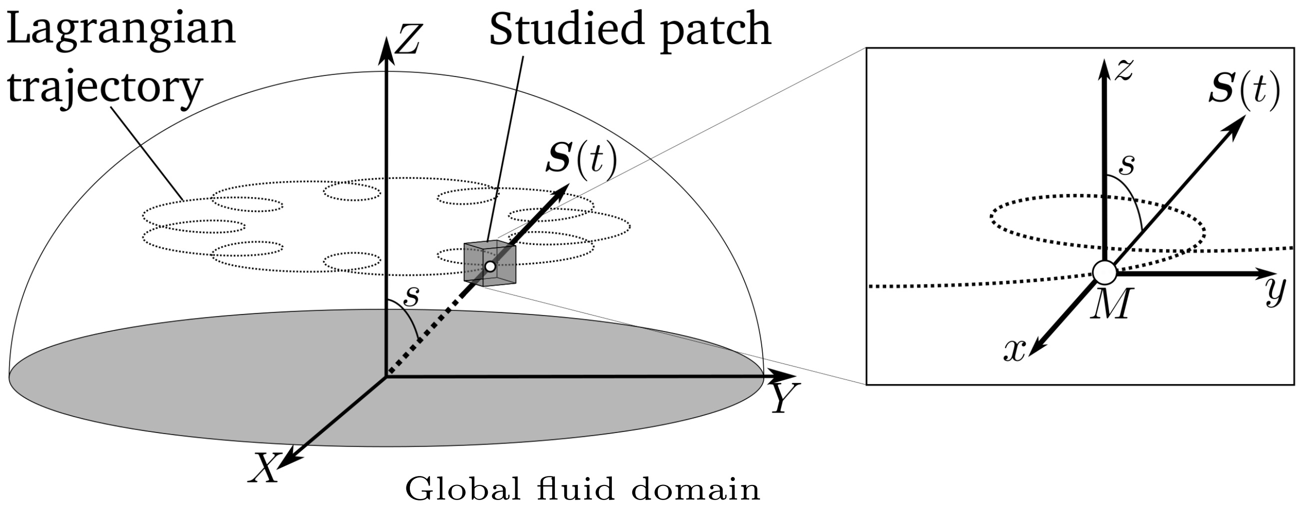


Global fluid domain



Lagrangian trajectory of a small fluid patch

Local model of tidally-driven elliptic instabilities



Perturbation equations in the Lagrangian frame moving with the base flow

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{A}(t)\boldsymbol{x} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{A}(t)\boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + 2\boldsymbol{e}_z \times \boldsymbol{u} &= -\boldsymbol{\nabla} \Pi + E \boldsymbol{\nabla}^2 \boldsymbol{u} \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 \end{aligned}$$

Local model of tidally-driven elliptic instability

- Shearing box approach in a tri-periodic domain (Kelvin mode decomposition)

$$\{\boldsymbol{u}, \Pi\} \;=\; \sum_{\boldsymbol{k}} \left\{ \hat{\boldsymbol{u}}_{\boldsymbol{k}}(t), \hat{\Pi}_{\boldsymbol{k}}(t) \right\} \; e^{i\boldsymbol{k}(t)\cdot\boldsymbol{x}}$$

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- We can now reach $\beta \approx 10^{-3}$ and $E \approx 10^{-7}$
  ($\beta \approx 0.1$ and $E \approx 10^{-4}$ in global models!)

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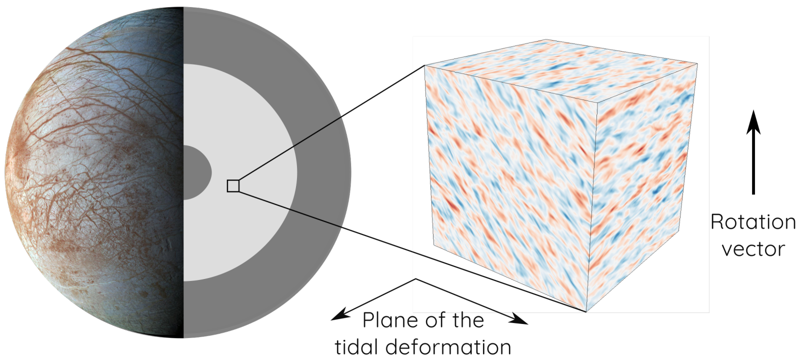
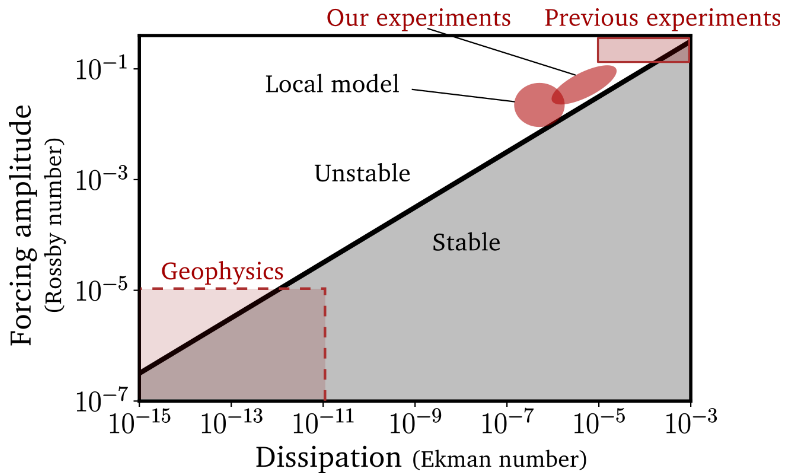




Outline

Introduction

Base flows, instabilities and parameter regimes

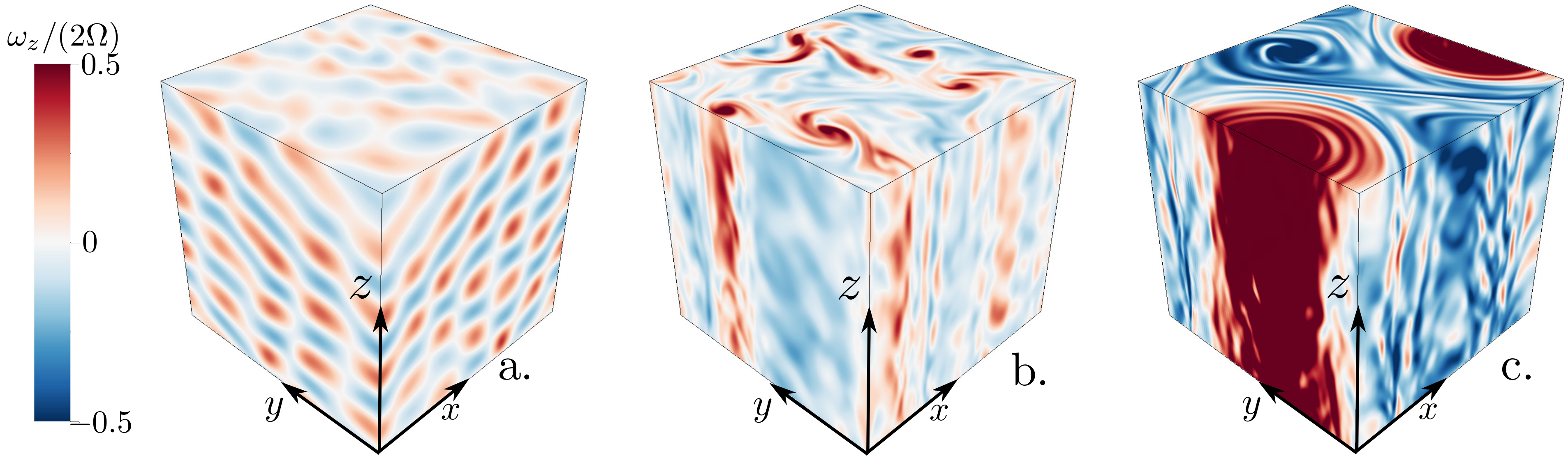
Towards the asymptotic regime in the lab?

Towards asymptotic regimes: the local approach

- Rotating case
- Stratified case

Conclusions and Perspectives

A typical simulation ($\beta = 0.05$, $E = 10^{-6}$)



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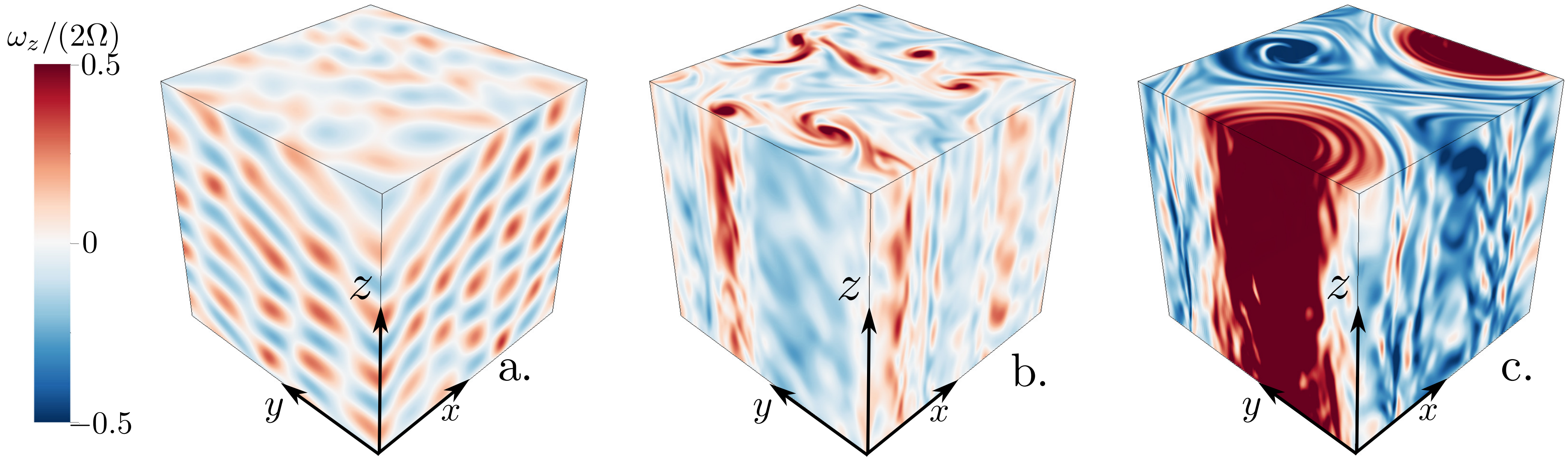
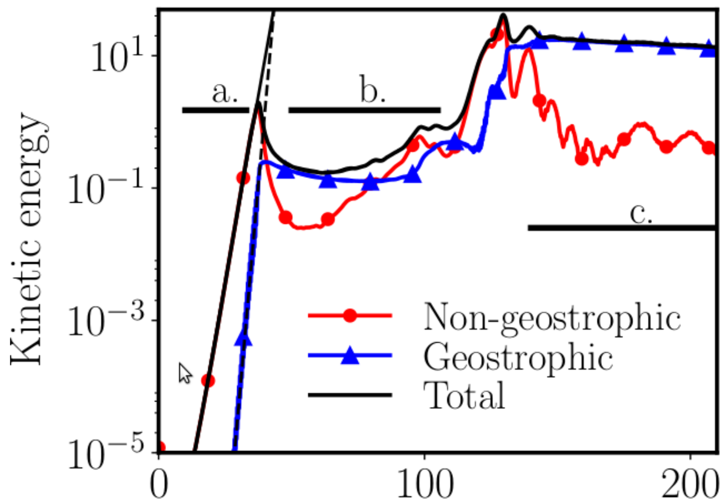
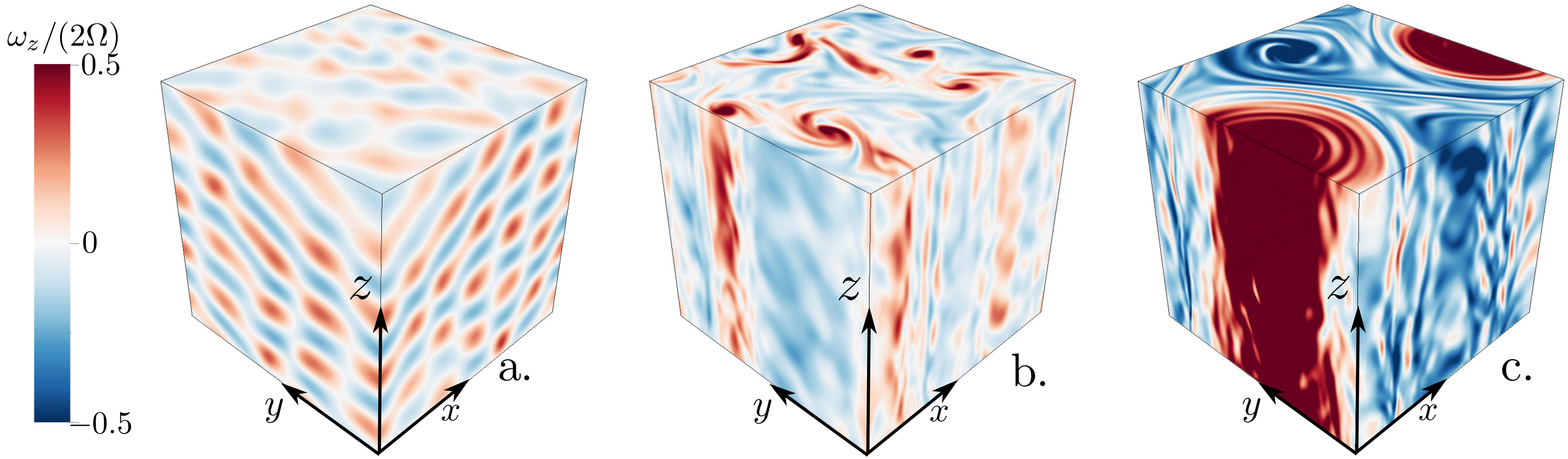




Geostrophic $\rightarrow k_z = 0$

Non-geostrophic $\rightarrow k_z \neq 0$

A typical simulation ($\beta = 0.05$, $E = 10^{-6}$)

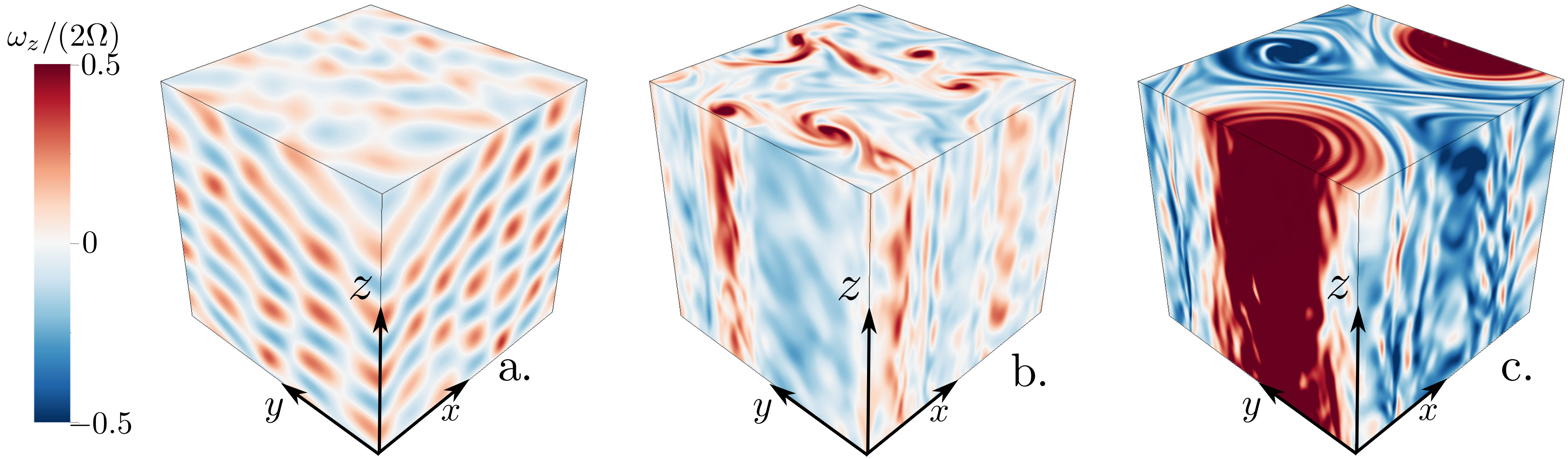


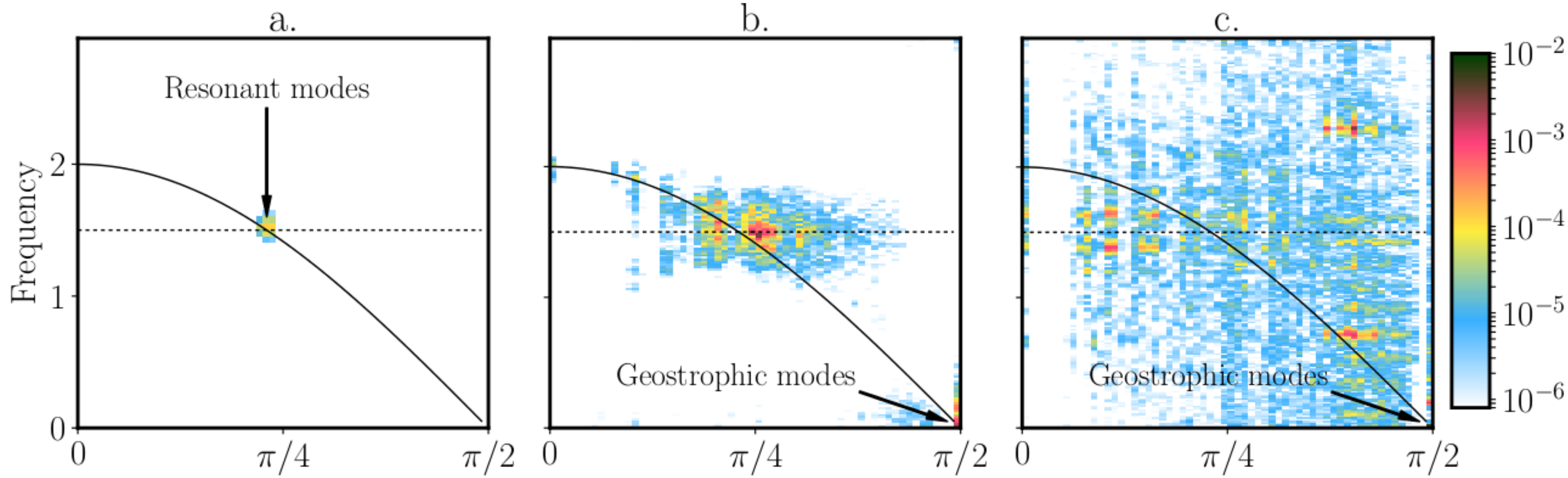
Spatio-temporal decomposition

$$\boldsymbol{u}(\boldsymbol{x}, t) \quad \rightarrow \quad \hat{\boldsymbol{u}}(\boldsymbol{k}, \omega) \quad \rightarrow \quad E(\theta, \omega)$$

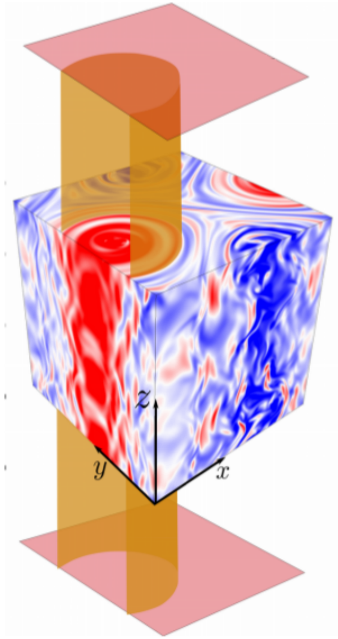
Dispersion relation of inertial waves: $\omega = 2\Omega\cos\theta$

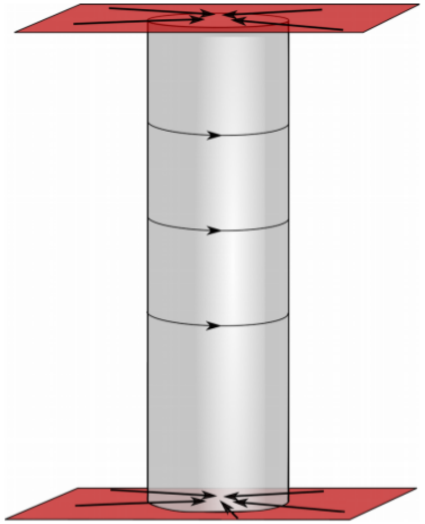
A typical simulation ($\beta = 0.05$, $E = 10^{-6}$)





What are we missing?





$\boxed{\text{Ekman friction term}}$

$$-f_r E^{1/2} \boldsymbol{u}_G$$

$-f_r E^{1/2} \hat{\boldsymbol{u}}(k_z = 0)$ in spectral space

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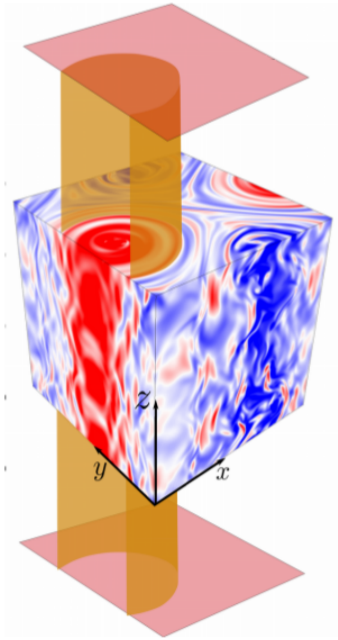
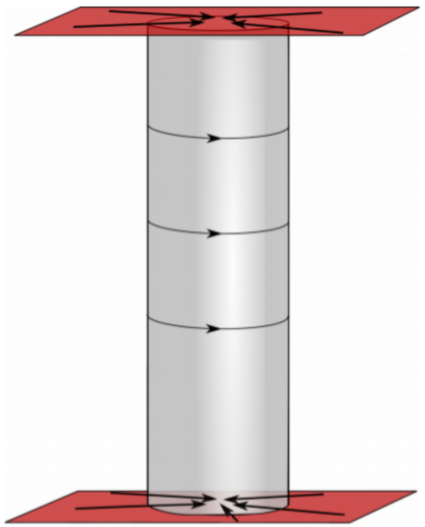




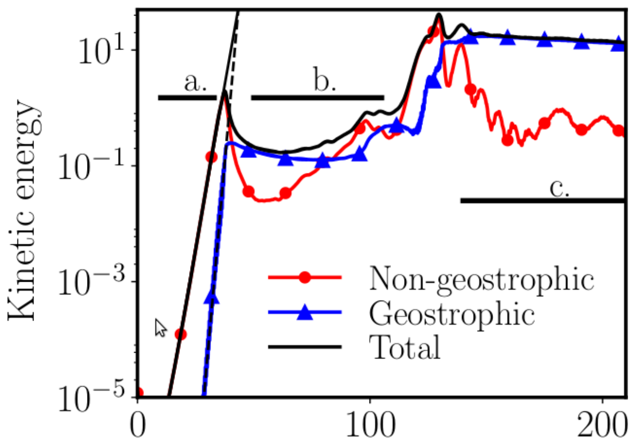
$\boxed{\text{Ekman friction term}}$

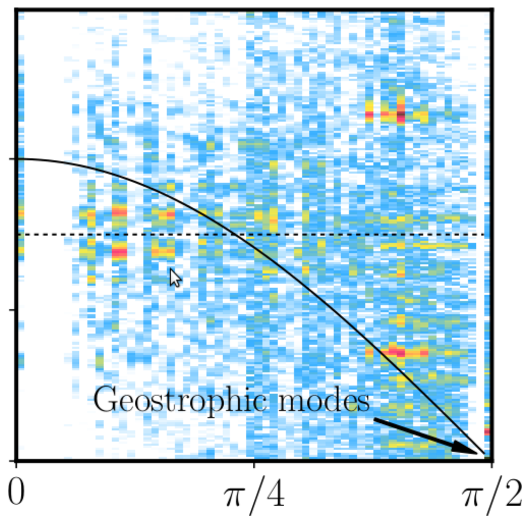
$$-f_r E^{1/2} \boldsymbol{u}_G$$

$-f_r E^{1/2} \hat{\boldsymbol{u}}(k_z = 0)$ in spectral space

- Analogous to 2D, QG and WT in a channel (Scott 2014)
- Asymptotically, the geostrophic flow cannot be forced by exactly-resonant inertial modes (Greenspan 1969)
- Magnetic field damps geostrophic flows (Barker2013, Guervilly2015)

Simulations with frictional damping





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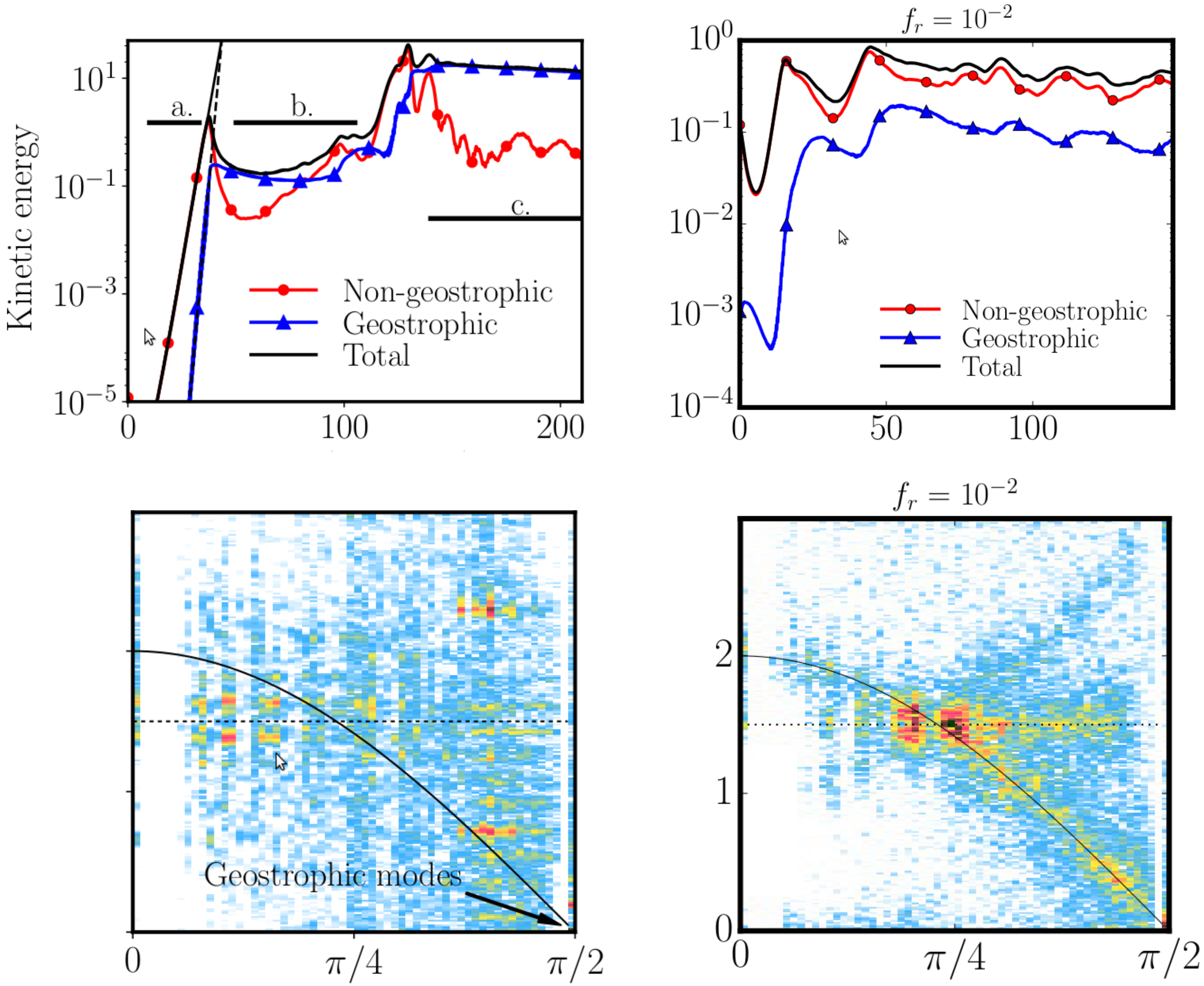

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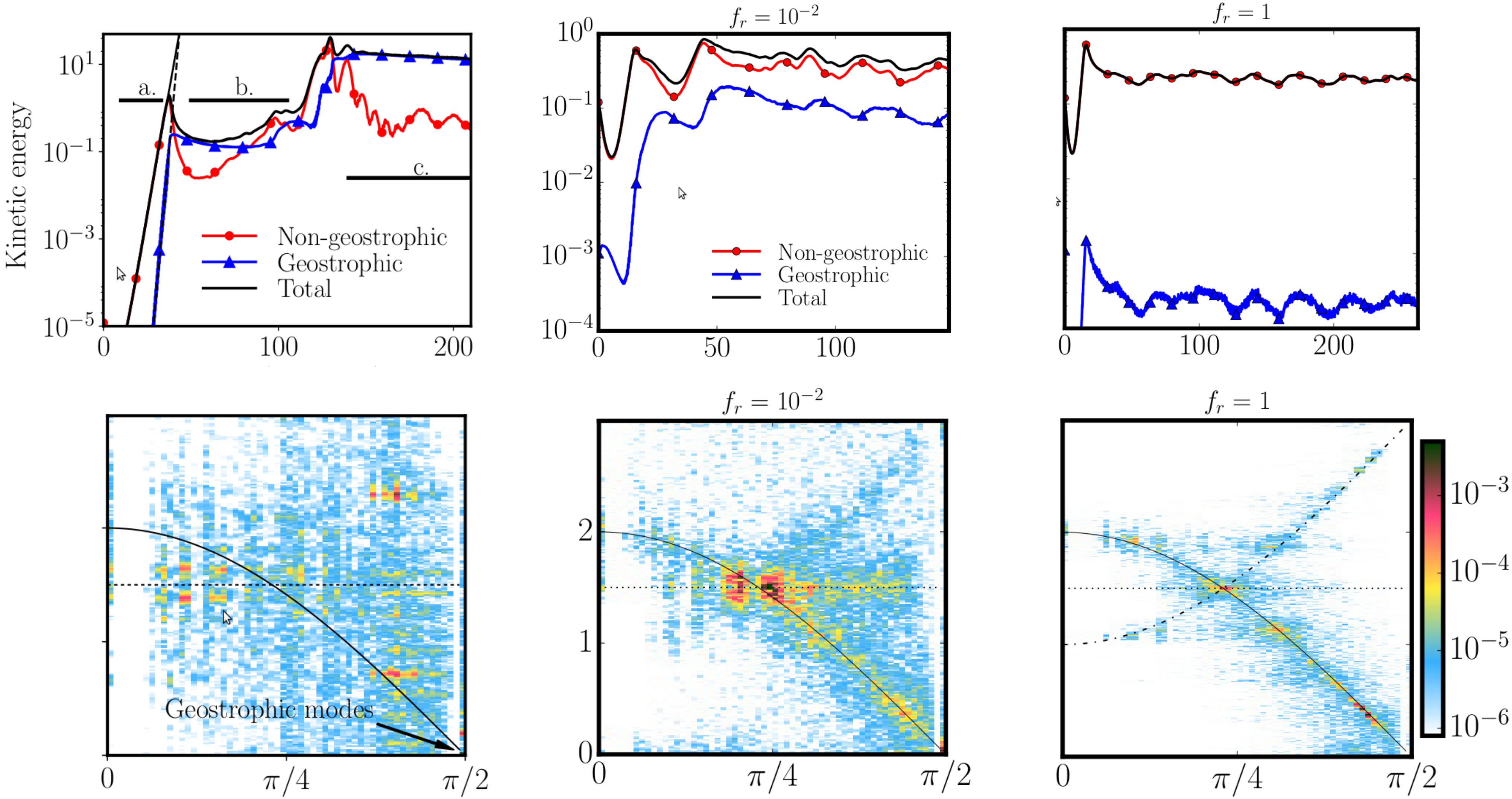
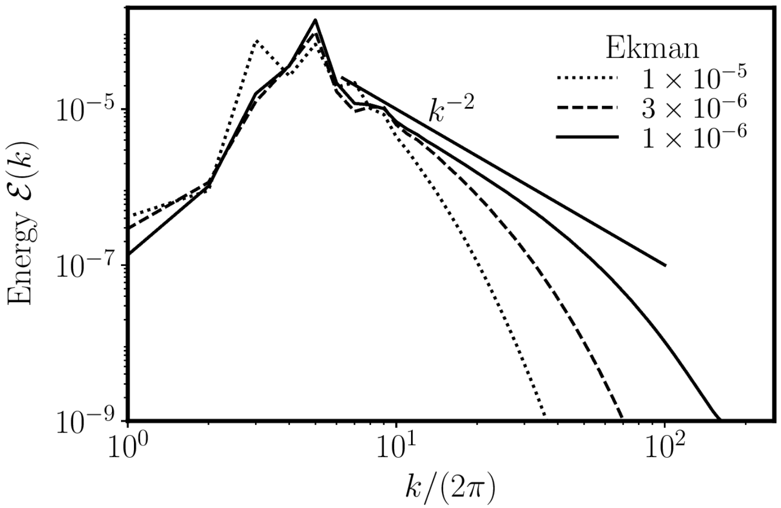
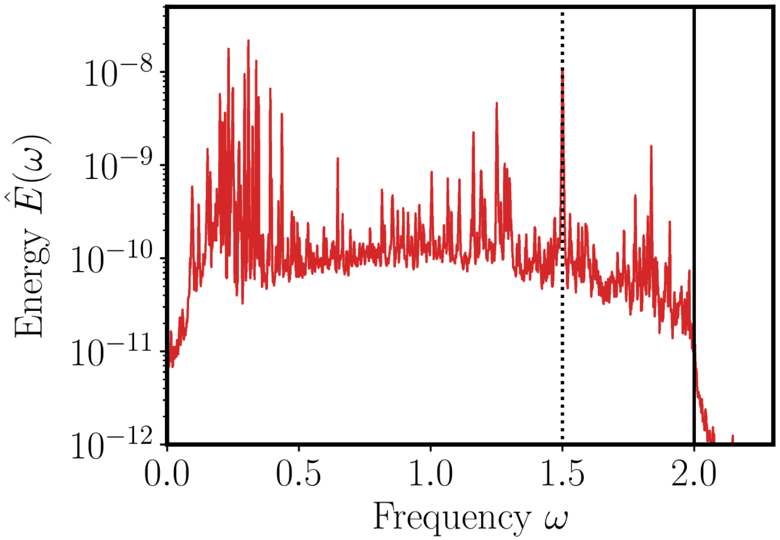


Spatial and temporal spectra for the most extreme case





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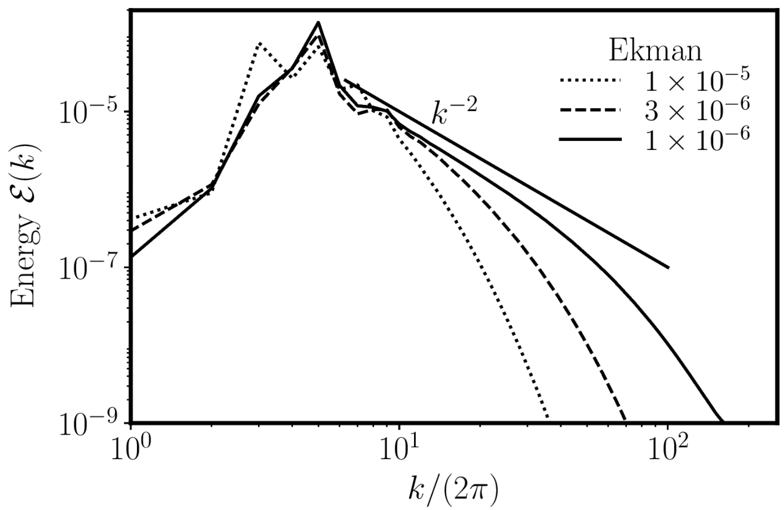
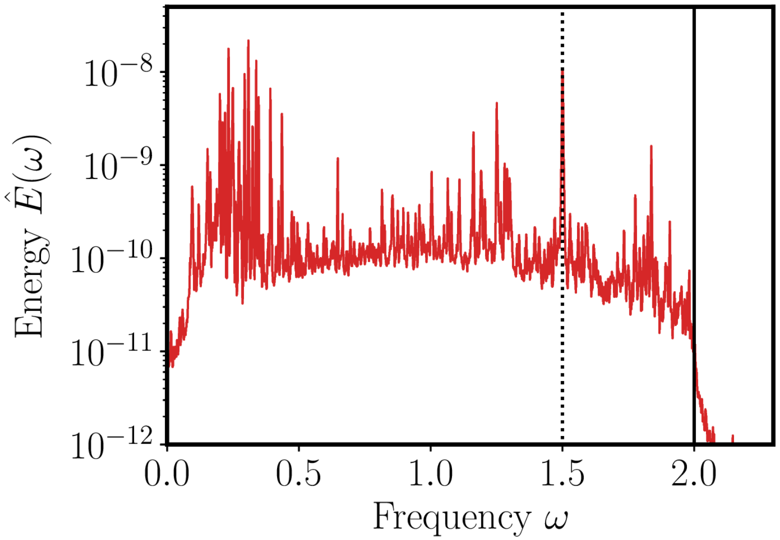
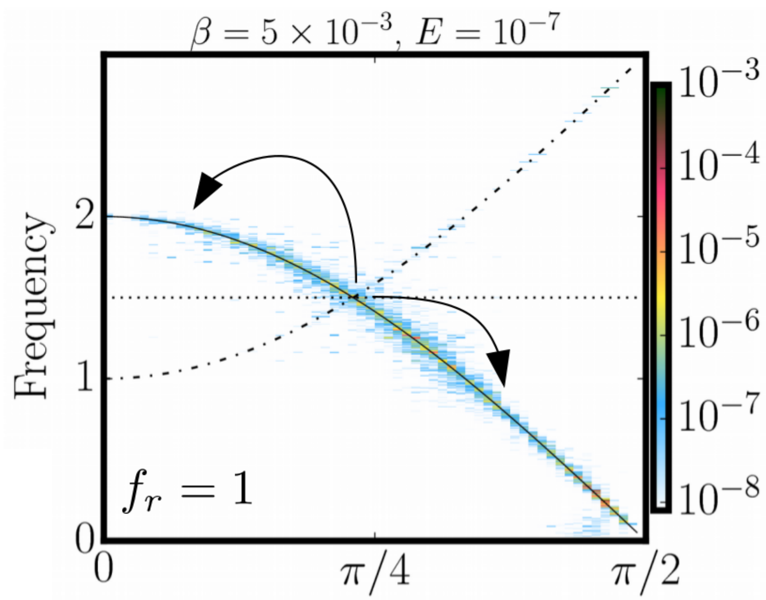
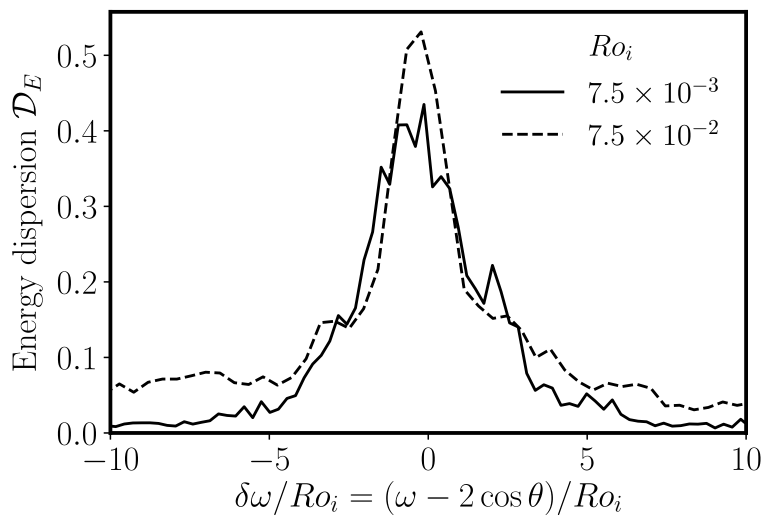
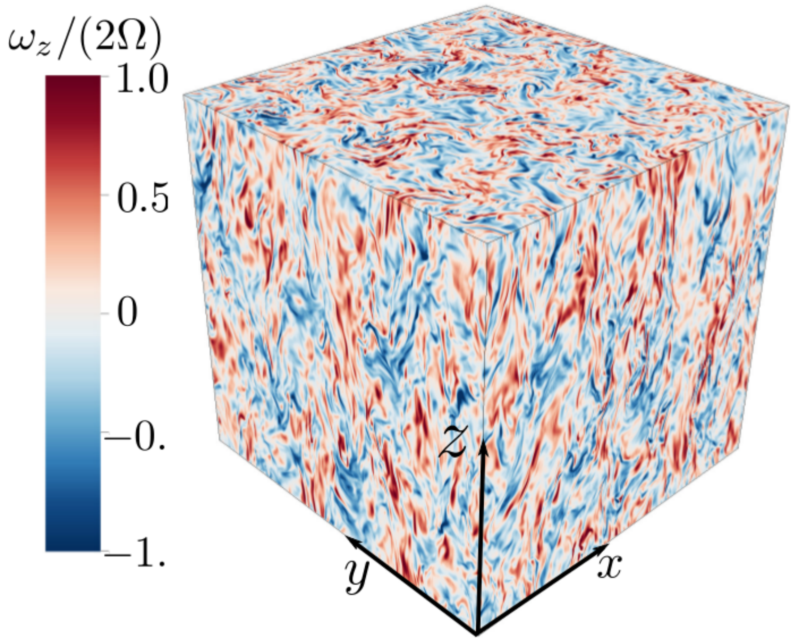


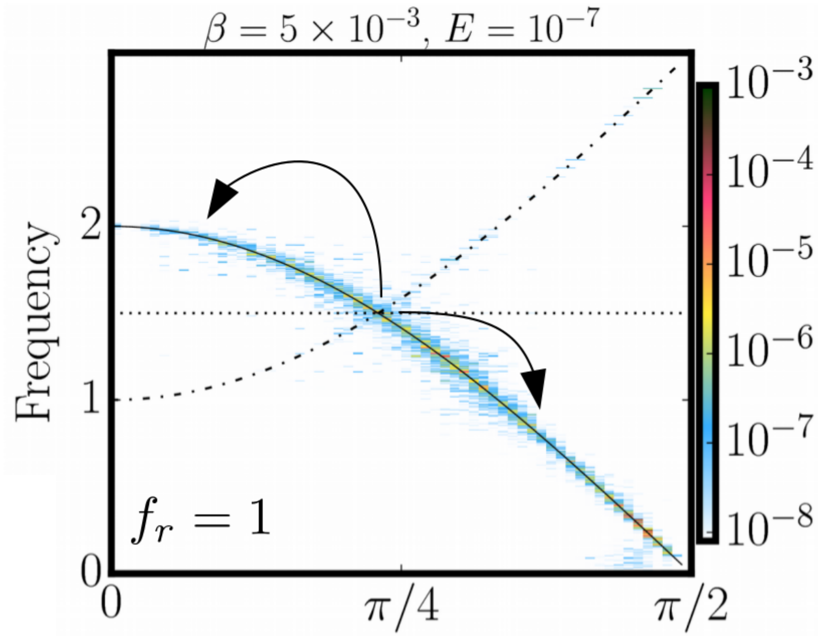






Towards inertial wave turbulence?





- Most of the energy is contained in inertial waves
- Weak nonlinear transfers from the resonant frequency only
- Small finite size effects, very low E

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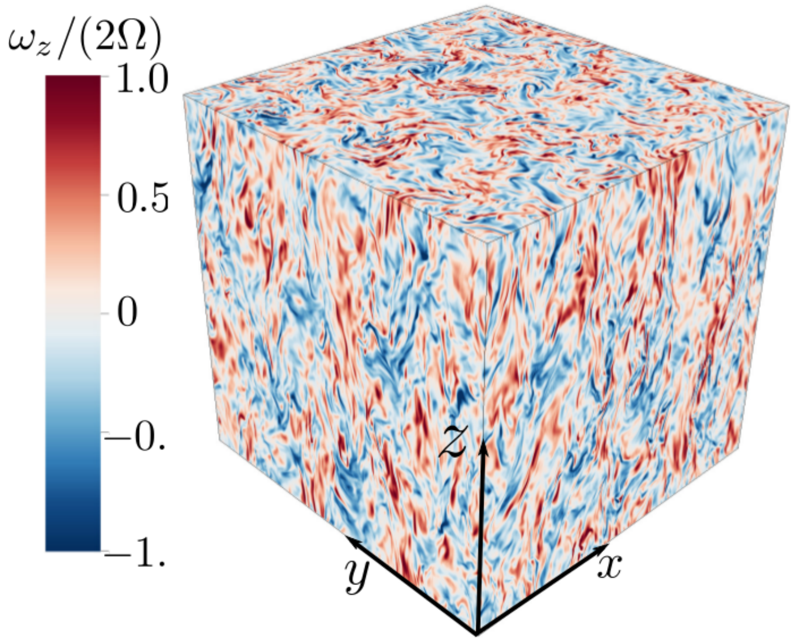

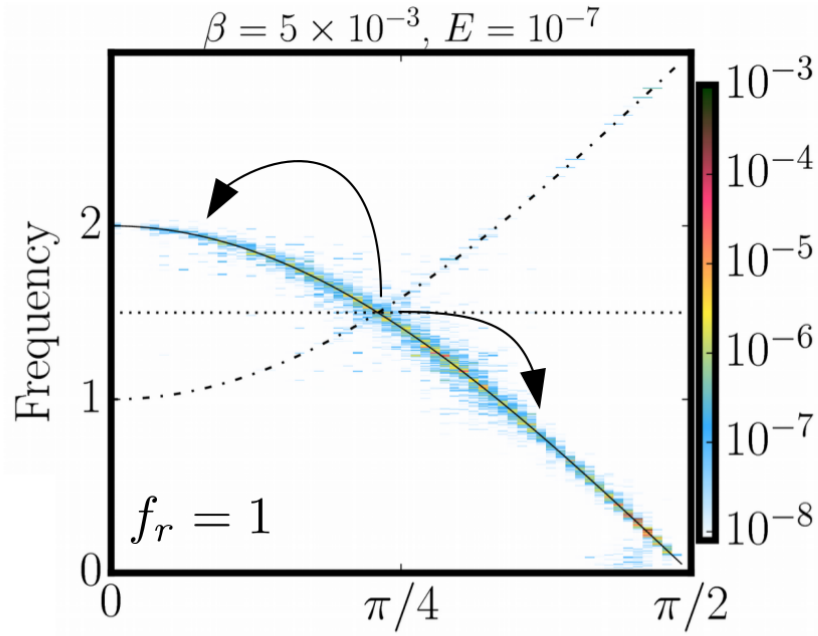

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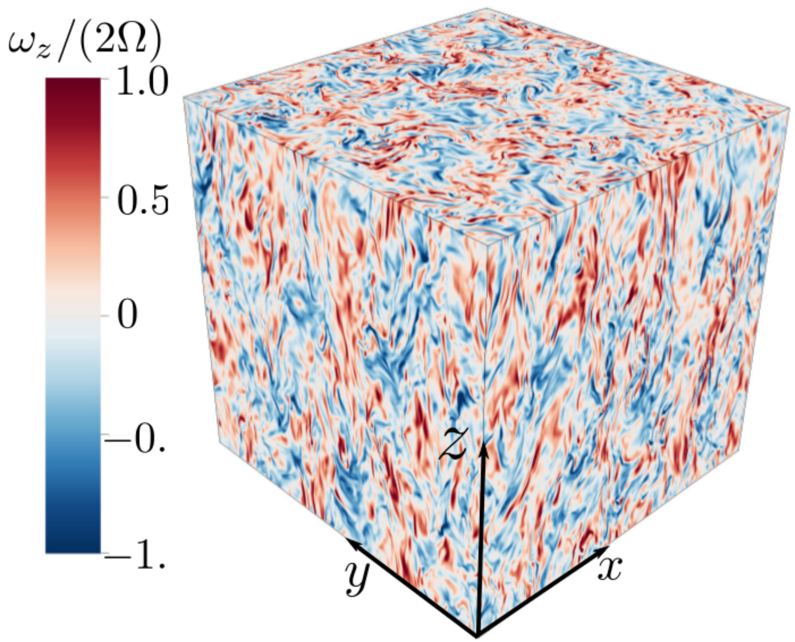

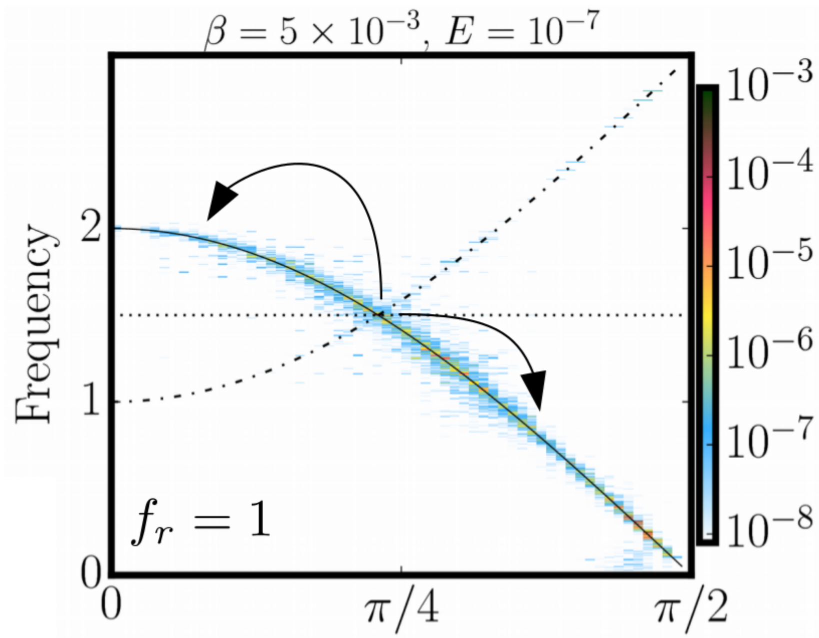

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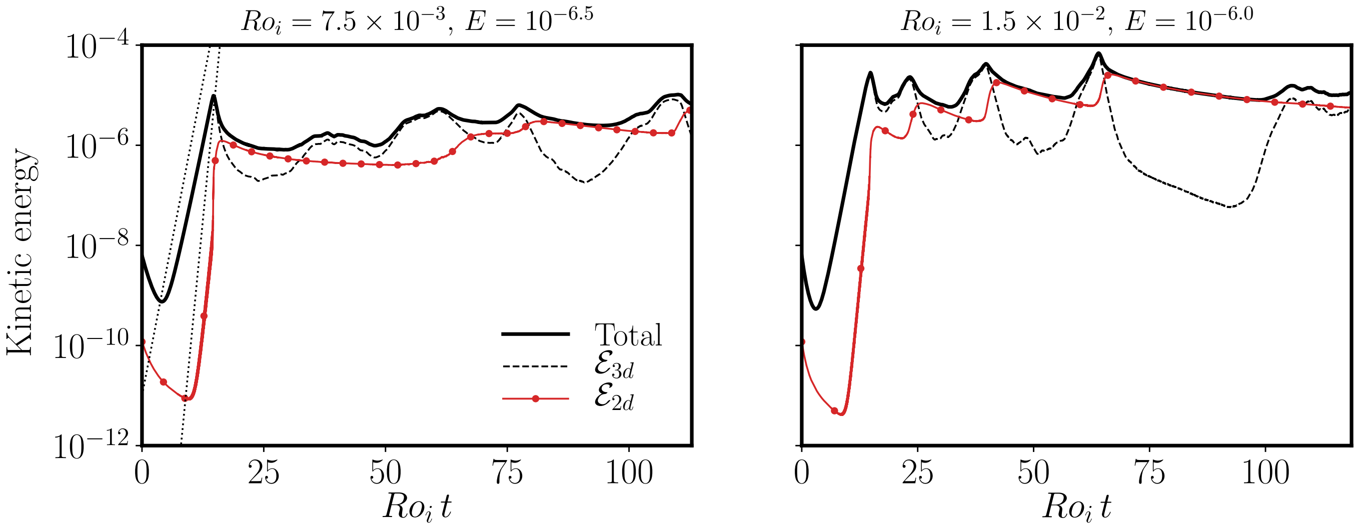
$\boxed{\Rightarrow \text{Weak inertial wave turbulence}}$

Key quantity: $|\boldsymbol{u}_G|/|\boldsymbol{u}_{3D}|$

Sub-dominant geostrophic mode even without friction...



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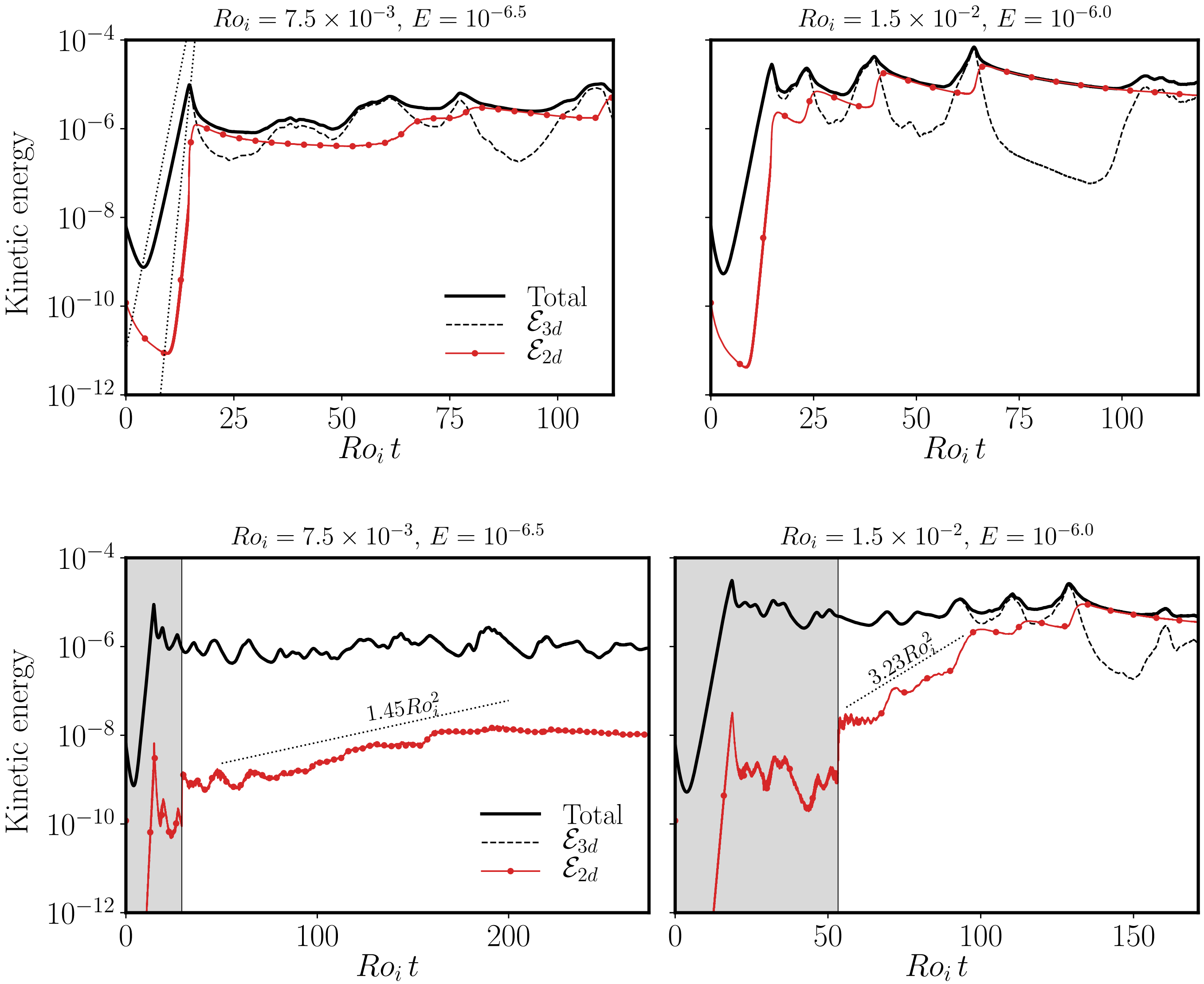

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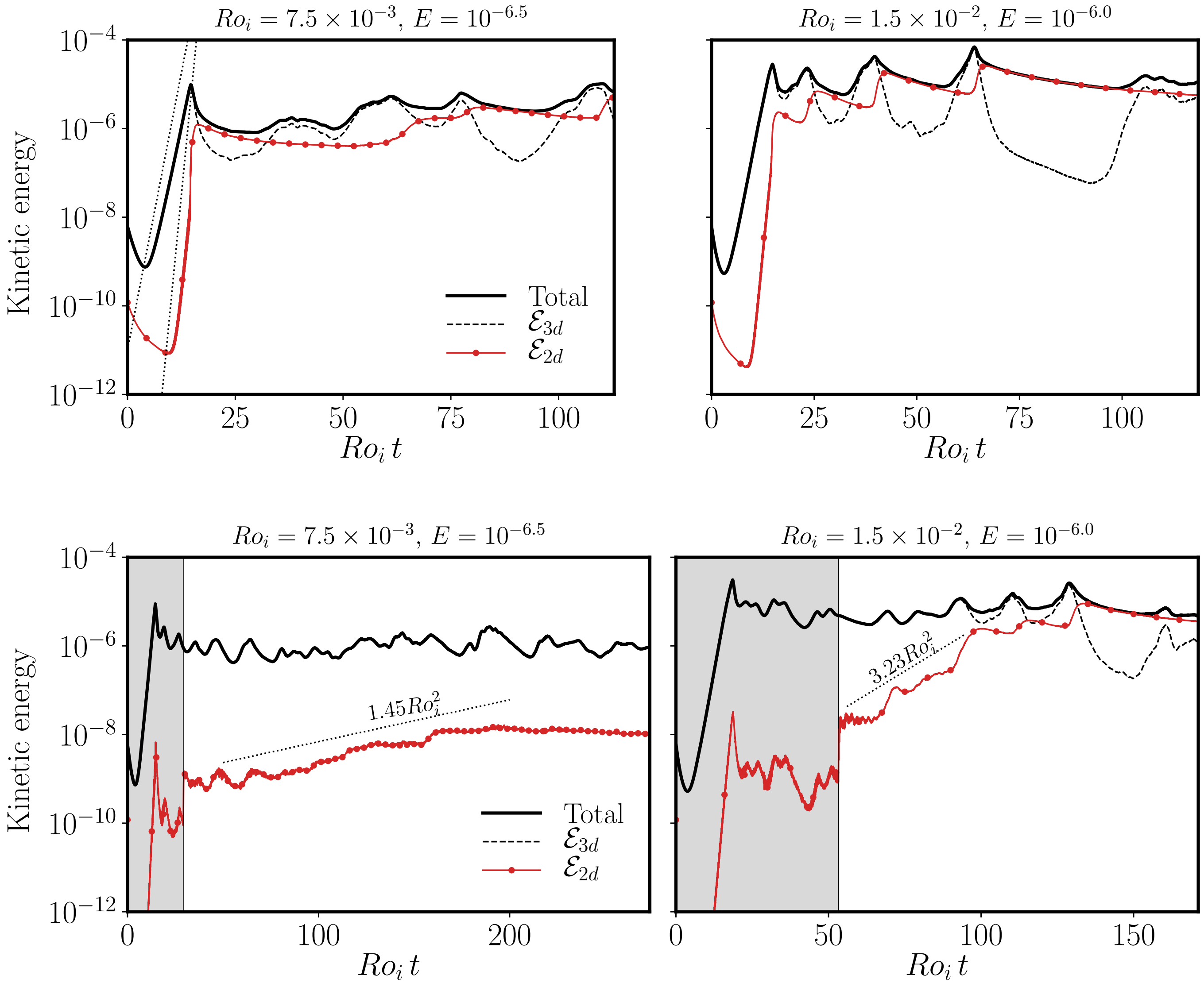


Bi-stability? Sub-criticality?

Outline

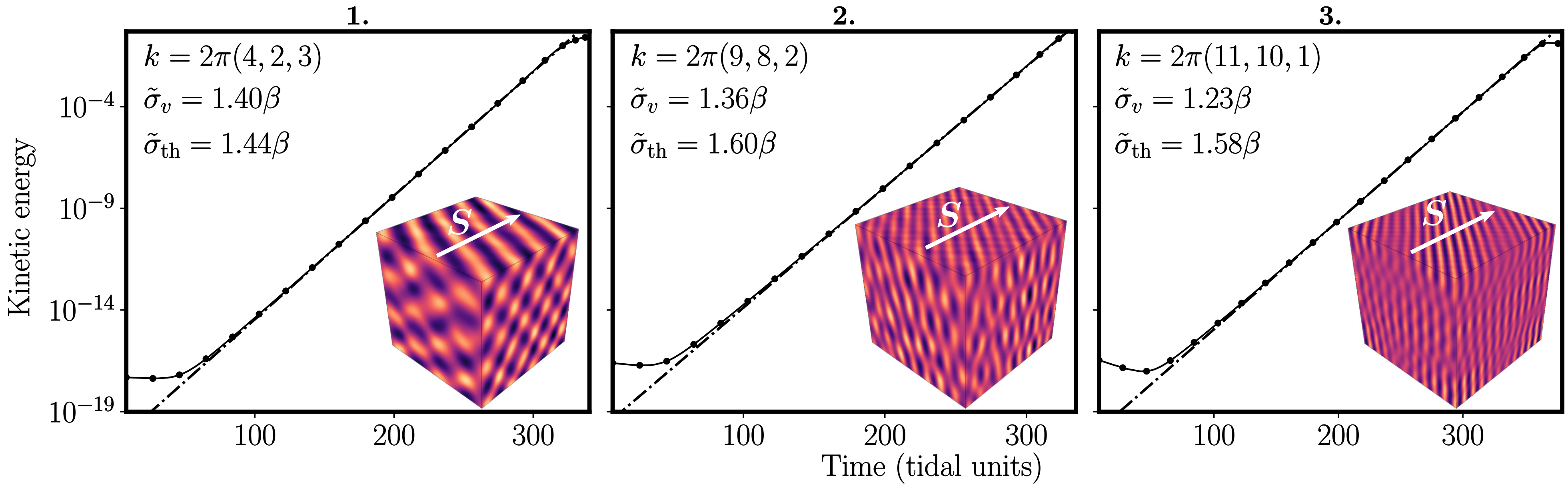
Generalization to stratified fluids

- Elliptic instabilities can be generalized to stratified fluids (Miyazaki&Fukumoto 1992)
- Similar to PSI in stratified tanks (McEwan 1975, Benielli&Sommeria 1996, Bourget 2013) and in the ocean (MacKinnon 2013)
- Resonance mechanism based on internal waves instead of inertial waves

$$\frac{\partial^2 \nabla^2 u_z}{\partial t^2} + N^2 \nabla_H^2 u_z = 0$$

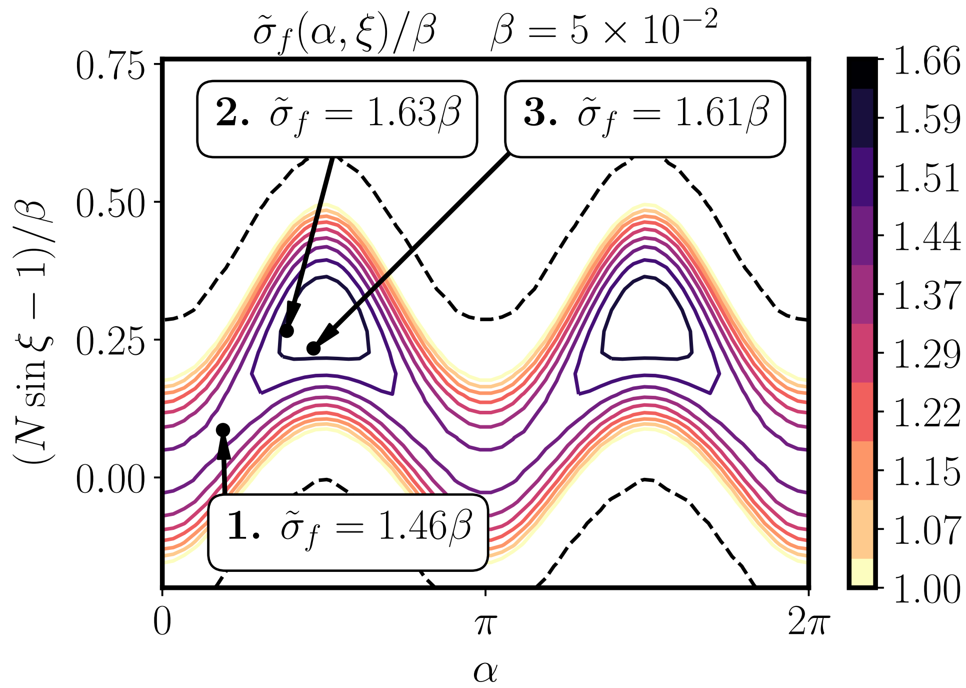
$$\omega = \pm N \sin\theta$$

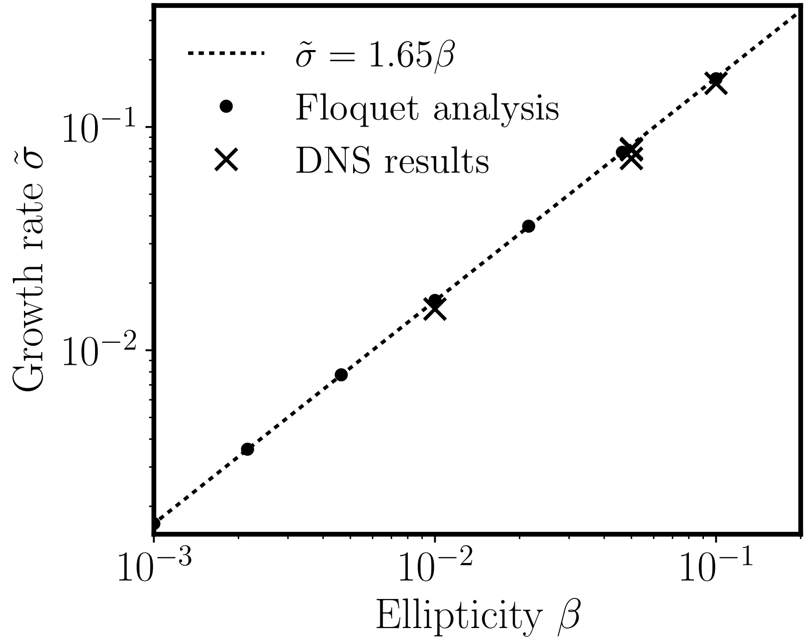
Comparison between DNS and Floquet analysis



WKB + Floquet analysis for the linear inviscid limit

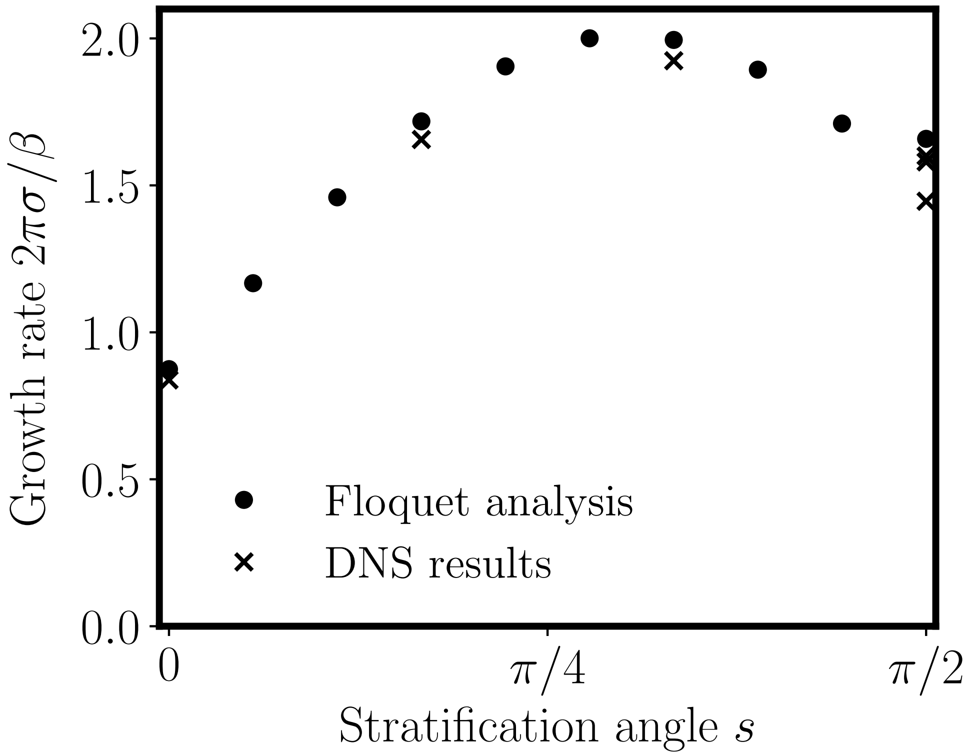
Comparison between DNS and Floquet analysis

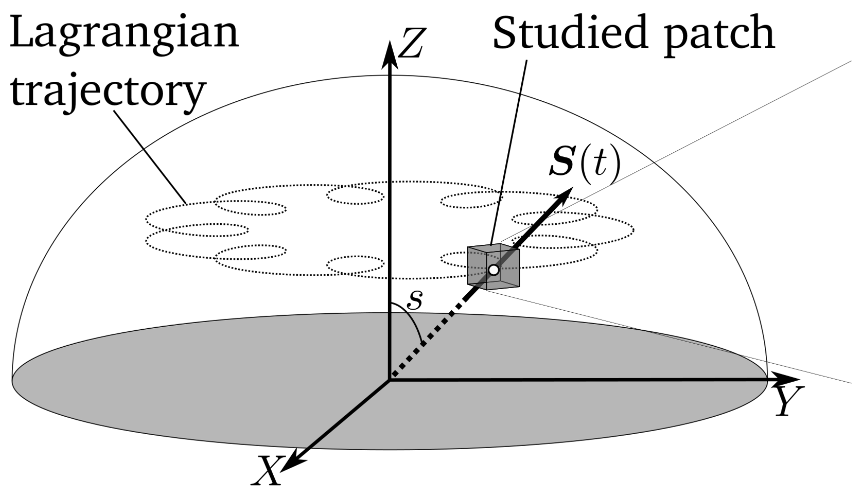




Growth rate scales linearly with the amplitude of the forcing β

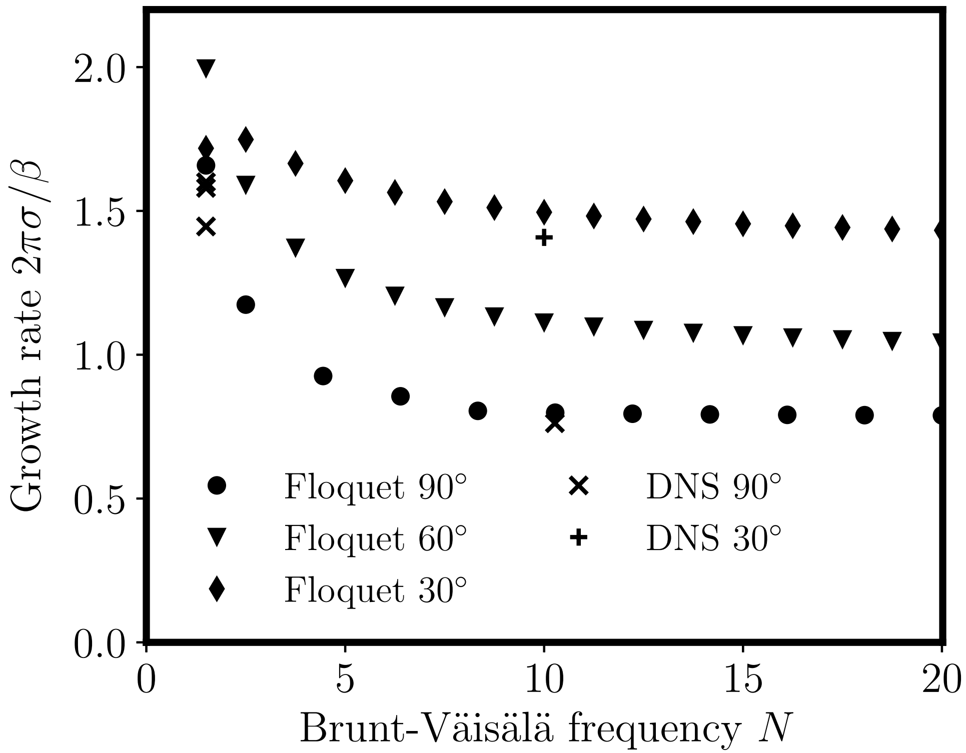
Comparison between DNS and Floquet analysis





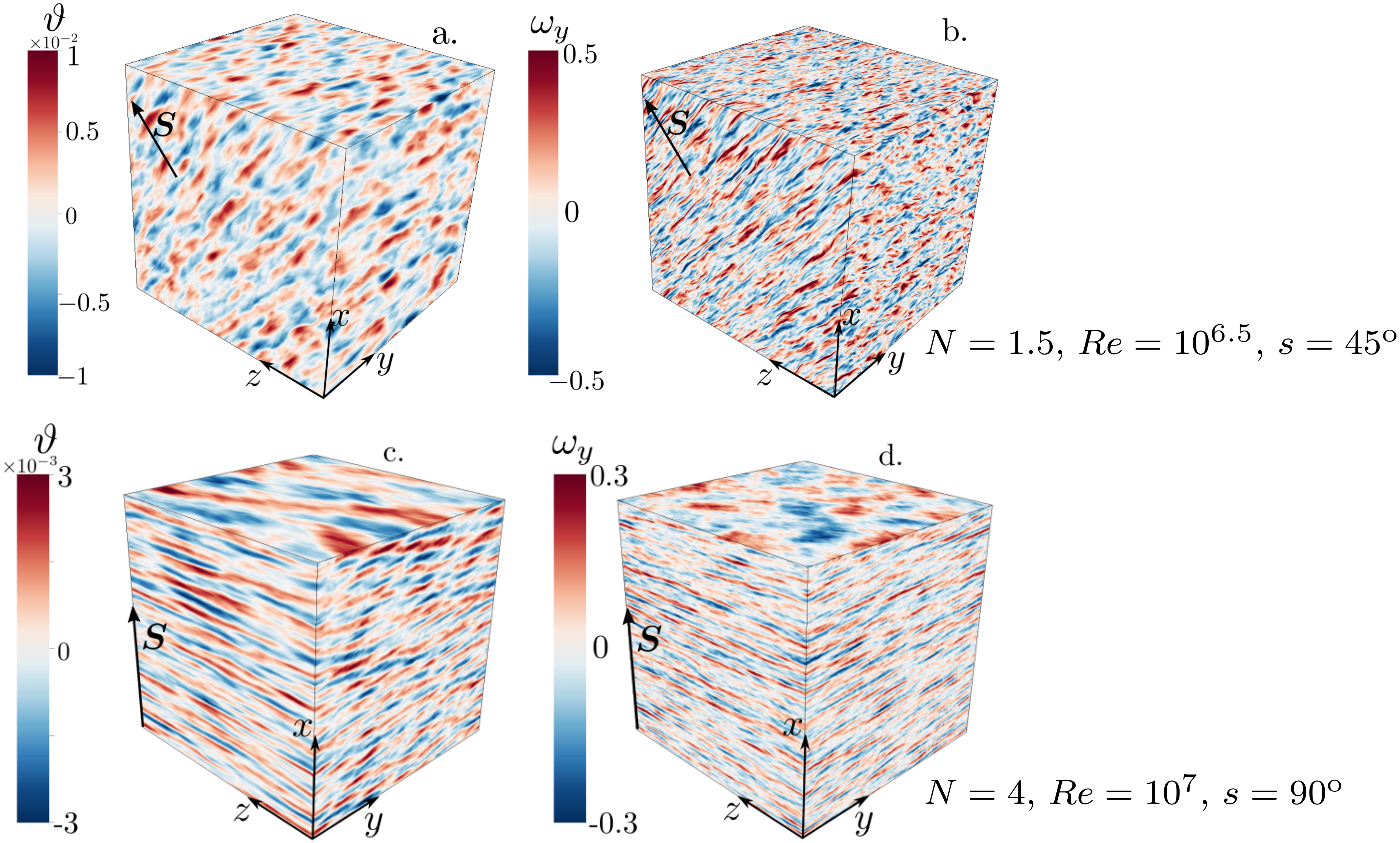
Instability at all latitude, optimum around $\pi/4$

Comparison between DNS and Floquet analysis

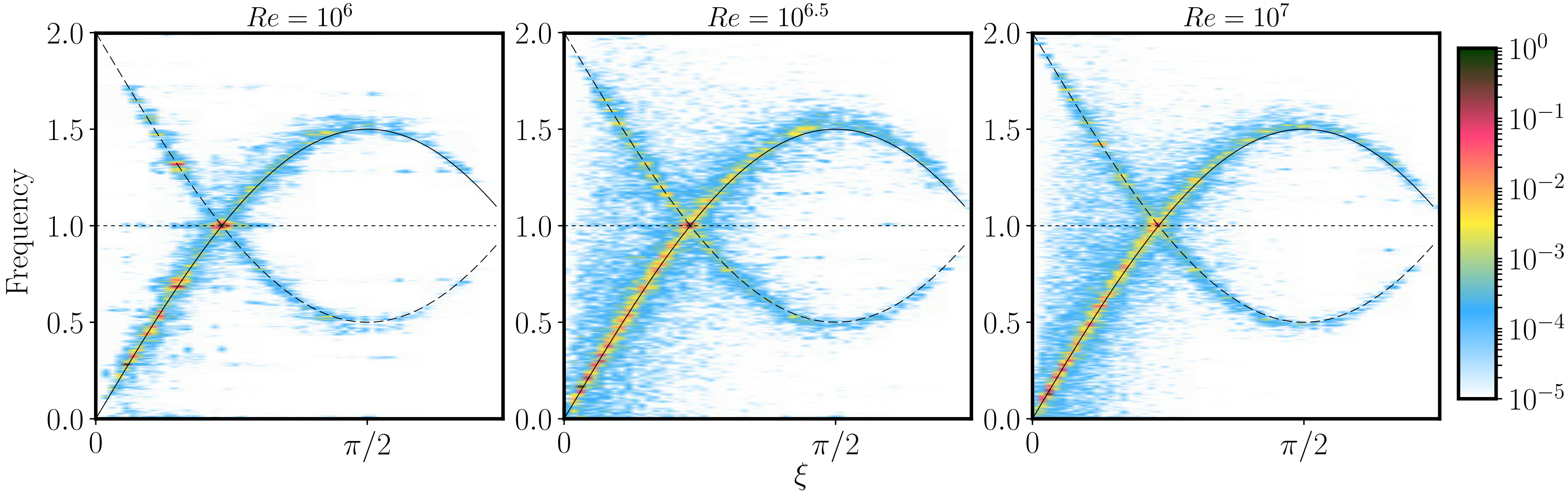


Growth rate saturates for $N \gg 1$

Nonlinear saturation via DNS

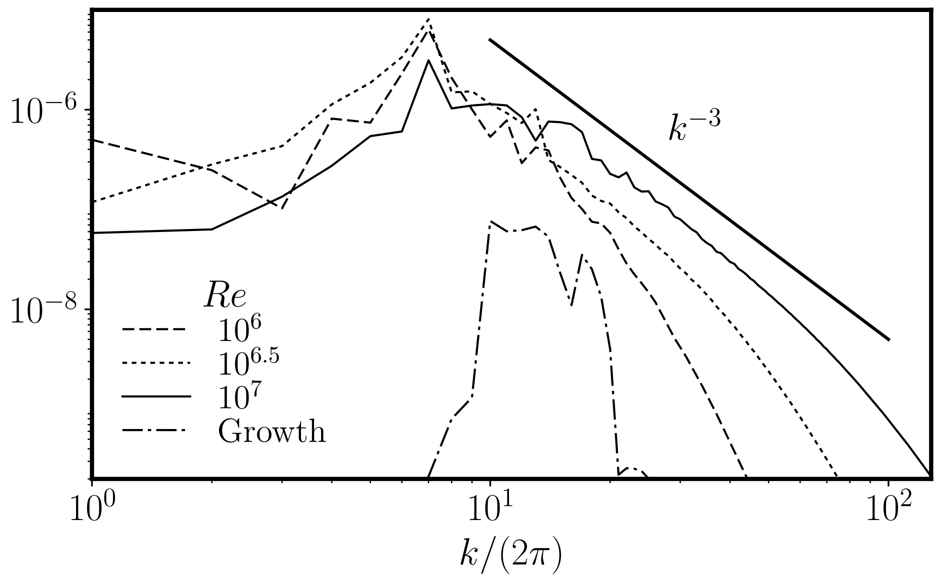


Spatio-temporal spectra

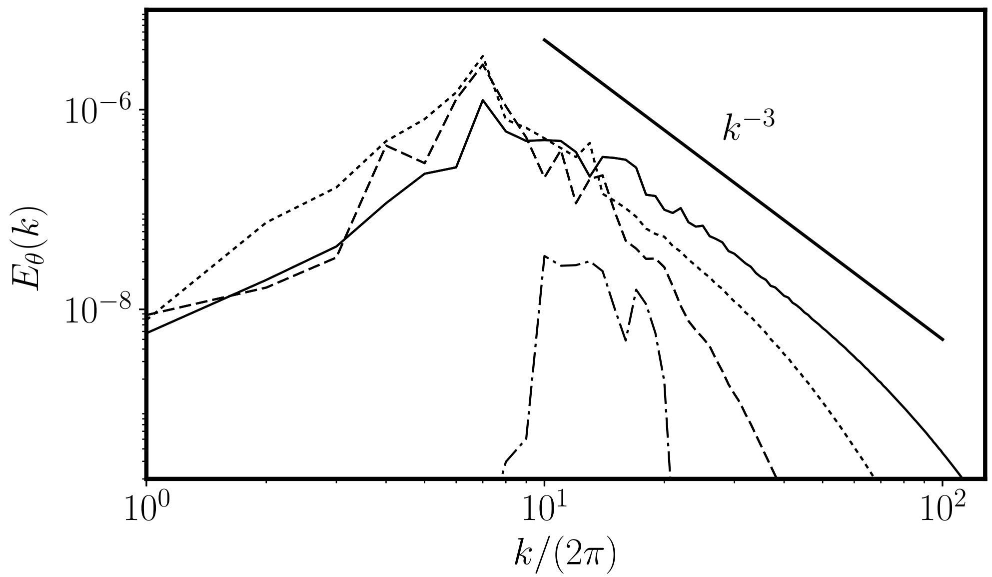


Isotropic energy spectra

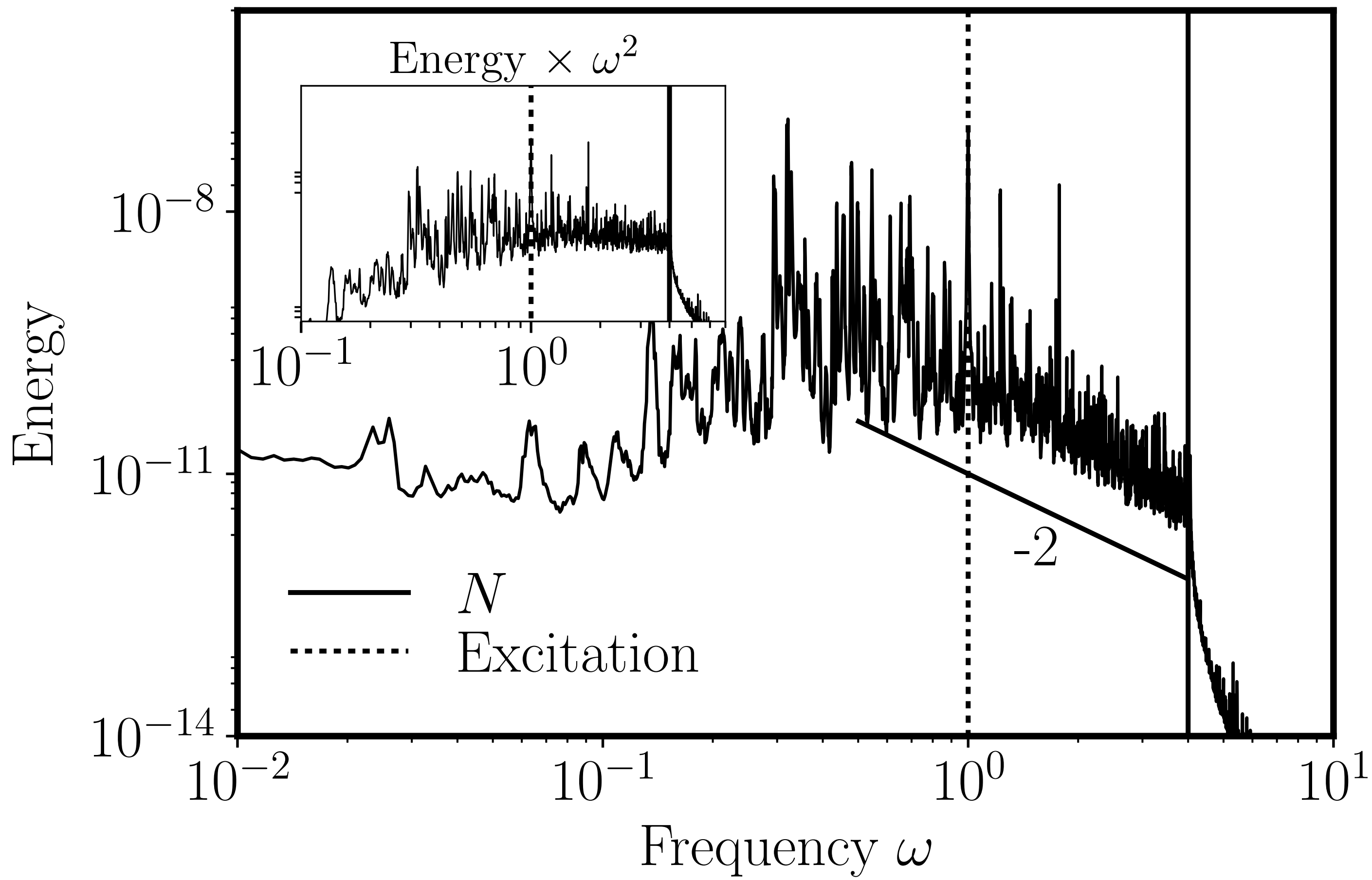
Velocity



Buoyancy



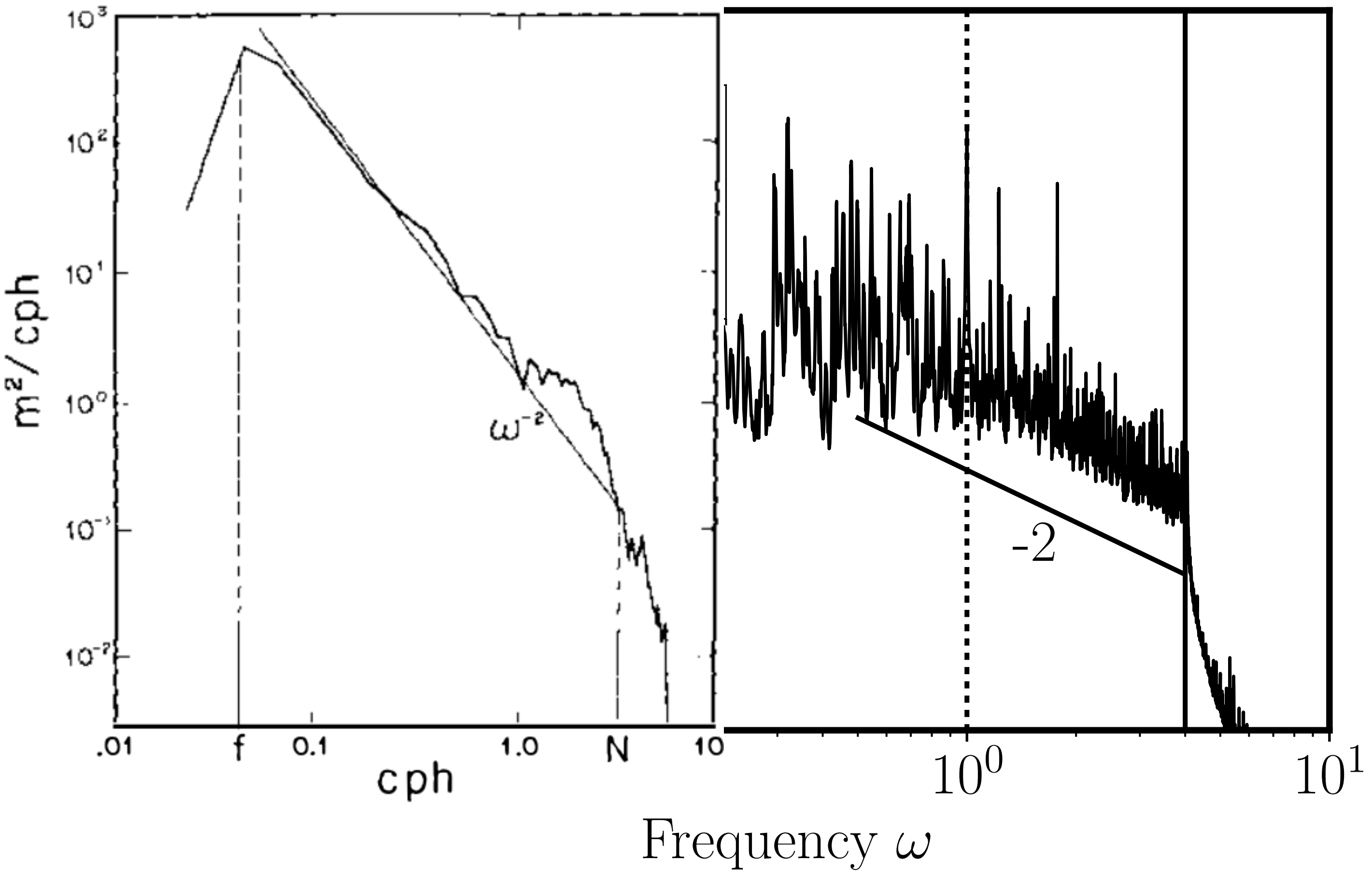
Temporal spectra



$N = 4,\ \beta = 0.05,\ Re = 10^7,\ s = 90^o$

Temporal spectra

Garret&Munk 1975



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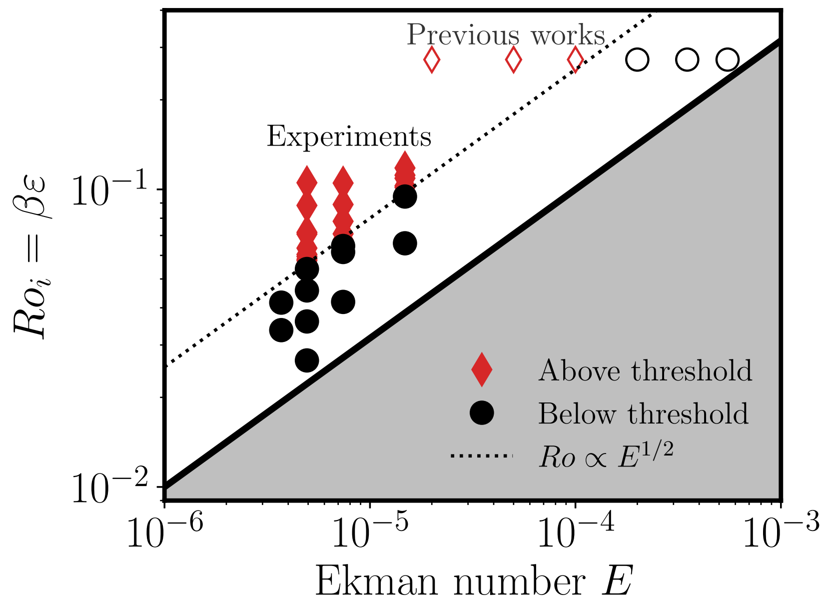
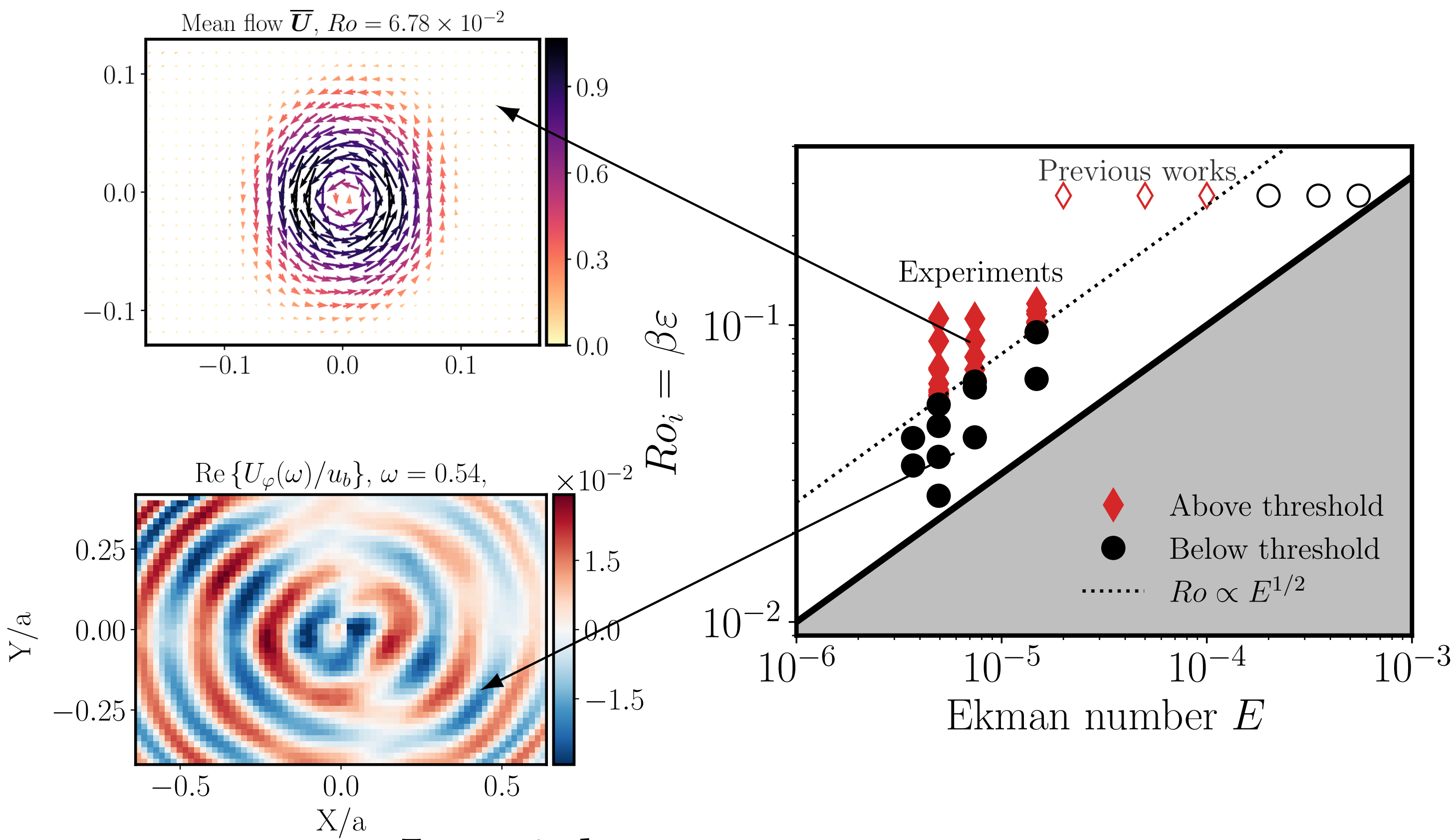
Two regimes of saturation



Two regimes of saturation

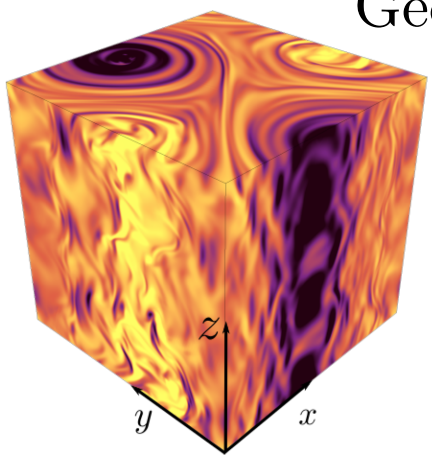
Geostrophic flow



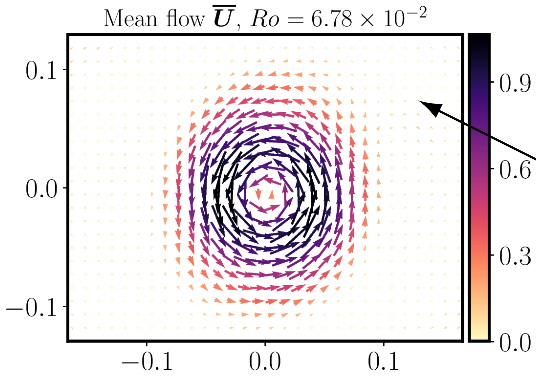
Inertial wave resonances

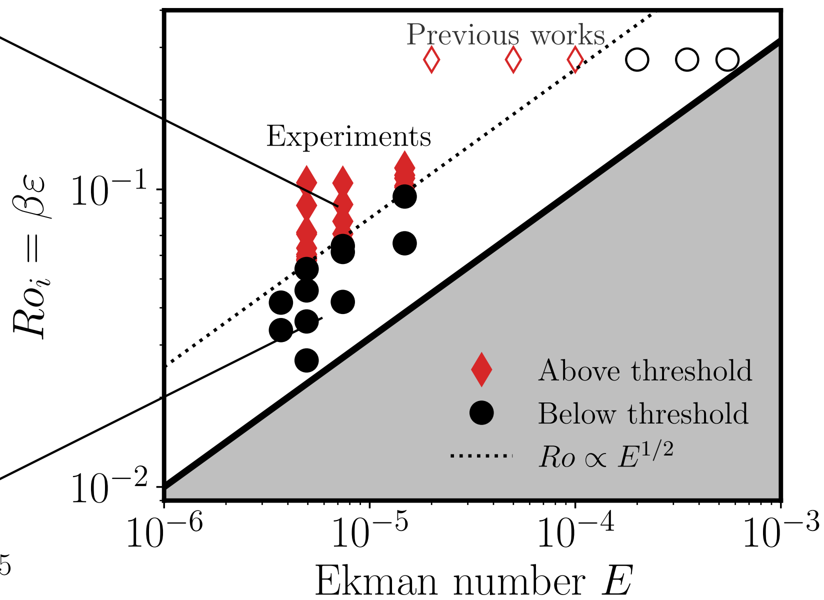
Two regimes of saturation

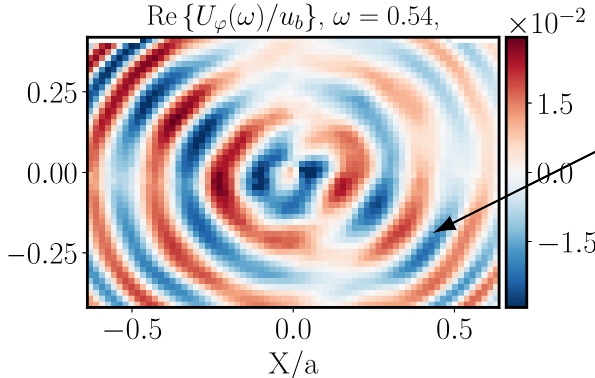
Geostrophic turbulence?



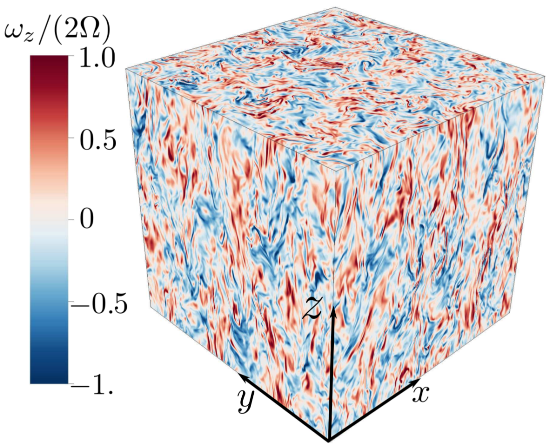
Geostrophic flow





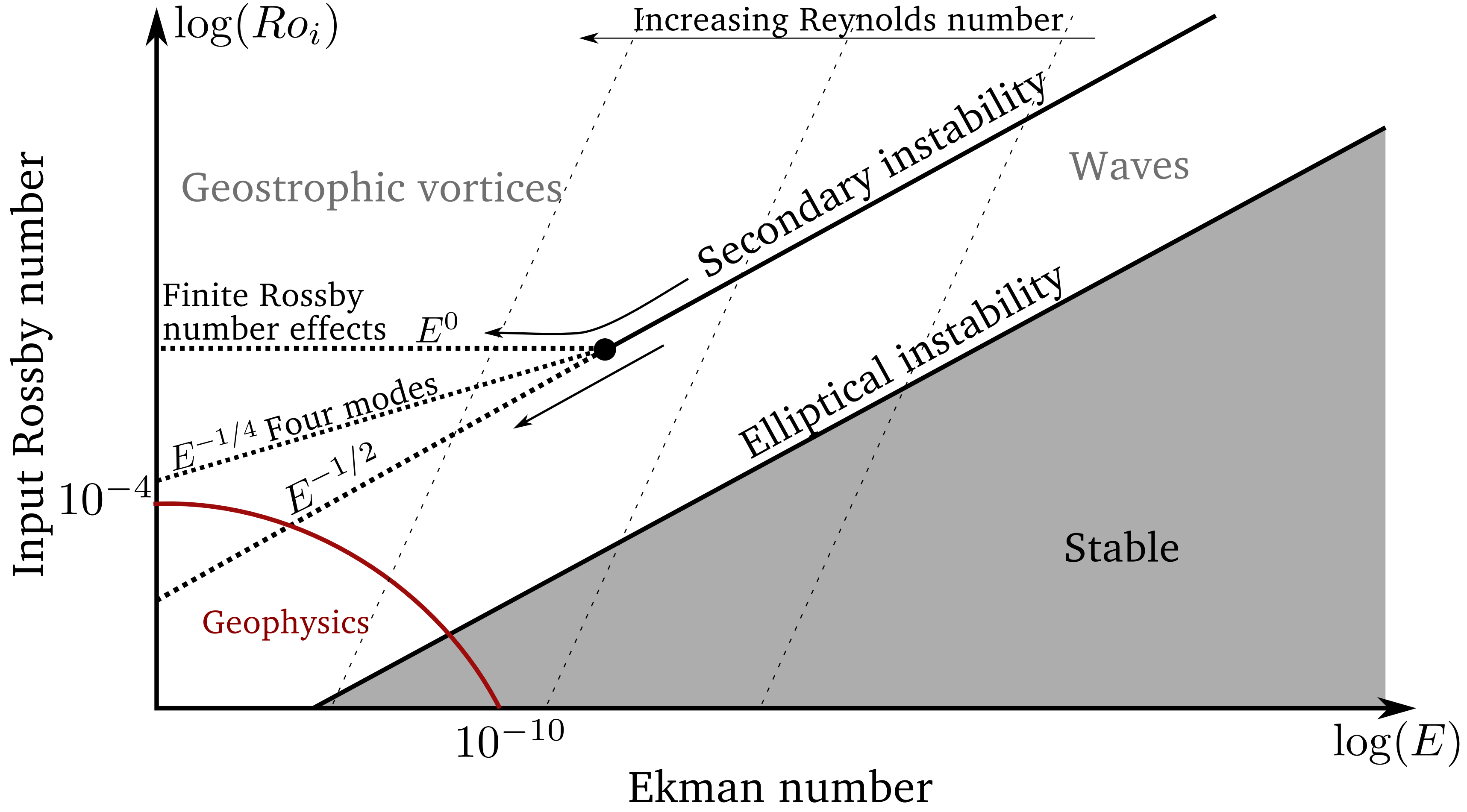


Inertial wave resonances



Inertial wave turbulence?

Towards inertial wave turbulence in planetary cores?



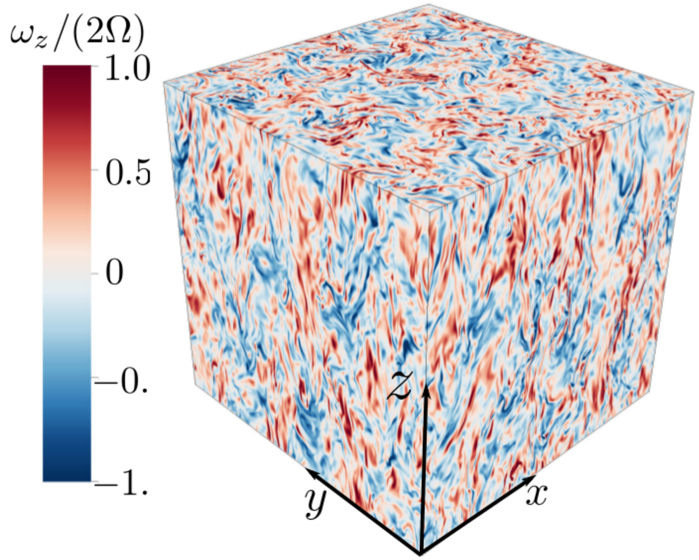
Conclusion

- Large-scale mechanical forcings can drive small-scale turbulence in rotating fluids
- Nonlinear saturation can lead to :
  - dominant geostrophic flows
  - inertial wave cascade
- Which regimes are more relevant to geophysics?
- The generation of geostrophic modes by quasi-resonant interactions still needs to be clarified

Perspectives

- More detailed statistical description of the wave turbulence regime
  - Comparison with AQNM (Bellet *et al.* 2006) and kinetic theory (Galtier 2003, Gelash *et al.* 2017, Gamba *et al.* 2017)
- Can we isolate small-scale inertial waves “surfing” on the geostrophic mode?
  - Systematic study varying the amplitude of the geostrophic flow
  - Critical layers?
- Can these flows drive a dynamo (Moffatt 1970, Davidson 2014)?

Thank you for your attention!



Generation and maintenance of bulk turbulence by libration-driven elliptical instability. B. Favier, A.M. Grannan, M.Le Bars & J.M. Aurnou, *Phys. Fluids* **27** (2015)

Inertial wave turbulence driven by elliptical instability. T. Le Reun, B. Favier, A. Barker & M. Le Bars, *Phys. Rev. Lett.* **119** (2017)

Parametric instability and wave turbulence driven by tidal excitation of internal waves. T. Le Reun, B. Favier & M. Le Bars, *J. Fluid Mech.* **840** (2018)

Base flow driven by libration

Time-dependent rotation rate:

$$\boldsymbol{\Omega}(t) = \Omega_0 \left(1 + \epsilon \cos(f \Omega_0 t)\right) \boldsymbol{e}_z$$

Vorticity equation in the frame rotating at $\Omega(t)$

$$\dot{\boldsymbol{\omega}} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \left[(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \cdot \nabla\right] \boldsymbol{u} + 2\dot{\boldsymbol{\Omega}}$$

Looking for uniform vorticity solution of the form $\boldsymbol{\omega} = (0, 0, \omega_z)$ leads to

$\boxed{\text{Mantle frame rotating at } \Omega(t)}$

$$\boldsymbol{U}_b = \Omega_0 \epsilon \sin(\Omega_0 f t) \begin{bmatrix} -(1+\beta)Y \\ (1-\beta)X \\ 0 \end{bmatrix}$$

$\boxed{\text{Frame rotating at } \Omega_0}$

$$\boldsymbol{U}_b = \Omega_0 \beta \epsilon \sin(\Omega_0 f t) \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Equations and growth rates

Let us decompose the velocity field as

$$\boldsymbol{U} = \boldsymbol{U}_b + \boldsymbol{u}$$

The equation for the perturbations is

$$\partial_t \boldsymbol{u} + \boldsymbol{U}_b \cdot \nabla \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{U}_b + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u}$$

Rewriting the base flow as $\boldsymbol{U}_b = \boldsymbol{\mathcal{A}}(t)\boldsymbol{x}$, and taking the linear inviscid limit leads to

$$\underbrace{\partial_t \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} + \nabla p}_{\text{Linear oscillator}} = \underbrace{-\left[\boldsymbol{\mathcal{A}}(t)\boldsymbol{u} + \boldsymbol{\mathcal{A}}(t)\boldsymbol{x} \cdot \nabla \boldsymbol{u}\right]}_{\text{Parametric forcing}}$$

which is similar to a Mathieu equation.

Equations and growth rates

The velocity perturbation can be decomposed onto two inertial modes

$$\boldsymbol{u}(\boldsymbol{x}) = A_1(t)\boldsymbol{\Psi}_1(\boldsymbol{x}) + A_2(t)\boldsymbol{\Psi}_2(\boldsymbol{x})$$

where each inertial mode satisfies

$$2\boldsymbol{\Omega} \times \boldsymbol{\Psi}_i + \nabla\Pi_i = i\omega_i\boldsymbol{\Psi}_i \quad \text{and} \quad \nabla \cdot \boldsymbol{\Psi}_i = 0$$

Using the orthogonality relation $\langle \boldsymbol{\Psi}_i | \boldsymbol{\Psi}_j \rangle = \delta_{ij}$ leads to the amplitude equations for $a_j(t) = \exp(i\omega_j t)A_j(t)$

$$\dot{a}_1 = \langle \boldsymbol{\Psi}_1 | L(t)\boldsymbol{u} \rangle \, e^{-i\omega_1 t}$$
$$\dot{a}_2 = \langle \boldsymbol{\Psi}_2 | L(t)\boldsymbol{u} \rangle \, e^{-i\omega_2 t}$$

where $L(t)$ is the parametric forcing operator

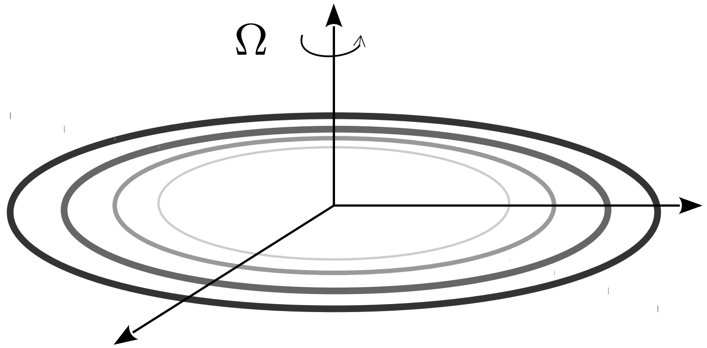
$$L(t)\boldsymbol{u} = \boldsymbol{\mathcal{A}}(t)\boldsymbol{u} + \boldsymbol{\mathcal{A}}(t)\boldsymbol{x} \cdot \nabla\boldsymbol{u} \quad \text{with} \quad \boldsymbol{U}_b = \boldsymbol{\mathcal{A}}(t)\boldsymbol{x}$$

Equations and growth rates

The interaction coefficients are non-zero providing that the following resonance conditions are satisfied
This can readily be done assuming a scale separation between short wavelength resonant modes and the large-scale base flow (WKB).

These are inviscid growth rates which are, in practice, corrected by

The forcing: base flow with elliptic streamlines



Base flow with elliptic streamlines

- Strained vortices (Bayly 1986)
- Vortex pair (Pierrehumbert 1986)
- Planetary cores (Malkus 1989)

- $\boldsymbol{u} = (Ay, -Bx, 0)$ (Uniform vorticity $A + B$ and Strain $A - B$)
- Elliptic instability: resonance of a pair of inertial modes with the underlying strain field (Kerswell 2002)
- Observed experimentally in ellipsoidal containers (Le Bars 2015)

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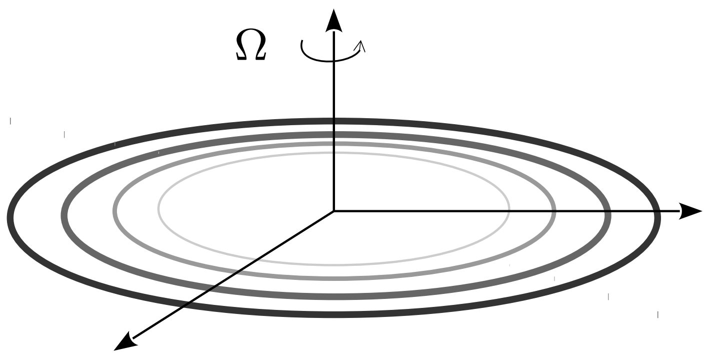

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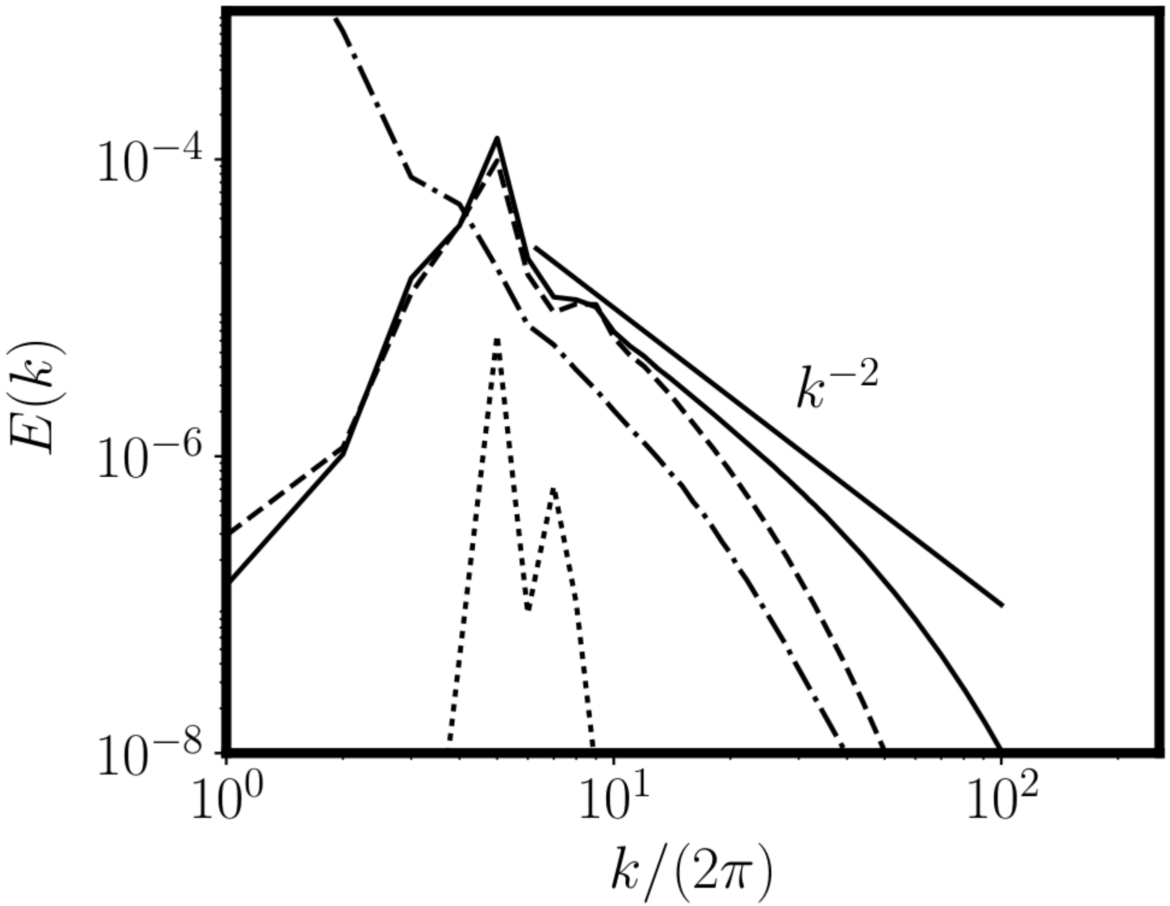
$\Rightarrow$ Energy is injected in resonant inertial waves only

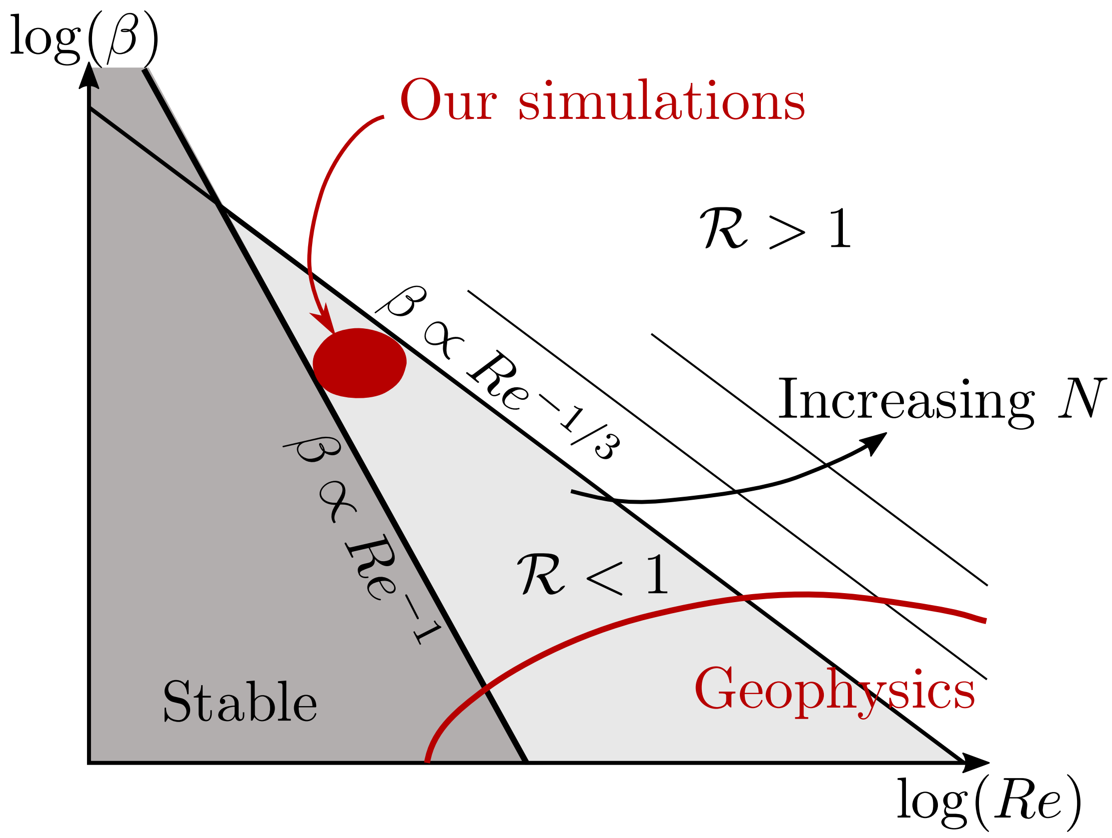
An example of tidally-driven instability

Isotropic energy spectra- Rotating case



Toward wave turbulence driven by tidal forcing?

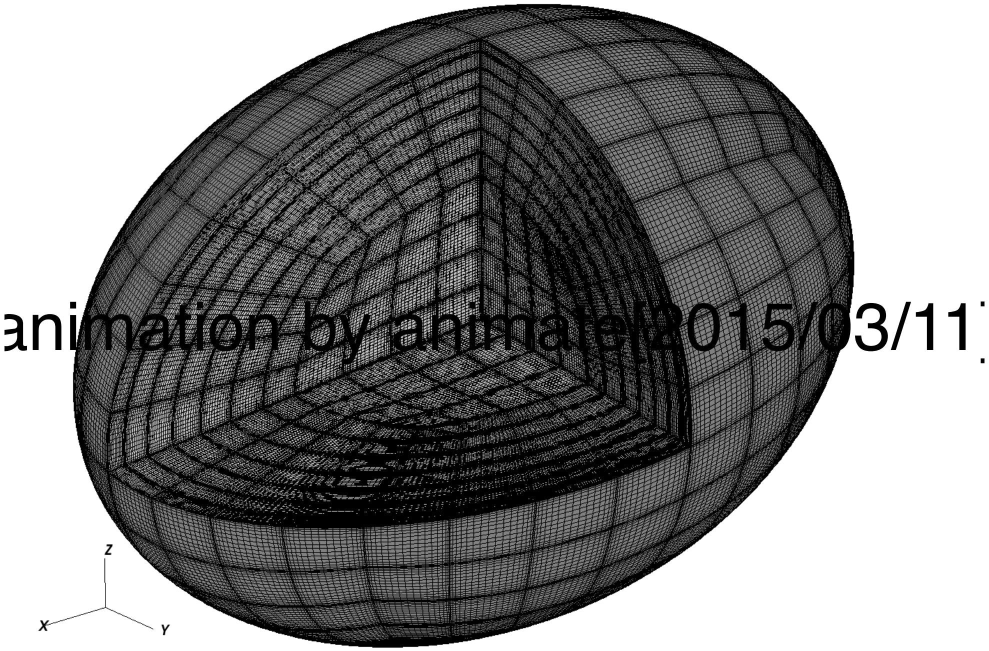
Stratified case



Numerical approach

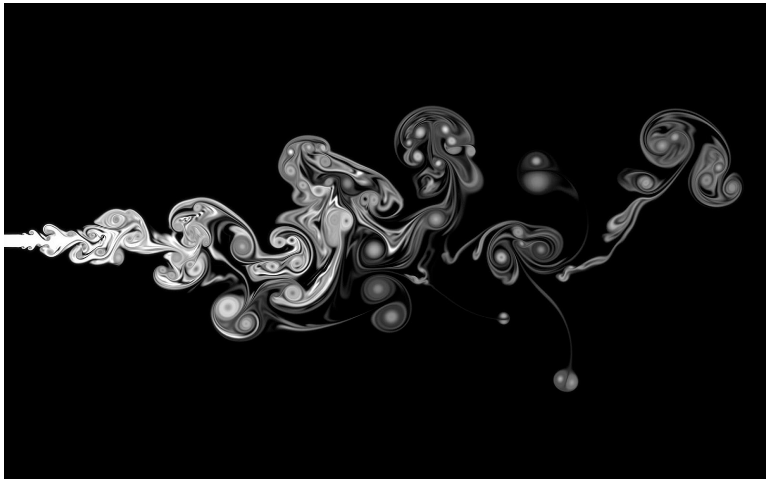
Spectral element code **Nek5000**

`http://nek5000.mcs.anl.gov`



- E hexahedral elements
- N^3 tensor-product Gauss-Lobatto Legendre collocation points
- Algebraic convergence with E
- Exponential convergence with N
- 3rd order explicit Adams-Bashforth scheme for convective terms
- 3rd order implicit Backward Differentiation scheme for diffusive and pressure terms

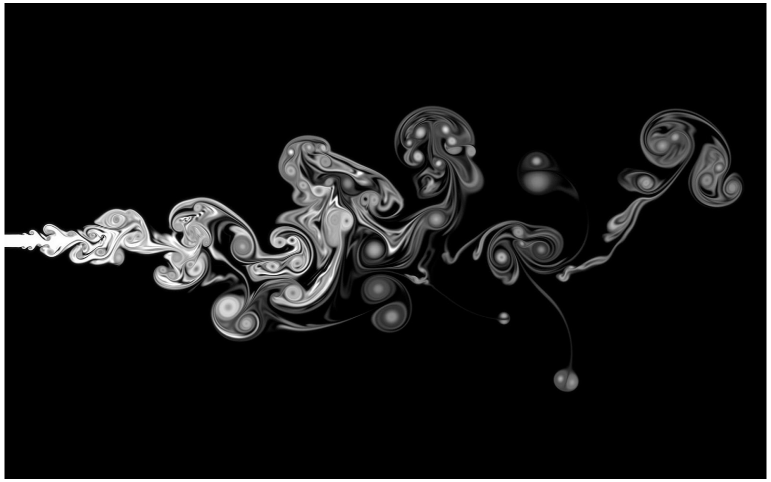
From strong to weak turbulence?



Turbulent jet

- Fast and strong nonlinearities
- Intermittency and coherent structures
- No complete statistical theory

From strong to weak turbulence?



Turbulent jet

Surface gravity waves

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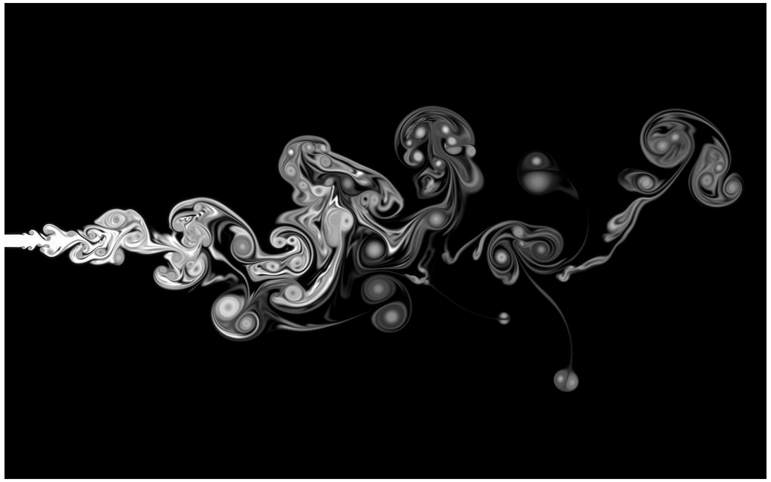
- Slow and weak nonlinearities
- Only dispersive waves
- Weak turbulence theory (Zakharov spectrum)

From strong to weak turbulence?



Turbulent jet

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Which regime is more relevant to planetary settings?