Dynamo theory and experiments

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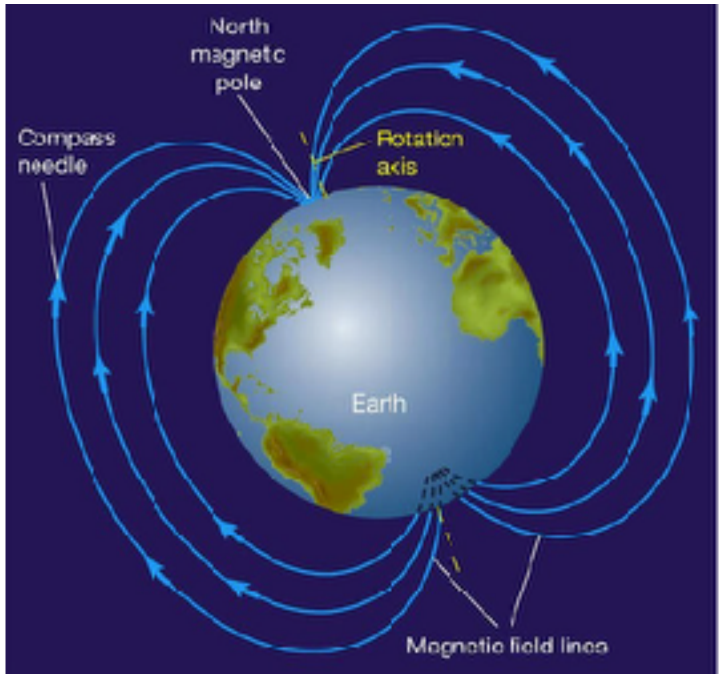
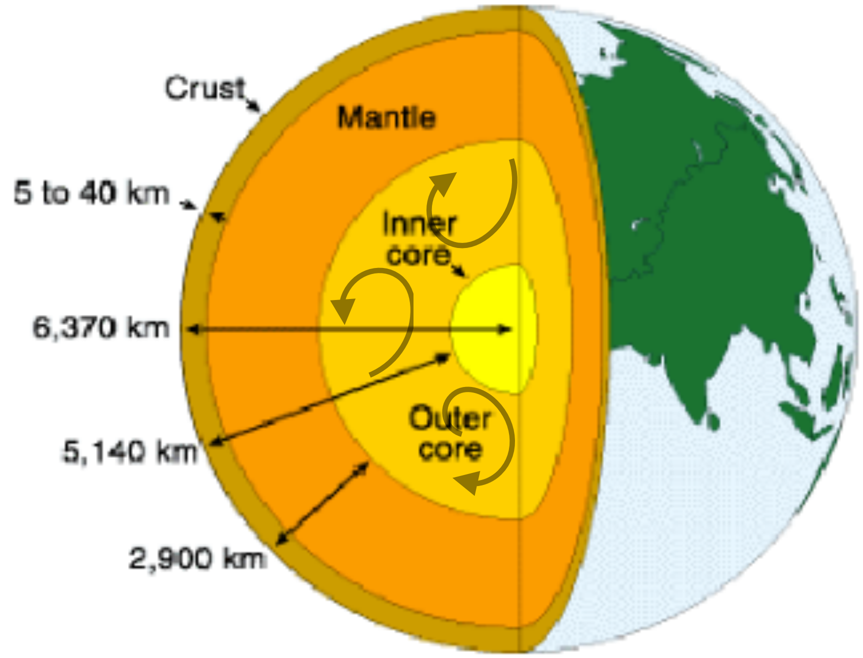
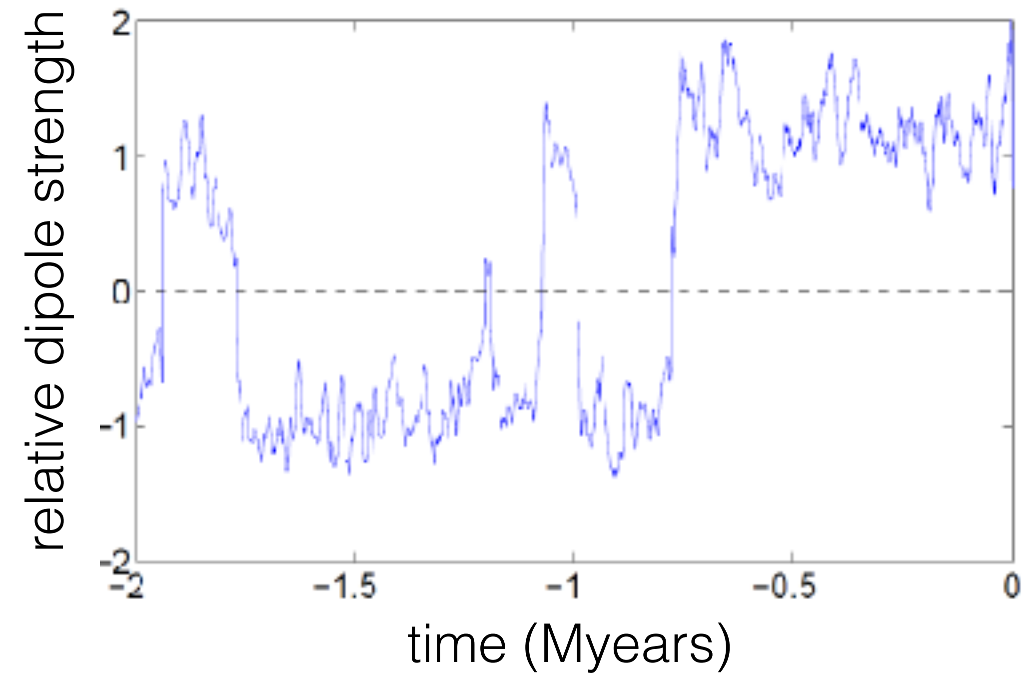
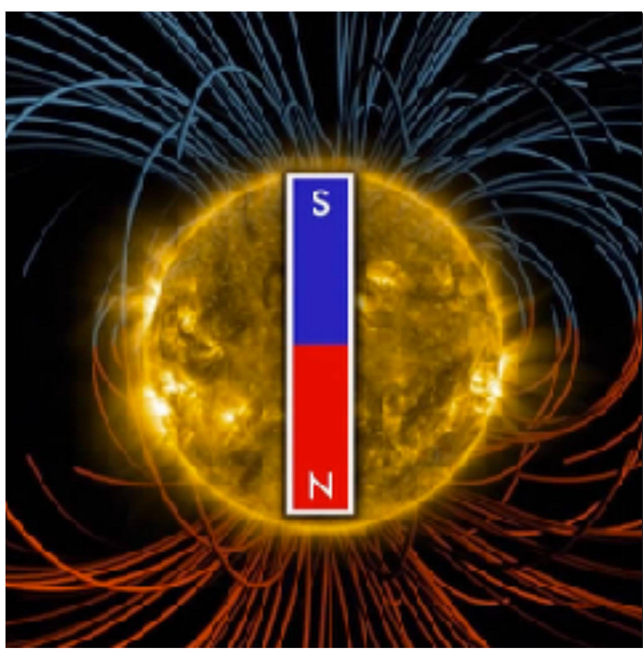
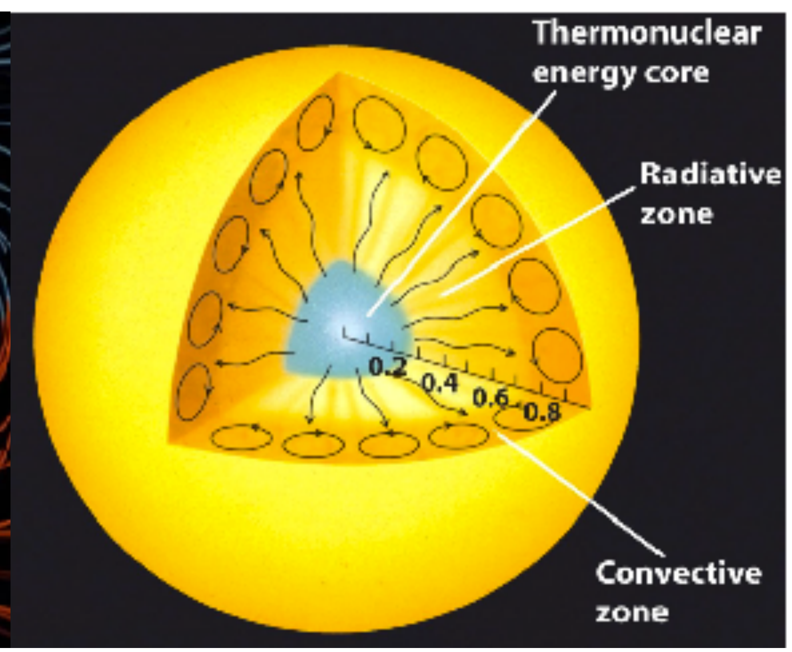
Dynamo effect

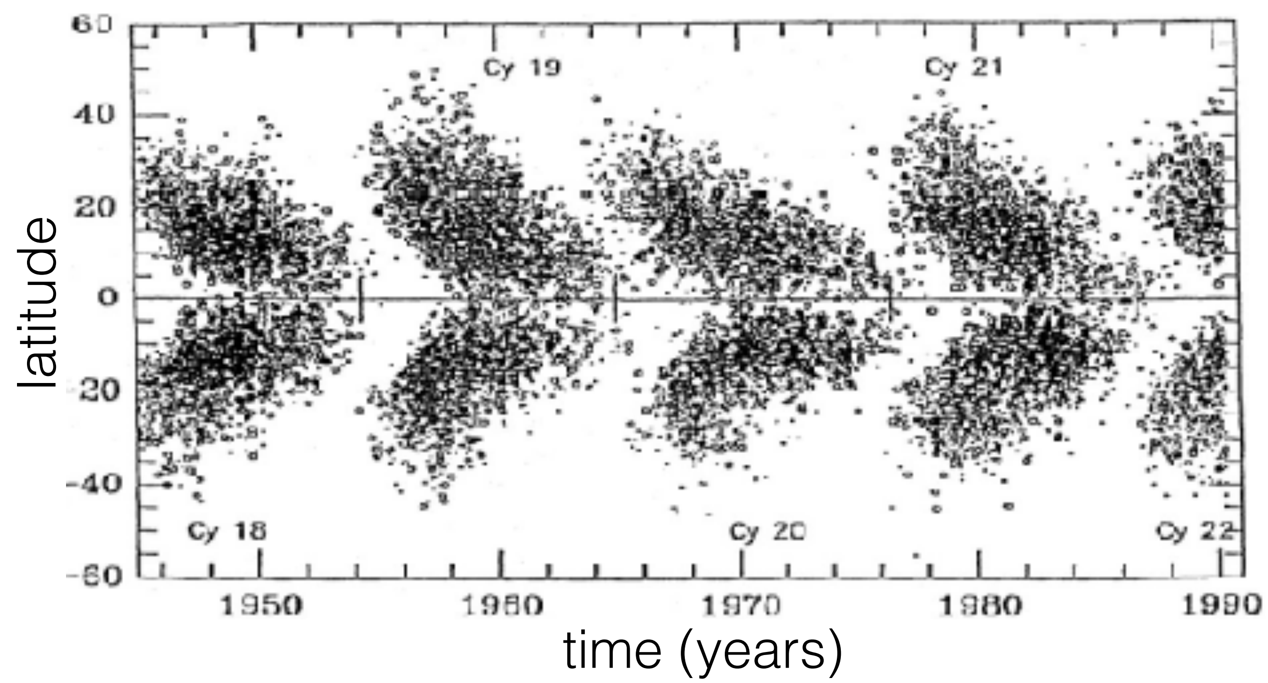






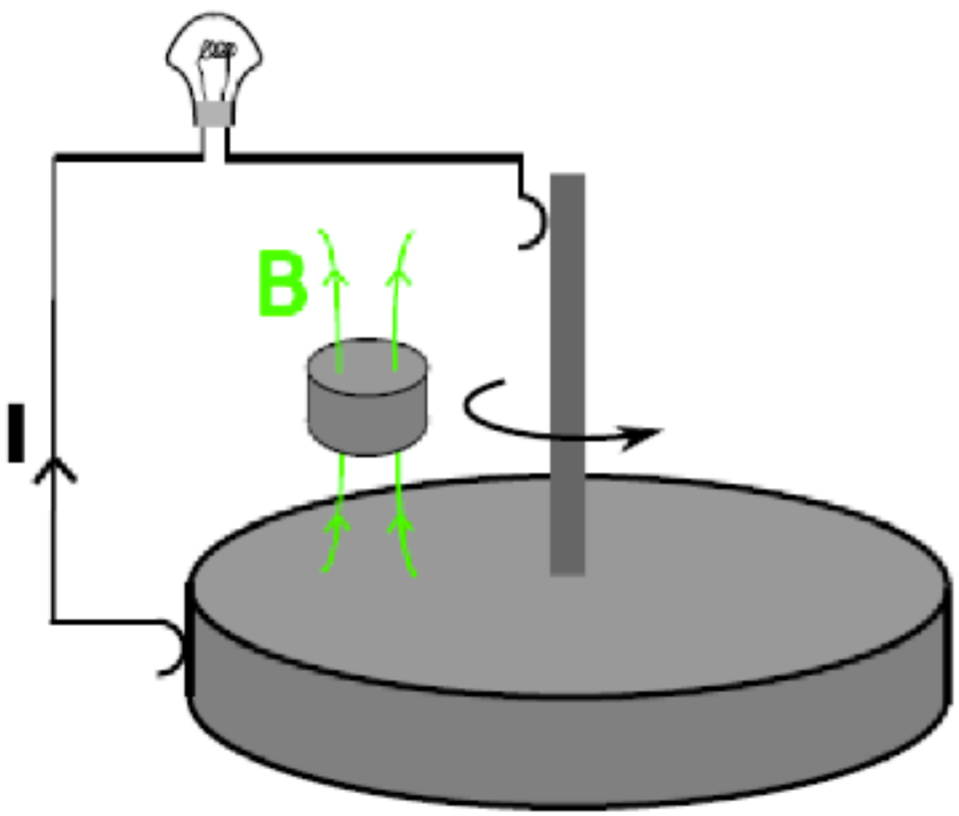




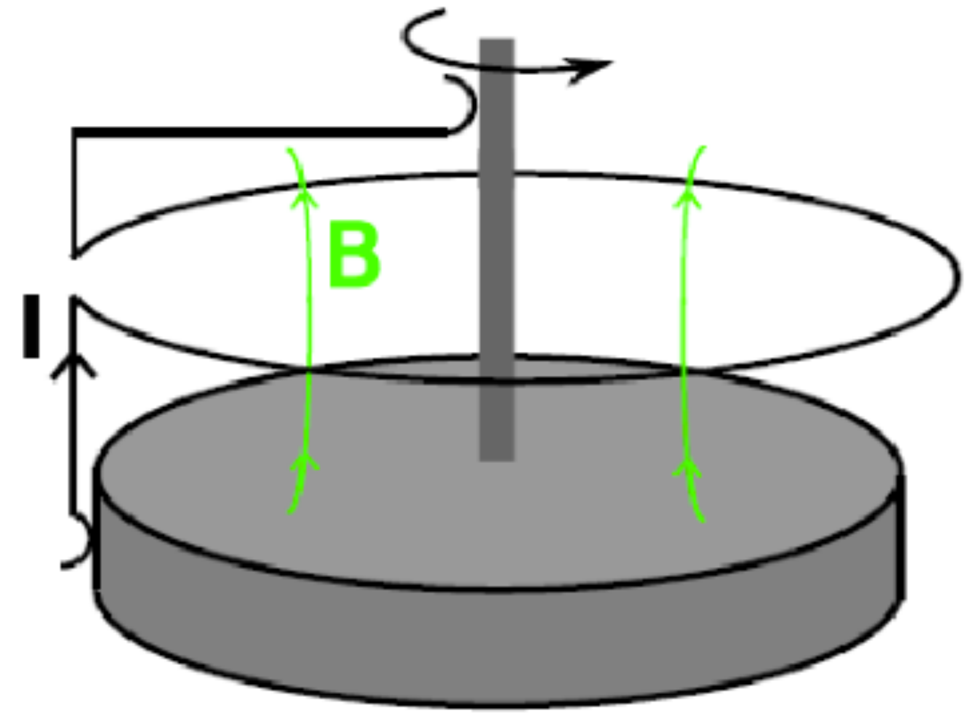


Larmor 1919: magnetic field perturbations can be amplified by the flow. **Dynamo instability.**

Dynamo instability: self-amplification of B



The rotation of the disk induces electrical current.

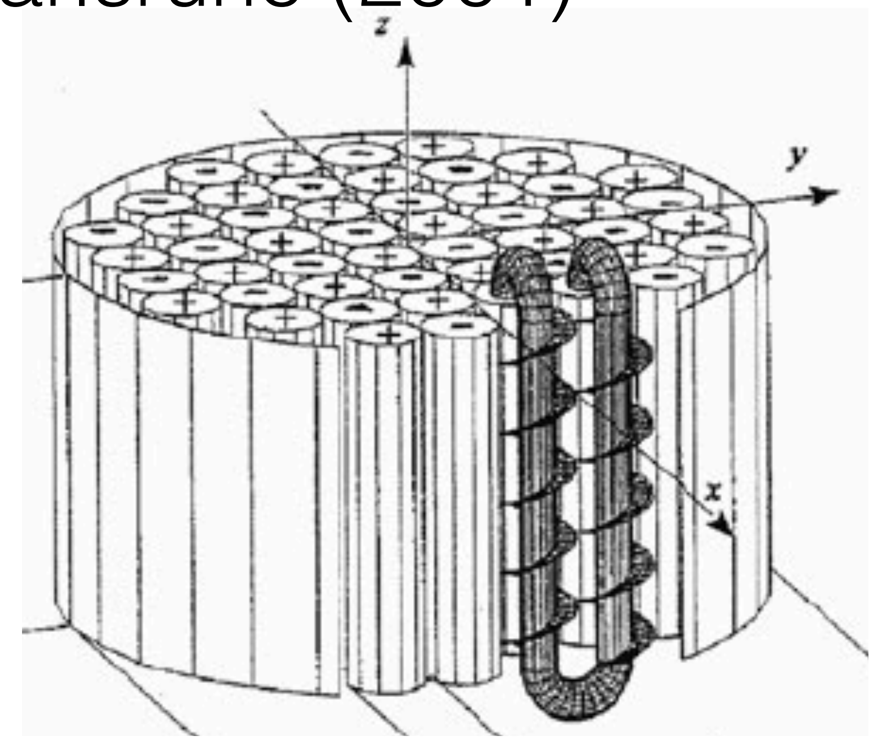


If the disk spins fast enough: B grows **spontaneously**.

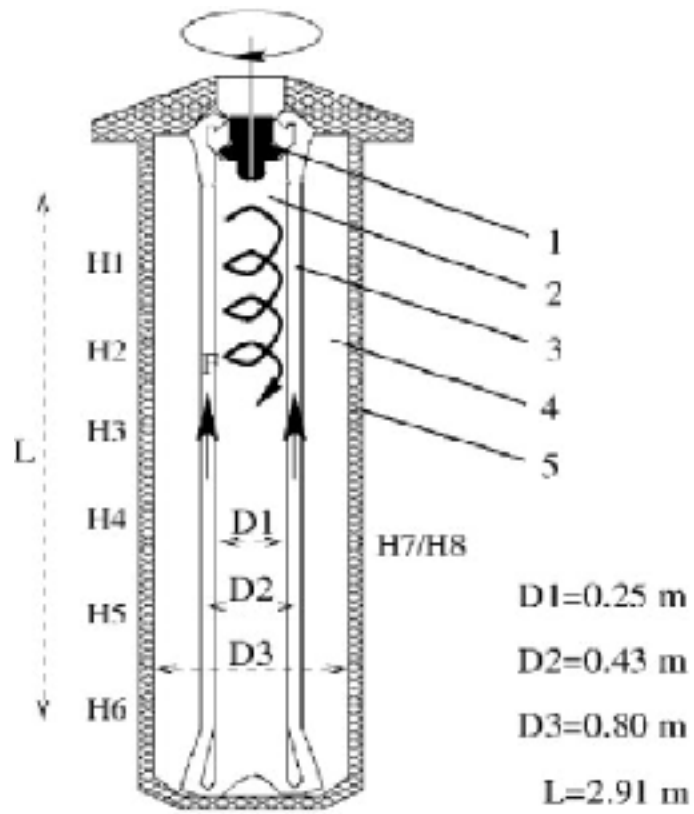
- Instability from which most industrial electricity is produced!
- But can this instability arise in an electrically conducting fluid?

Dynamo experiments

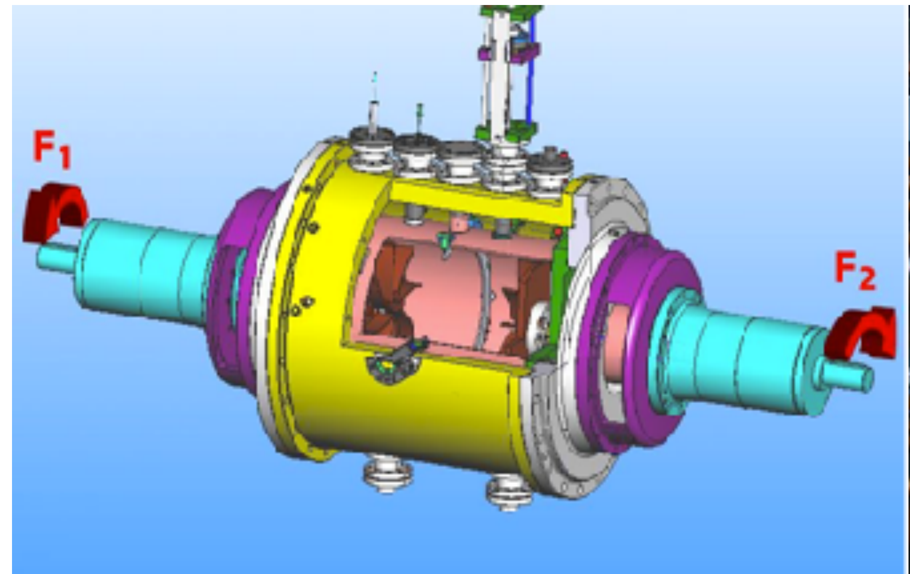
Karlsruhe (2001)

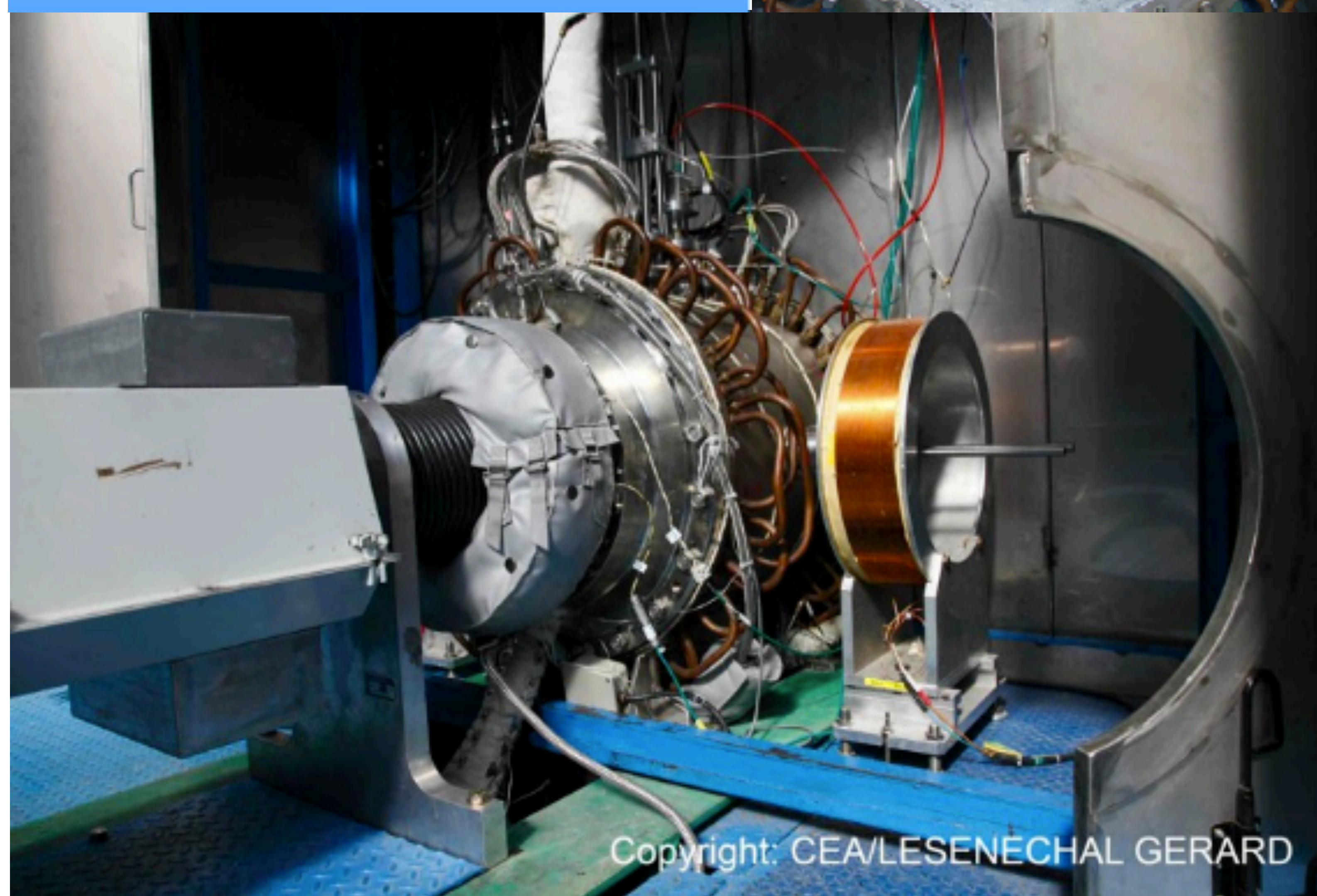


Riga (2001)



Von Karman Sodium (2006)



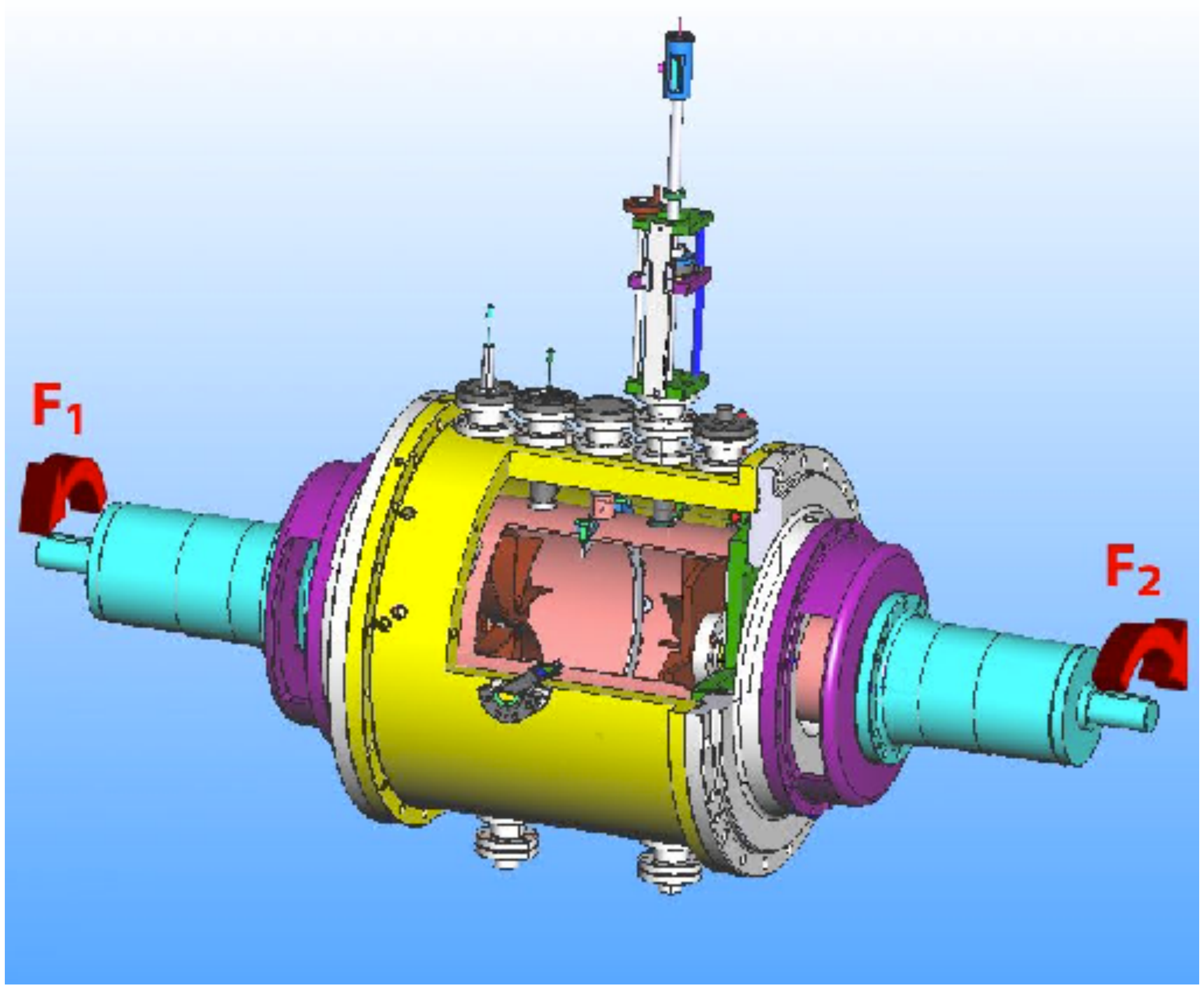


Dynamo instability

- Incompressible MHD equations:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{u} + \frac{(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}}{\rho \mu_0}$$

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) + \eta \boldsymbol{\nabla}^2 \mathbf{B}$$



$$Rm = \frac{U\ell}{\eta}$$

dynamo for $Rm > Rm_c$.

$$Pm = \frac{\nu}{\eta}$$

$Pm \ll 1$ for liquid metals.

Kinematic vs dynamic studies

- **Kinematic dynamo** problem: which flows produce a dynamo? What is the threshold for dynamo action?

Imposed velocity field

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) + \eta \boldsymbol{\nabla}^2 \mathbf{B}$$

linear in **B** ➡ linear stability analysis to find Rm_c.

- **Dynamic** problem: How does the instability saturate? Intensity of the resulting magnetic field?

feedback through Lorentz force

$$\partial_t \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{u} + \frac{(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}}{\rho \mu_0}$$

Outline

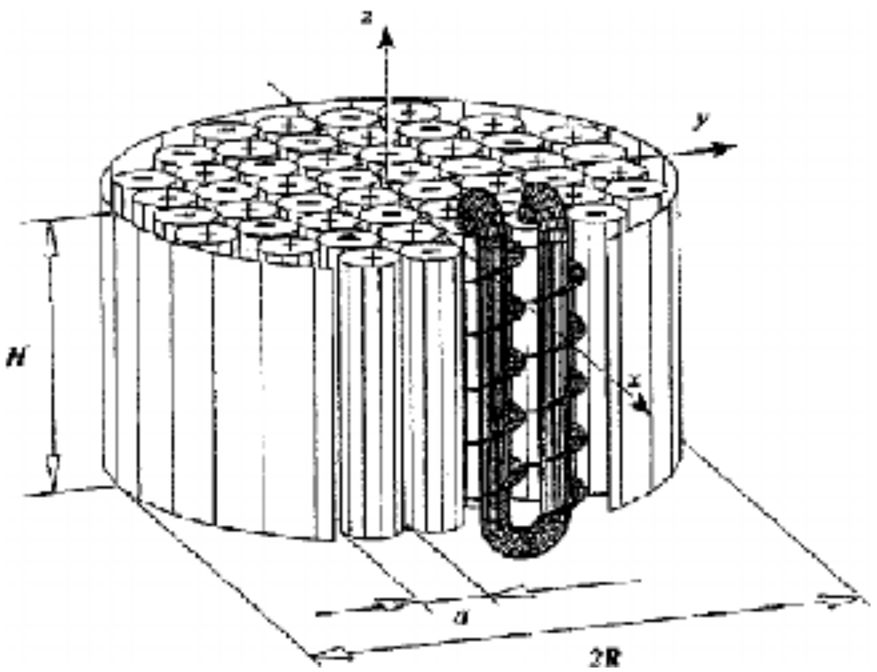
- **Kinematic dynamo problem:** magnetic field generation mechanisms.
    - alpha-effect mechanism: Roberts flow
    - helicity at large scale: Ponomarenko flow
    - omega-effect mechanism: Herzenberg dynamo
    - alpha-omega mechanism: Von Karman Sodium dynamo

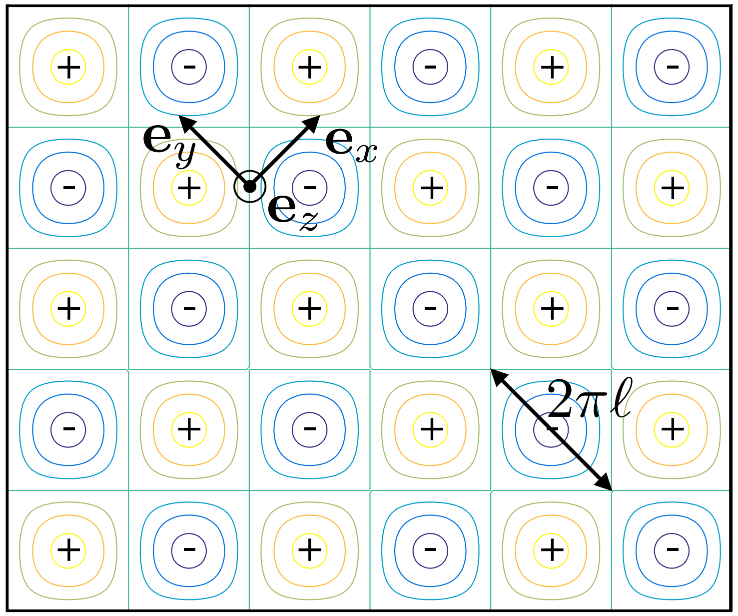
- **Dynamic problem:** magnetic field saturation mechanisms.
    - Scaling-laws for the saturation of turbulent dynamos
    - The predominance of viscosity in DNS
    - Global rotation and the magnetostrophic scaling regime

Part I: kinematic dynamo problem

- Which flows generate a dynamo?
- How fast need the flow be?
- Can we isolate magnetic field generation mechanisms?

Roberts flow: alpha effect





$$\mathbf{v} = V \begin{cases} \cos(y/\ell) \\ \cos(x/\ell) \\ \sin(y/\ell) - \sin(x/\ell) \end{cases}$$

$Rm^{(\ell)} = V\ell/\eta \ll 1$
$\ell \ll \lambda$, size along z

scale λ scale ℓ

$$\mathcal{B} = \mathbf{B} + \mathbf{b}$$

average over scale ℓ

with $\begin{cases} \langle \mathcal{B} \rangle = \mathbf{B} \\ \mathbf{b} \ll \mathbf{B} \end{cases}$

$$\eta \Delta \mathbf{b} = -(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{v}$$

$$\langle \mathbf{v} \times \mathbf{b} \rangle = \begin{cases} \alpha_{xx} B_x \\ \alpha_{yy} B_y \\ 0 \end{cases}$$

$$\alpha_{xx} = \alpha_{yy} = -Rm^{(\ell)} V$$

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times \langle \mathbf{v} \times \mathbf{b} \rangle + \eta \Delta \mathbf{B}$$

$$\mathbf{B} \sim e^{\sigma t} e^{i \frac{2\pi}{\lambda} z}$$

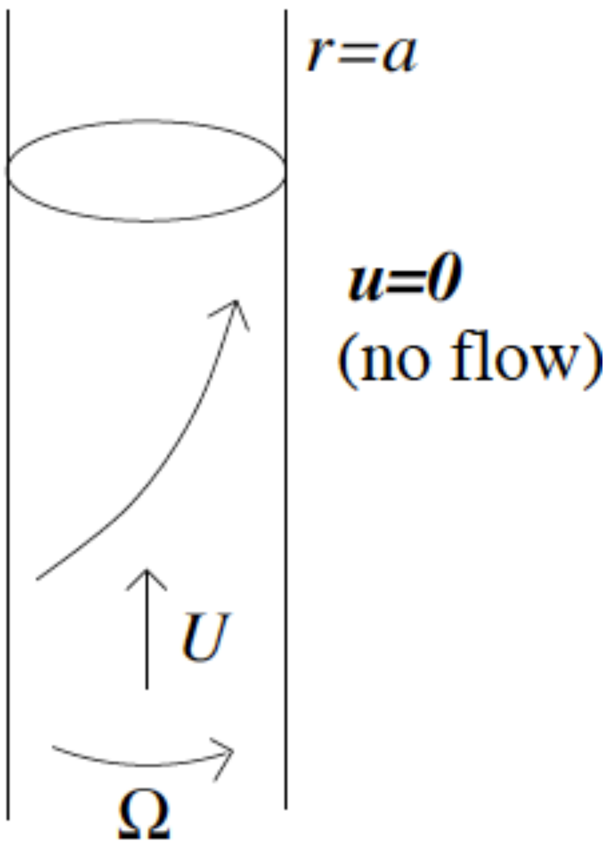
$$\sigma = |\alpha| \frac{2\pi}{\lambda} - \eta \frac{4\pi^2}{\lambda^2}$$

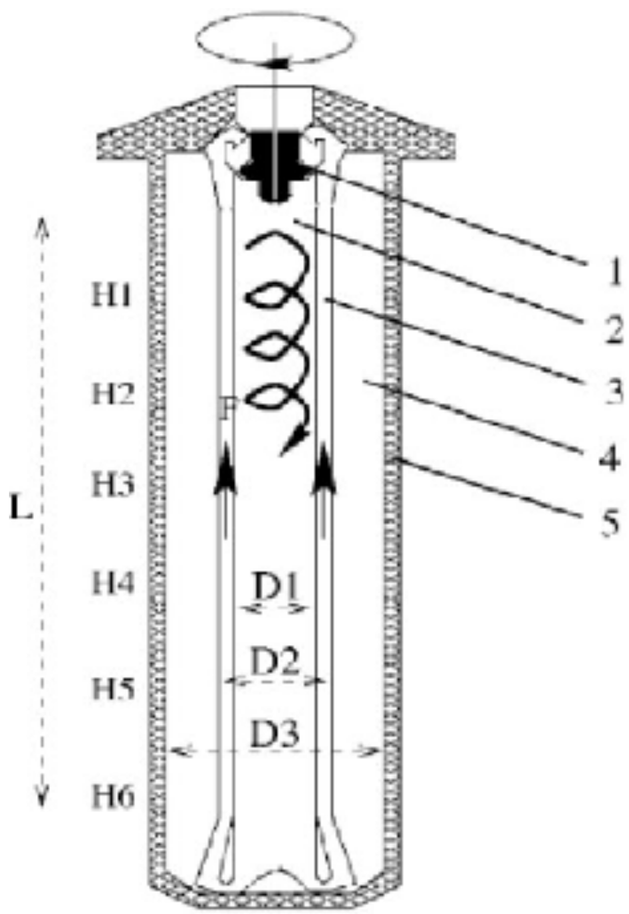
α^2 dynamo mechanism (helical flow).

Ponomarenko dynamo

A large-scale helical flow:

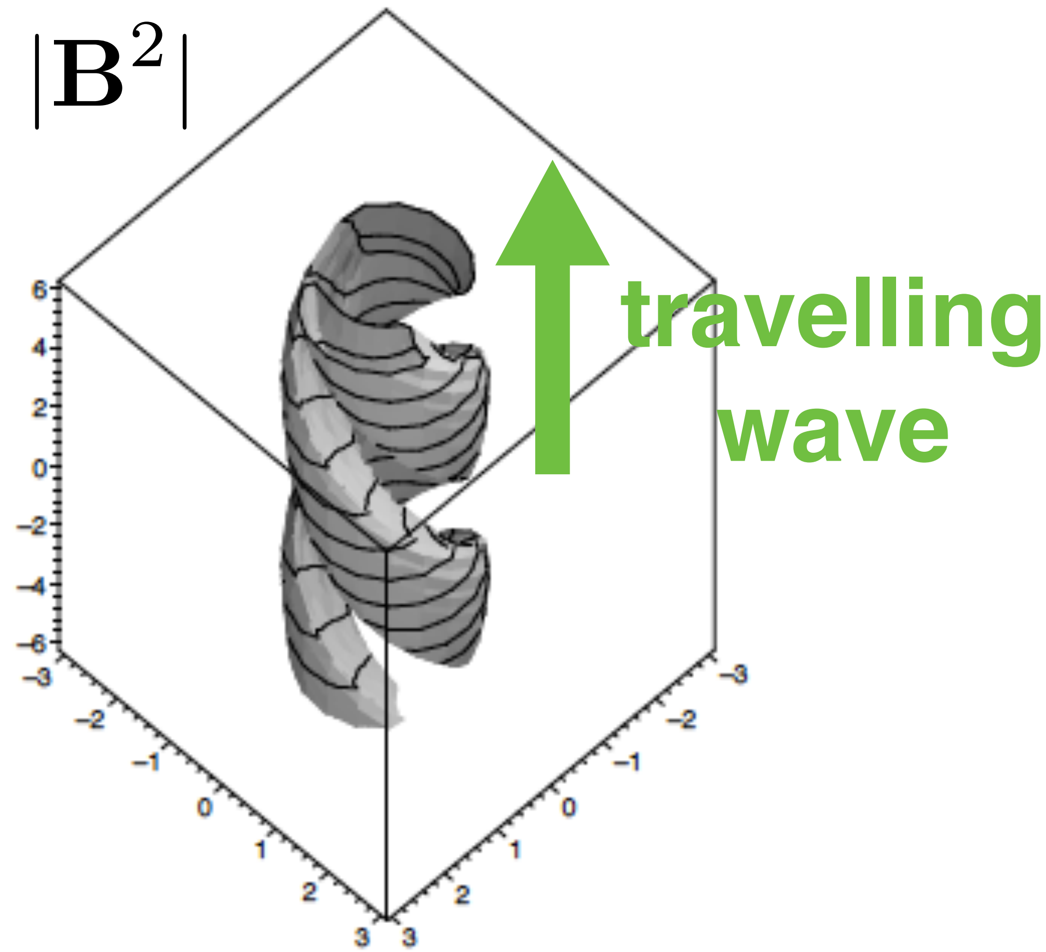
$$\boldsymbol{u} = \begin{cases} \hat{\boldsymbol{\theta}}\, r\Omega + \hat{\boldsymbol{z}}\, U & (r < a), \\ \boldsymbol{0} & (r > a), \end{cases}$$





Riga experiment

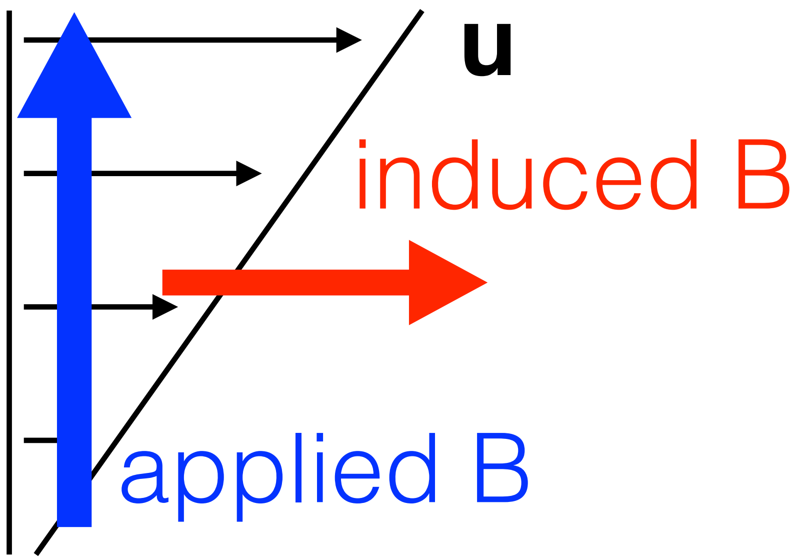
Oscillatory dynamo when Rm is above threshold:

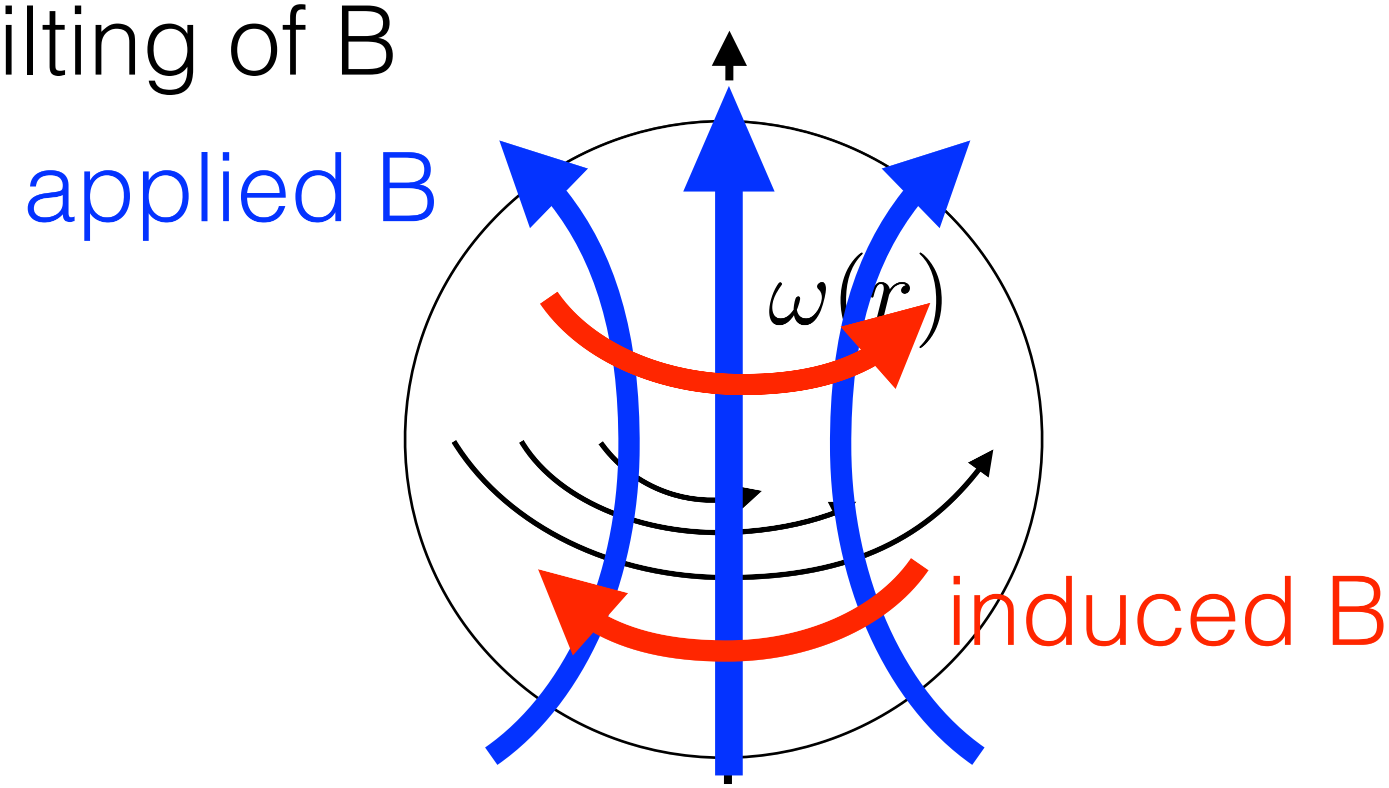


α^2 mechanism operating at large scale.

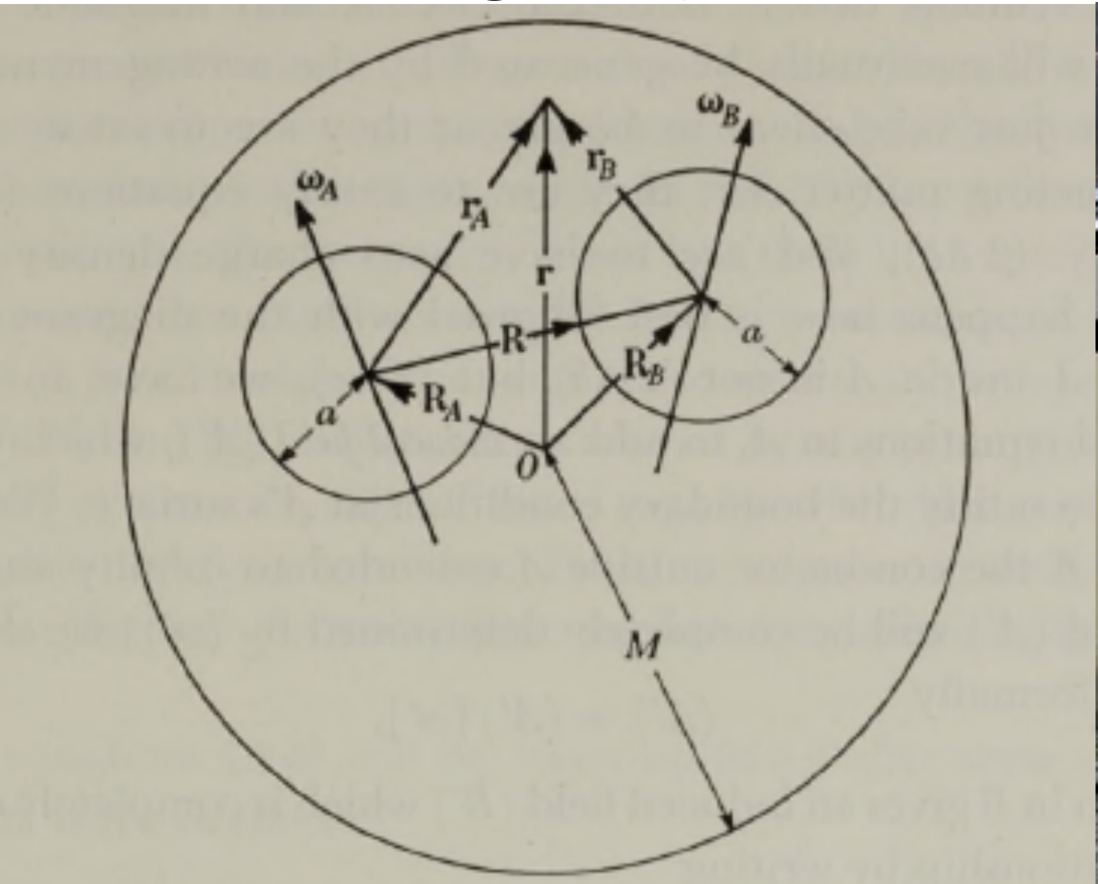
omega - effect

Shear flows: stretching and tilting of B



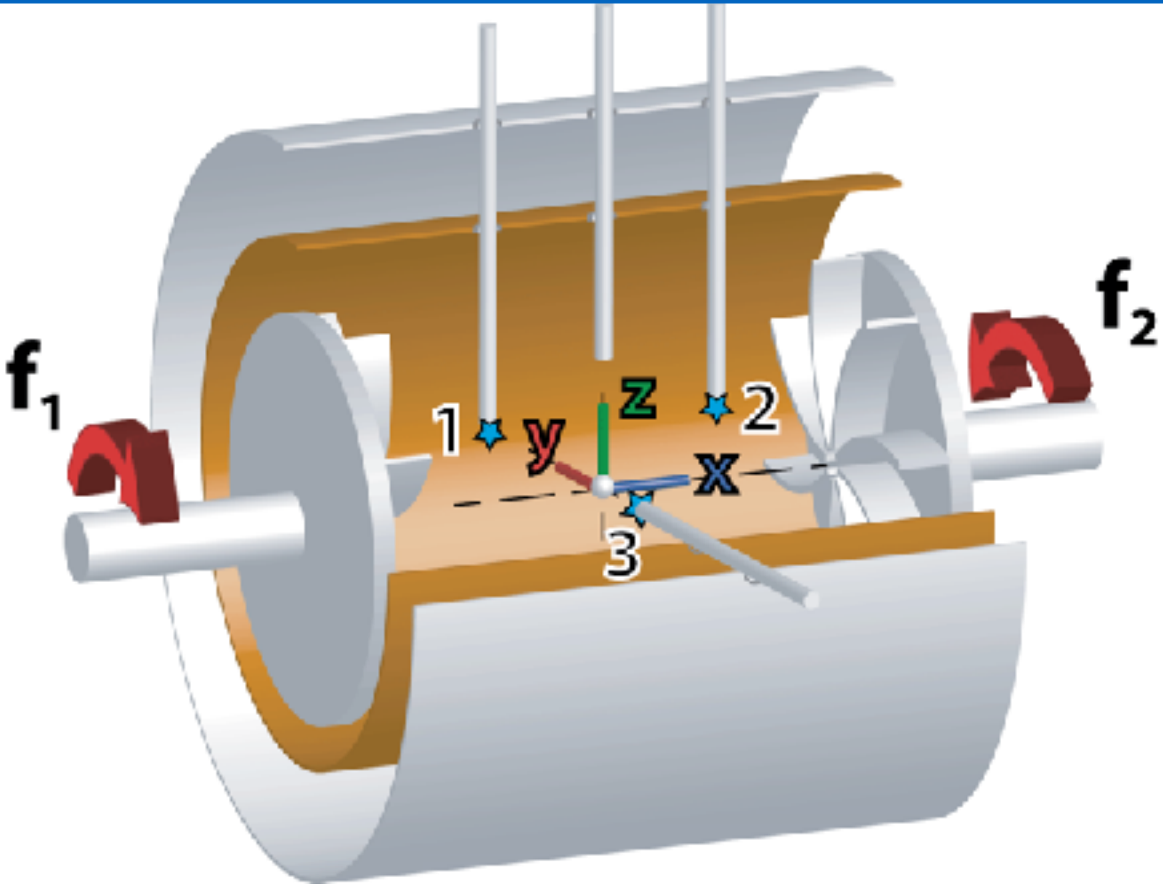


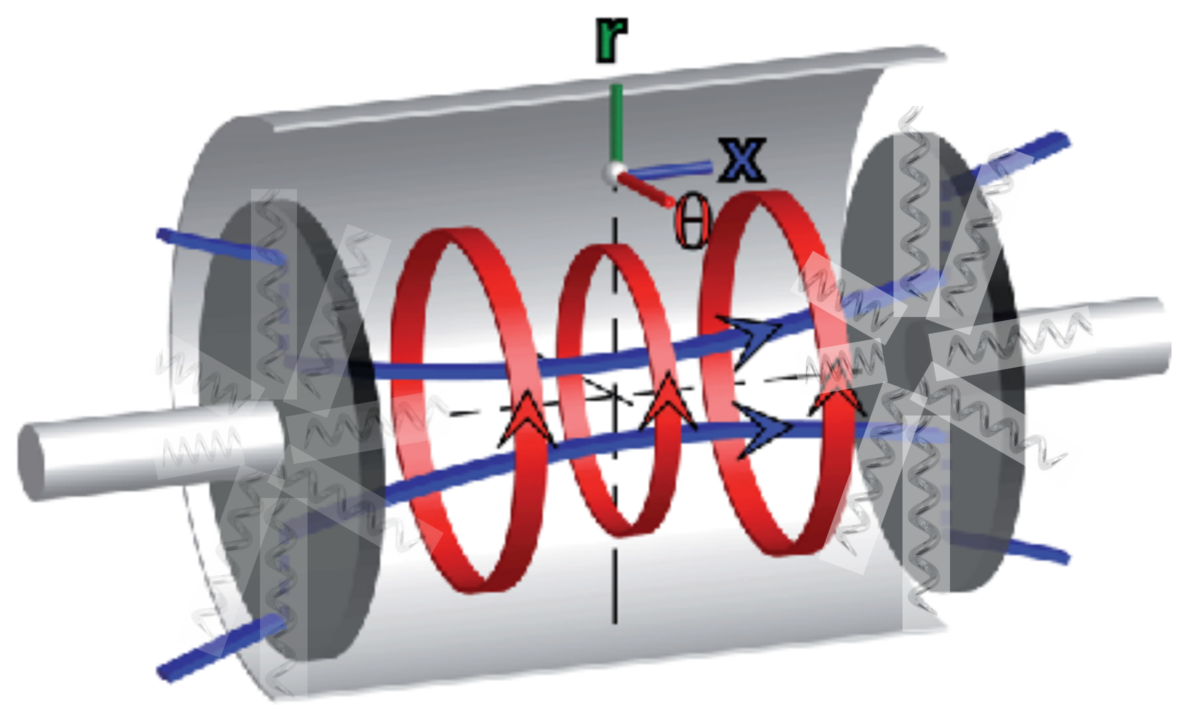
Herzenberg's dynamo: an ω^2 dynamo mechanism

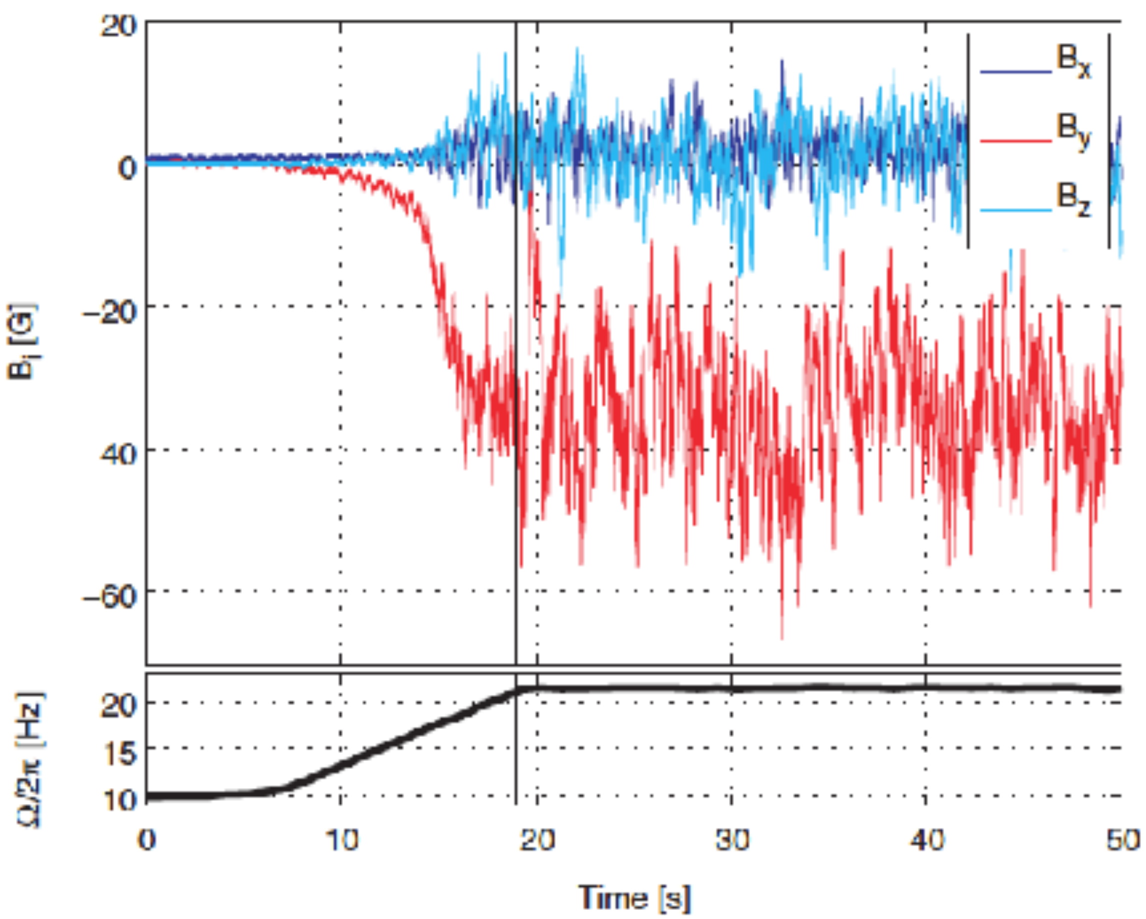


Experiment: Lowes & Wilkinson

VKS: alpha-omega mechanism







Counter-rotating disks:

ω-effect

axial B → azimuthal B

Radial Vortices near the blades:

α-effect

azimuthal B → axial B

Summary: magnetic field generation

Qualitatively:

- 2 main magnetic field amplification mechanisms:
  α-effect (helicity) and ω-effect (shear).
- A dynamo mechanism usually involves two steps:
  poloidal B ➡ toroidal B, and toroidal B ➡ poloidal B.
- In simple mechanisms, each step is achieved through alpha or omega effect:
  α^2, ω^2 and α - ω mechanisms.

Quantitatively:

one needs to solve the linear stability problem to find the unstable modes of B for a prescribed flow u.

Part II: dynamic problem

- How does the instability saturate?
- How intense is the resulting magnetic field?
- What is the influence of global rotation?

Kinematic vs dynamic studies

- **Kinematic dynamo** problem: which flows produce a dynamo? What is the threshold for dynamo action?

Imposed velocity field

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) + \eta \boldsymbol{\nabla}^2 \mathbf{B}$$

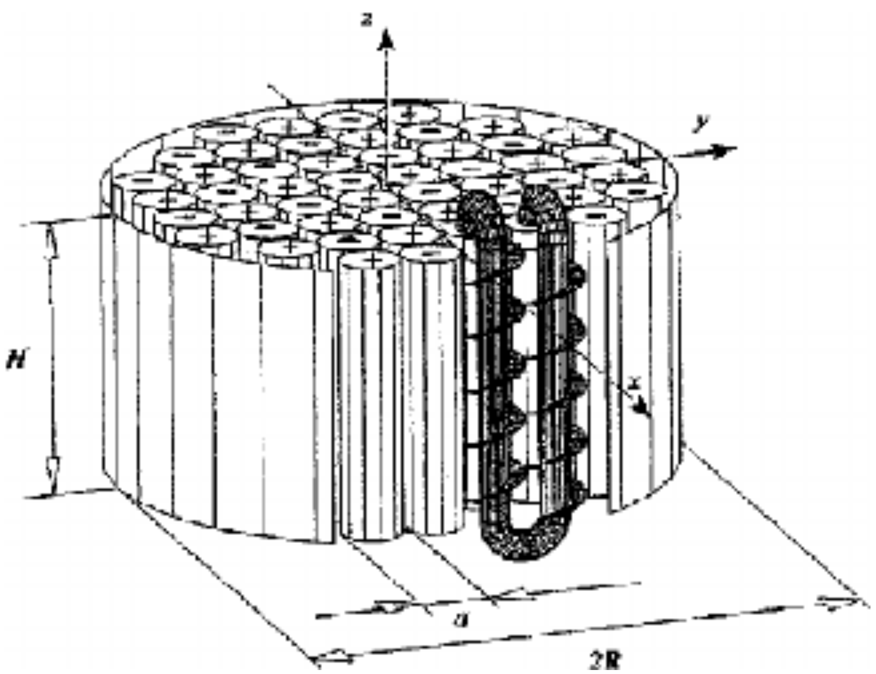
linear in **B** ➡ linear stability analysis to find Rm_c.

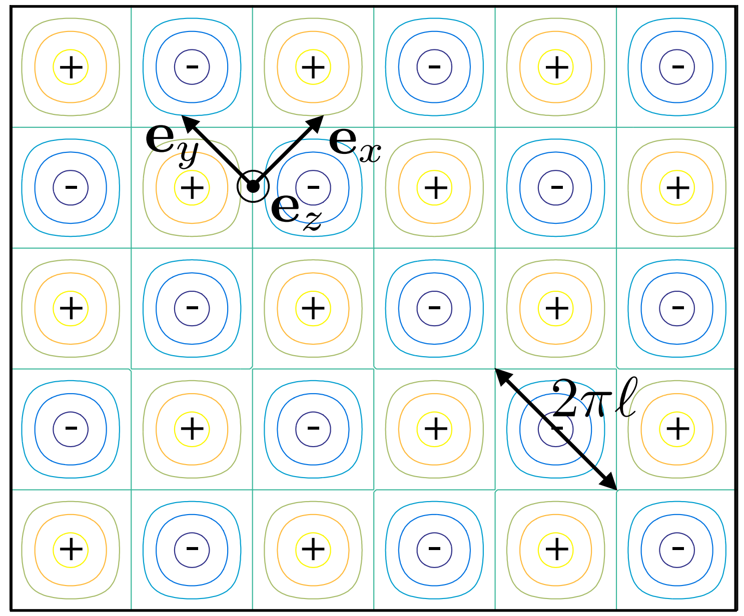
- **Dynamic** problem: How does the instability saturate? Intensity of the resulting magnetic field?

feedback through Lorentz force

$$\partial_t \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{u} + \frac{(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}}{\rho \mu_0}$$

Dynamic G.O. Roberts dynamo





$$\mathbf{F} = F \begin{cases} \cos(y/\ell) \\ \cos(x/\ell) \\ \sin(y/\ell) - \sin(x/\ell) \end{cases}$$

when B=0

$$\mathbf{v} = V \begin{cases} \cos(y/\ell) \\ \cos(x/\ell) \\ \sin(y/\ell) - \sin(x/\ell) \end{cases}$$

with $V = F\ell^2/\nu$ the flow speed of the purely hydro case (B=0).

Dynamic problem: velocity reduced by the Lorentz force

$$\partial_t \mathbf{v} + (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v} = -\boldsymbol{\nabla} p + \mathbf{F} + \frac{(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{b}}{\rho\mu_0} + \nu\Delta\mathbf{v}$$

Reduced alpha effect:

$$\alpha_{xx;yy} = -\frac{Rm^{(\ell)}V}{1 + \frac{\ell^2 B^2_{x;y}}{\rho\mu_0\eta\nu}}$$

Saturated state: alpha reduced to threshold value:

$$\frac{B^2\ell^2}{\rho\mu_0\eta^2} \sim Pm\left(Rm - Rm_c\right)$$

viscous scaling-law.

Dynamo saturation

- Dominant balance:

$$\partial_t \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \mathbf{u} + \frac{(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{B}}{\rho \mu_0}$$

viscous saturation: $\delta \mathbf{u} \sim \frac{\mathbf{B}^2 \ell}{\rho \mu_0 \nu}$

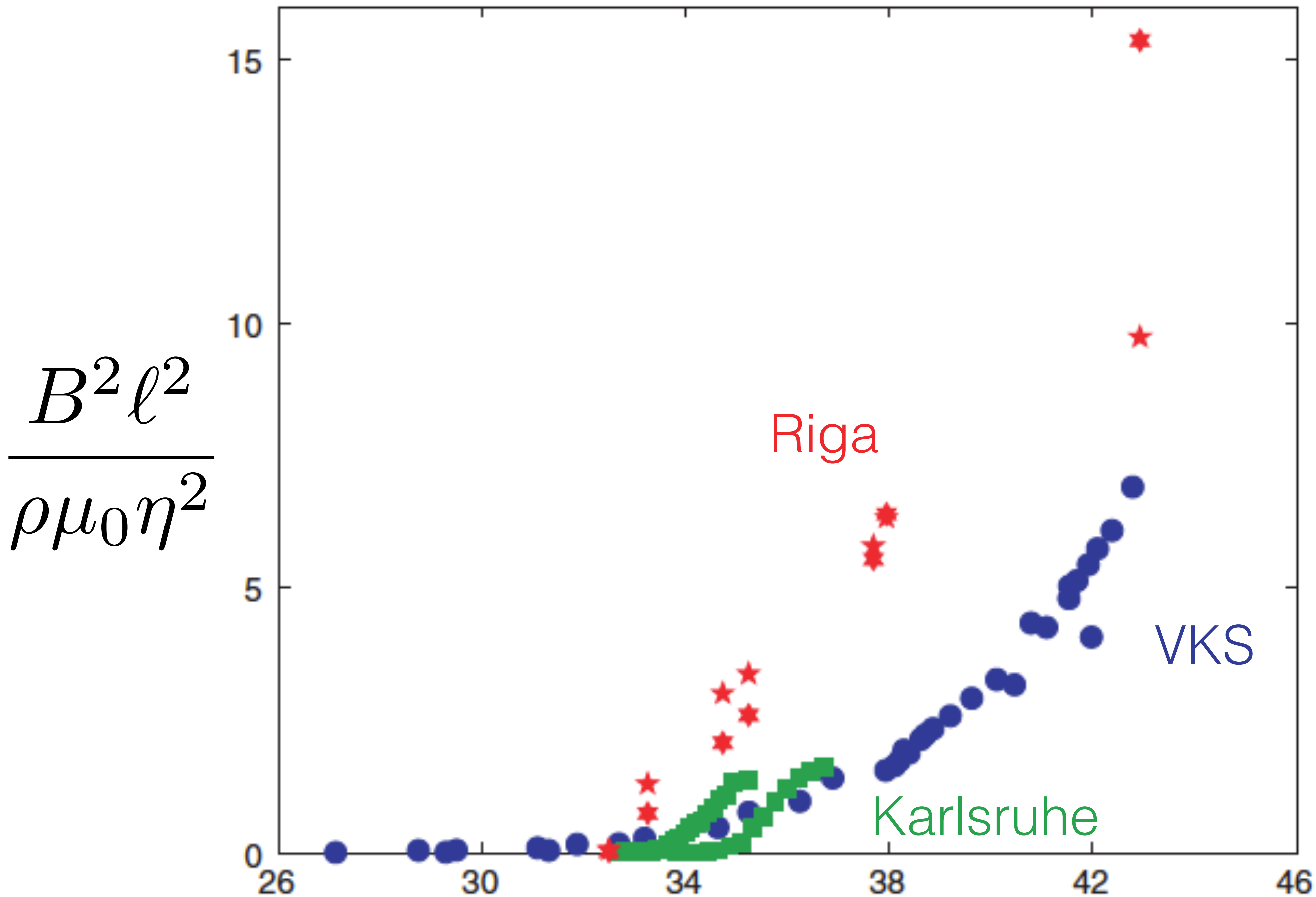
turbulent saturation: $\delta \mathbf{u} \sim \frac{\mathbf{B}^2}{\rho \mu_0 U}$

Coriolis saturation: $\delta \mathbf{u} \sim \frac{\mathbf{B}^2}{\rho \mu_0 \ell \Omega}$

- Magnetic energy at saturation [Pétrélis & Fauve, 2001]:

$$\frac{\delta \mathbf{u}\, \ell}{\eta} \sim Rm - Rm_c \quad \Rightarrow \quad \frac{B^2 \ell^2}{\rho \mu_0 \eta^2} \sim \begin{cases} Pm\,(Rm - Rm_c) & \text{viscous scaling} \\ Rm - Rm_c & \text{turbulent scaling} \\ \frac{\Omega \ell^2}{\eta}(Rm - Rm_c) & \text{magnetostrophic scaling} \end{cases}$$

Experimental data



[figure from Pétrélis et al. (2007)]

Experimentally, the magnetic energy is independent of ν.

Experimental vs theoretical data

- Experimentally: the magnetic energy is independent of ν.
- Theoretically, all model flows are driven viscously and produce a dynamo field obeying the viscous scaling regime.
- Numerically: turbulent saturation is not observed in 3D DNS, even at the lowest achievable Pm. [Oruba & Dormy (2014)]

➡ An important discrepancy between real flows on the one side, and numerical and theoretical models on the other side.

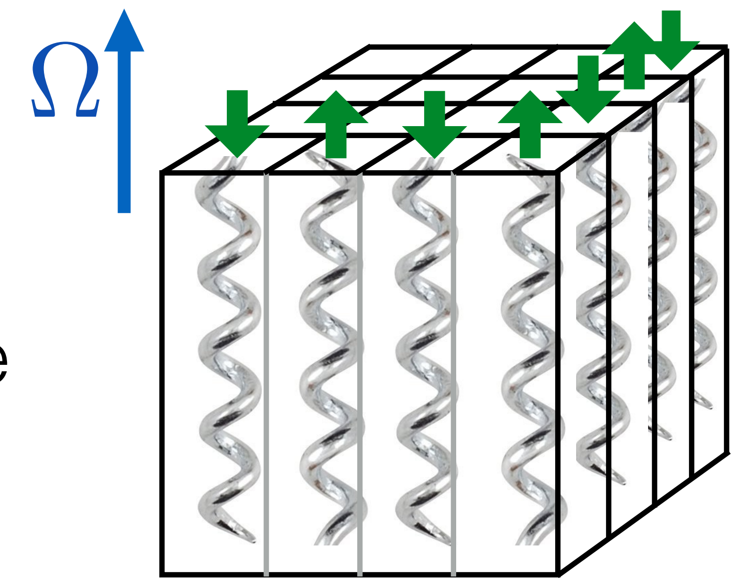
How to obtain the viscosity-independent scaling regime in a simple dynamo flow?

II.1) A numerical approach to observe the turbulent scaling regime.

with K. Seshasayanan and A. Alexakis

Bypassing the limitations of 3D DNS

- Rapidly rotating limit:



G.O. Roberts forcing in a rapidly rotating frame
➡ The flow becomes exactly 2D (but turbulent) above a critical value of Ω.

[BG, JFM 2015]

- Near the dynamo threshold, keep the first unstable mode only:

$$\mathbf{B}(x,y,z,t) = \mathbf{b}(x,y,t)e^{ikz} + c.c.$$

Lorentz force quadratic in B

→ ~~harmonic 2, balanced by Coriolis: velocity correction negligible for large Ω.~~

→ harmonic 0, balanced by advective term: feedback onto the 2D flow.

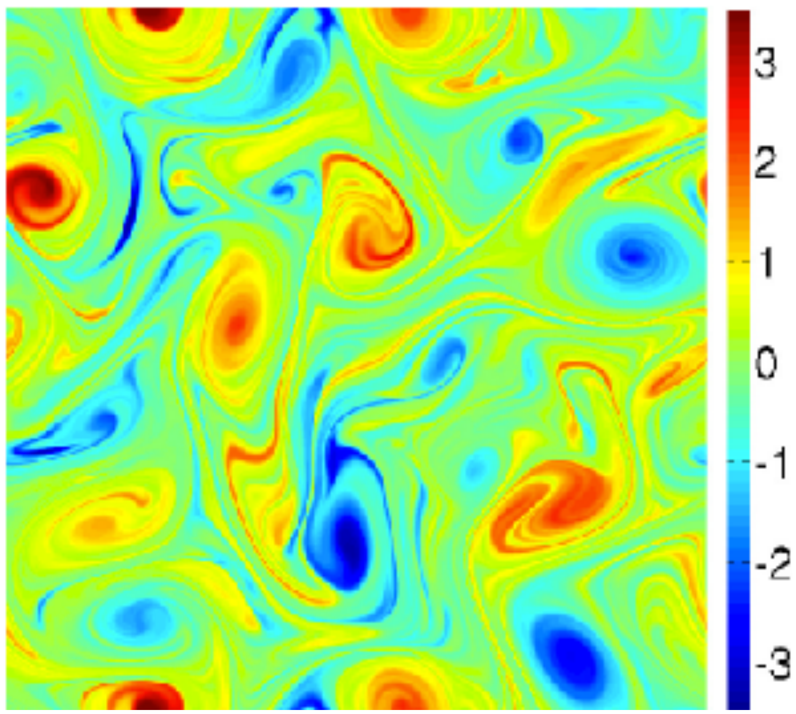
$$\partial_t \mathbf{b} = (\boldsymbol{\nabla}_\perp + ik\mathbf{e}_z) \times (\mathbf{u} \times \mathbf{b}) + \eta(\boldsymbol{\nabla}_\perp^2 - k^2)\mathbf{b}$$

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla}_\perp)\mathbf{u} = -\boldsymbol{\nabla}_\perp p - \gamma \mathbf{u}_\perp + \nu \boldsymbol{\nabla}_\perp^2 \mathbf{u} + \mathbf{f}(x,y) + \frac{1}{\rho\mu_0}[[(\boldsymbol{\nabla}_\perp + ik\mathbf{e}_z)\times\mathbf{b}]\times\mathbf{b}^* + c.c.]$$

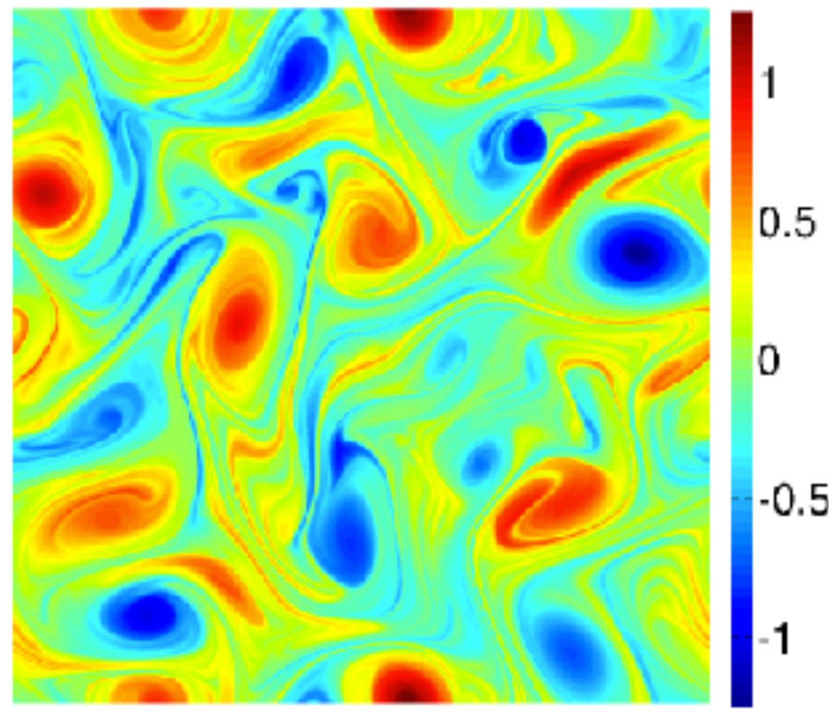
Quasi-2D equations (asymptotically valid).

Simulations at $Pm = 4.25\,10^{-5}$

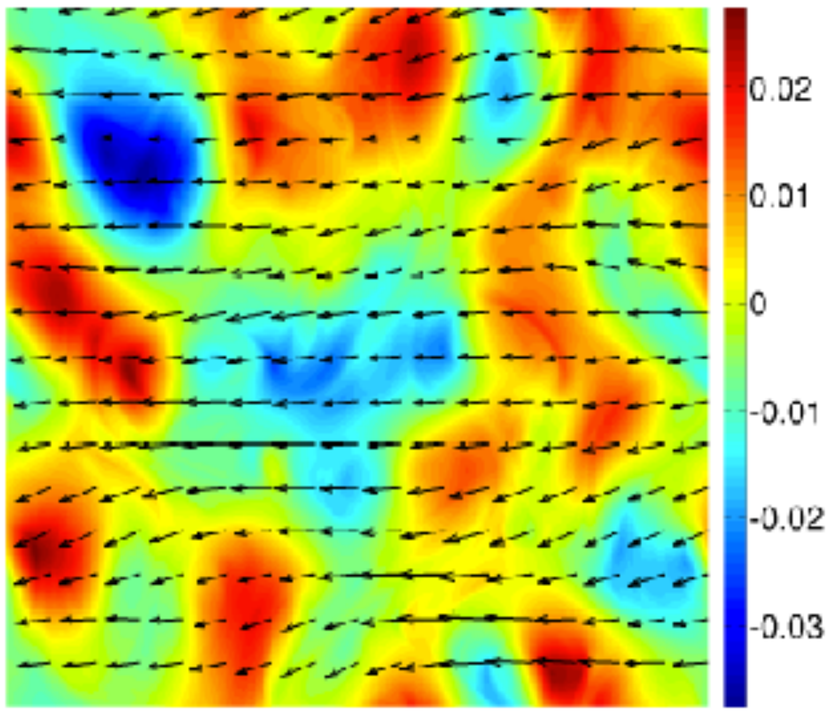
Vertical vorticity



Vertical velocity



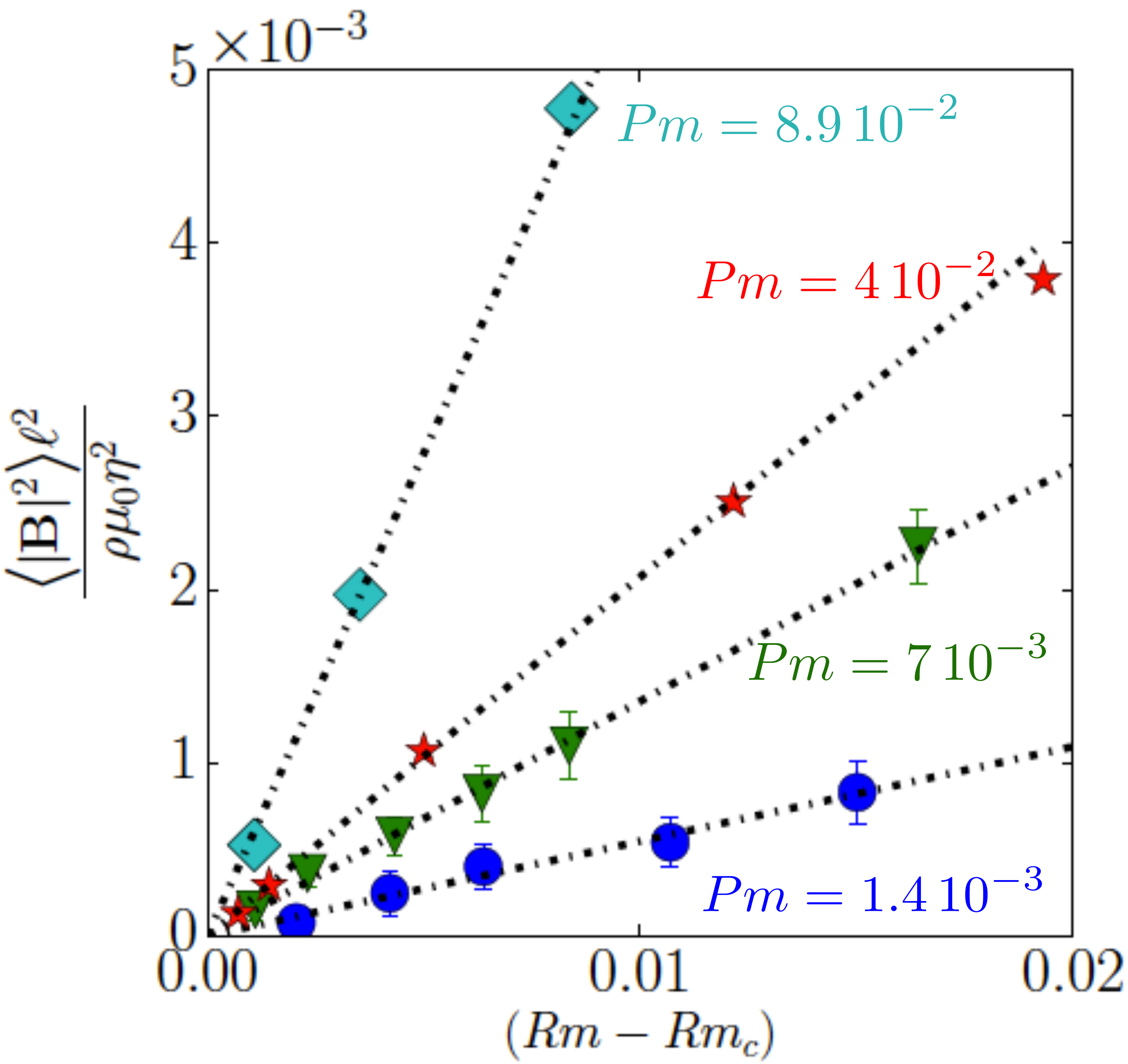
Magnetic field



α^2 dynamo mechanism:

- $\mathbf{B}$ almost uniform in a horizontal plane, with $\mathbf{B}_\perp \gg B_z$.
- Direction of $\mathbf{B}$ rotates with z.

Bifurcation curves



Standard supercritical bifurcation:

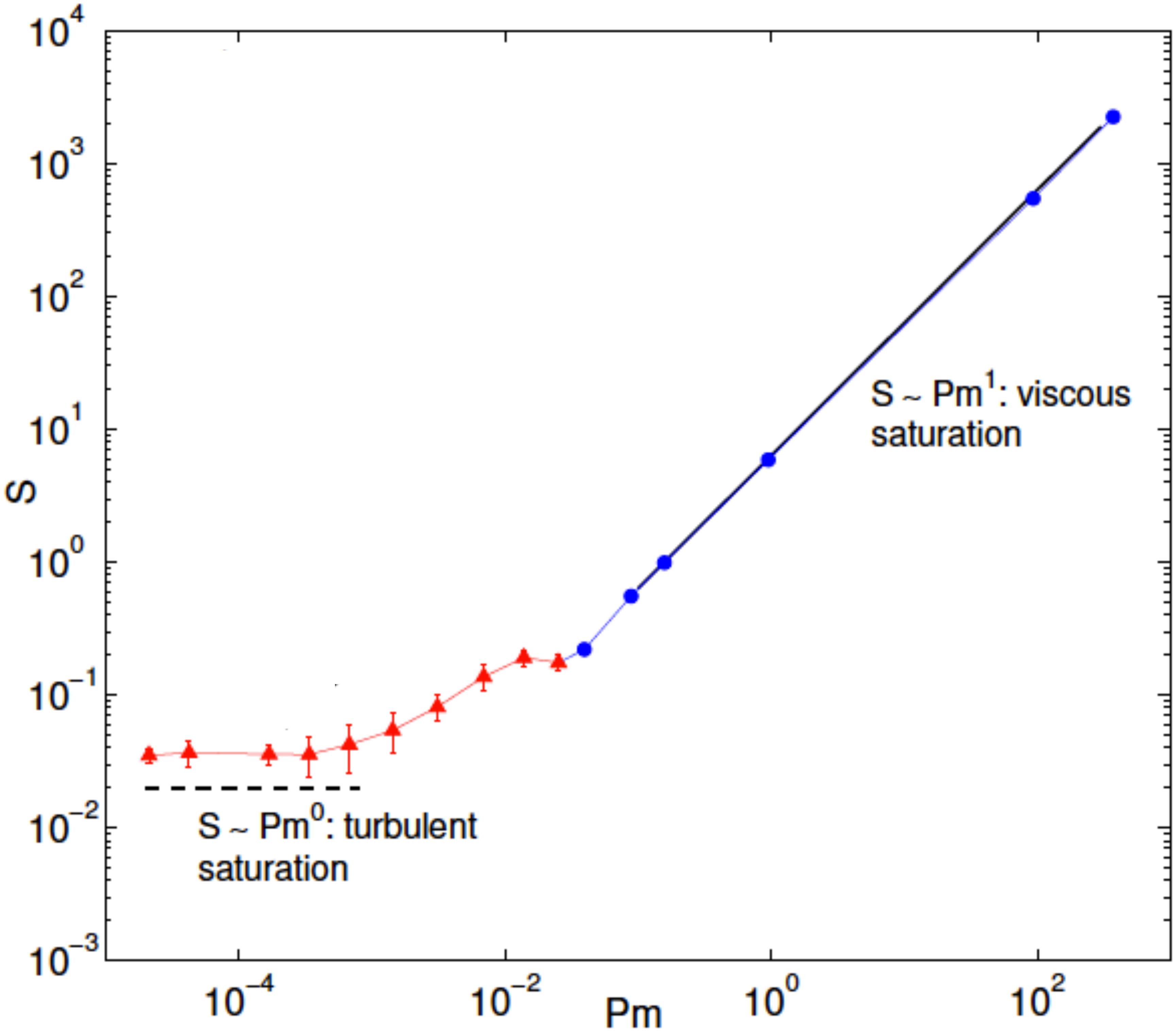
$$\frac{\langle|\mathbf{B}|^2\rangle\,\ell^2}{\rho\mu_0\eta^2} = S(Pm)\,(Rm - Rm_c)$$

viscous saturation: $S(Pm) \sim Pm$

turbulent saturation: $S(Pm) \sim 1$

➡ Extract S from each bifurcation curve and plot S vs Pm.

Transition to turbulent saturation



We can reach the Pm of liquid metals.

Very low Pm values for turbulent saturation: not achieved in current 3D DNS!

Summary of part II.1)

- Rapid rotation + vicinity of dynamo threshold
  → reduced set of equations, asymptotically exact.

- Can be simulated at very low Pm.
  → reaches the Pm of liquid metals.

- Observation of the turbulent saturation regime in a numerical solution of the MHD equations.

- This regime sets in at a surprisingly low value of Pm.
  → not yet observed in fully 3D DNS.

The role of global rotation

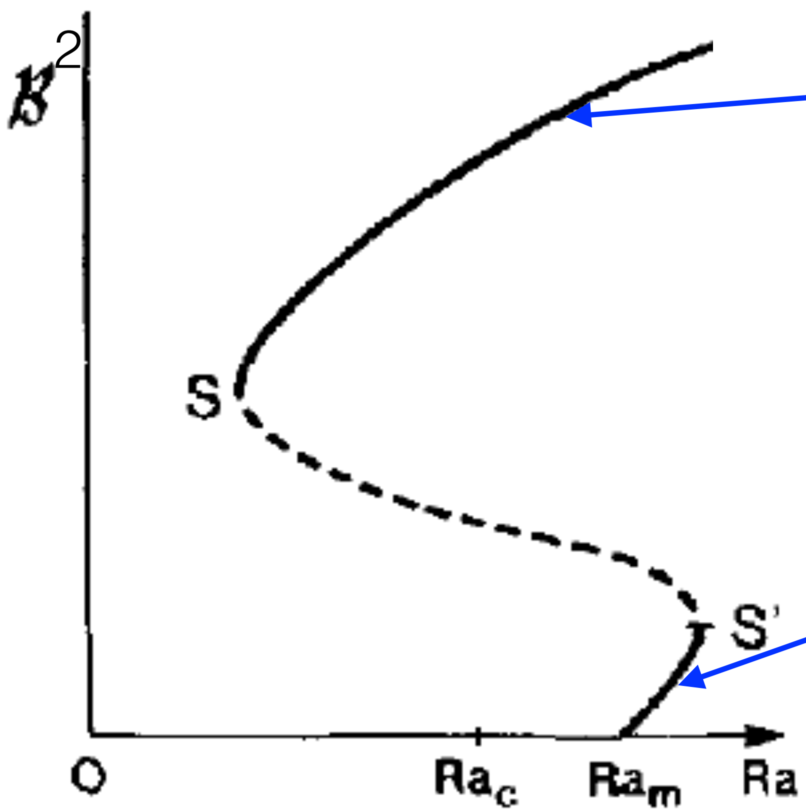
- On standard supercritical dynamos near threshold:

$$\frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{\rho\mu_0} \sim \mathbf{\Omega}\times\delta\mathbf{u} \quad \Rightarrow \quad \frac{B^2}{\rho\mu_0} \sim \eta\,\Omega\,(Rm - Rm_c)$$

Strong-field scaling regime

- On convective dynamos:

[after Roberts 1978]



Strong-field branch

$$\frac{B^2}{\rho\mu_0} \sim \eta\Omega$$

- Independent of ν
- « MAC balance »

Viscous weak-field branch

$$\frac{B^2}{\rho\mu_0} \sim \frac{\nu\eta}{L^2}$$

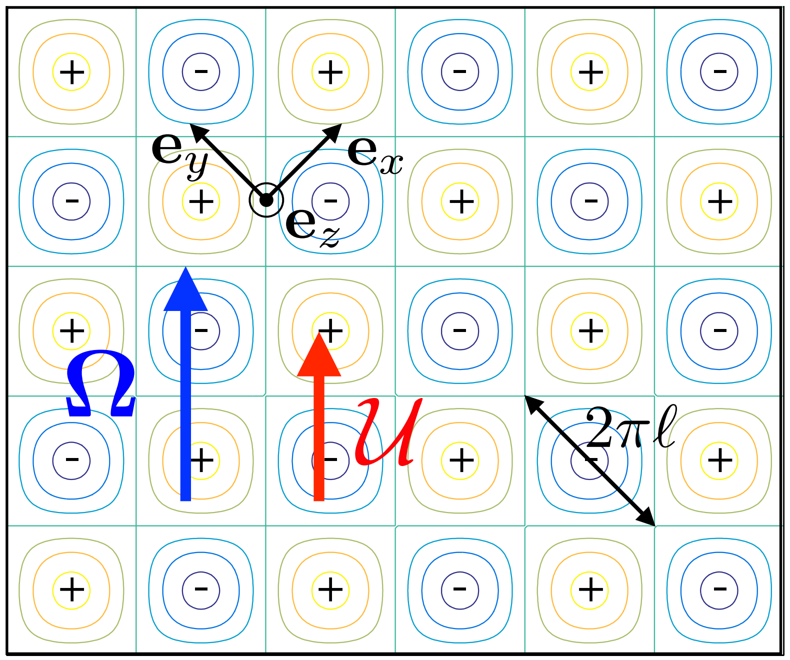
- In state-of-the-art DNS:

MAC balance locally observed in the bulk [Yadav et al. 2016, Schaeffer et al. 2017], but no clear strong-field scaling regime (B depends on ν) [Oruba & Dormy 2014].

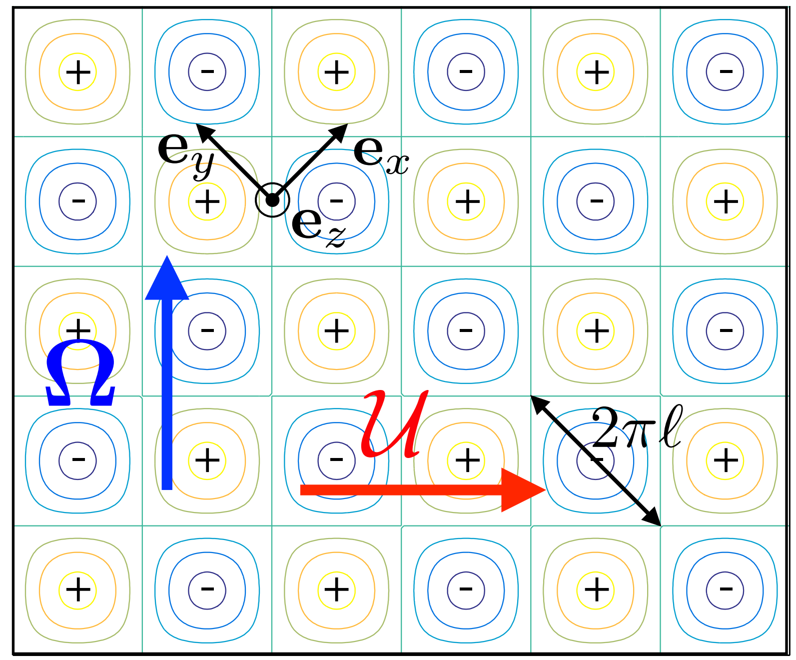
II.2) An analytical dynamo that achieves the magnetostrophic scaling regime

with K. Seshasayanan

Sweeping and rotation



Global rotation
+ « axial » flow



Global rotation
+ « zonal » flow

$$\mathbf{F} = F \begin{cases} \cos(y/\ell) \\ \cos(x/\ell) \\ \sin(y/\ell) - \sin(x/\ell) \end{cases}$$

Dimensionless parameters:

$$Re = \frac{\mathcal{U}\ell}{\nu} \qquad Ro = \frac{\mathcal{U}}{\ell\Omega} \qquad \mathcal{R} = \frac{\mathcal{U}\ell}{\eta}$$

New dimensionless numbers

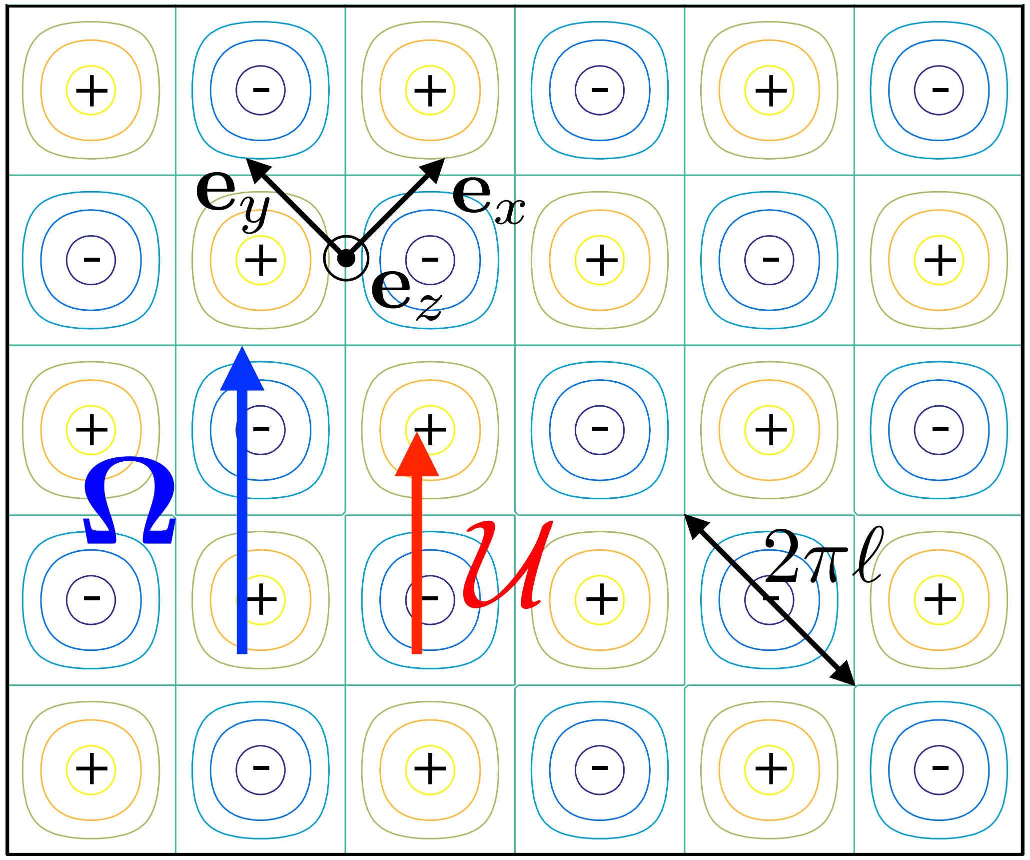
small-scale velocity of purely hydro state

$$Rm = \frac{V\ell}{\eta} \ll 1 \qquad \frac{\lambda}{\ell} \gg 1$$

standard separation of scales

Governing equations: axial case

magnetic field = $\mathbf{B}(z,t) + \mathbf{b}(x,y,z,t)$



Induction equation:

$$\mathcal{U}\left(\partial_x \mathbf{b} + \partial_y \mathbf{b}\right) = (\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{v} + \eta\boldsymbol{\nabla}^2\mathbf{b}$$

$$\partial_t \mathbf{B} = \boldsymbol{\nabla}\times\langle\mathbf{v}\times\mathbf{b}\rangle + \eta\boldsymbol{\nabla}^2\mathbf{B}$$

Navier-Stokes equation:

$$(\mathcal{U}\cdot\boldsymbol{\nabla})\mathbf{v} + 2\boldsymbol{\Omega}\times\mathbf{v} = -\boldsymbol{\nabla}p + \mathbf{F} + \frac{1}{\rho\mu_0}(\mathbf{B}\cdot\boldsymbol{\nabla})\mathbf{b} + \nu\boldsymbol{\nabla}^2\mathbf{v}$$

we compute once again the alpha effect coefficients.

Alpha effect for small viscosity

Alpha effect in terms of the dimensionless magnetic field $\mathcal{B}_{x;y} = \frac{B_{x;y}\,\ell}{\sqrt{\rho\mu_0}\eta}$

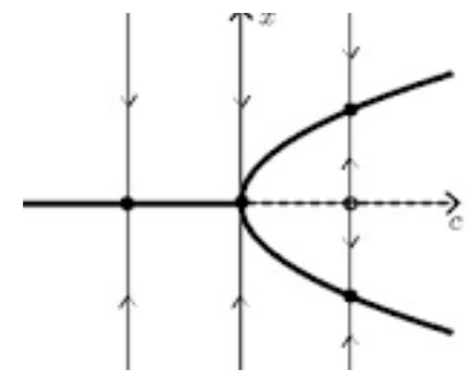
$$|\alpha_{xx;yy}| = \frac{\mathrm{Rm}\,V}{\frac{\mathcal{B}^4_{x;y}}{R^2(Ro^{-1}-1)^2} + 2\frac{\mathcal{B}^2_{x;y}}{Ro^{-1}-1} + 1 + R^2}$$

(for $Re \gg 1$)

We can guess the nature of the dynamo bifurcation:

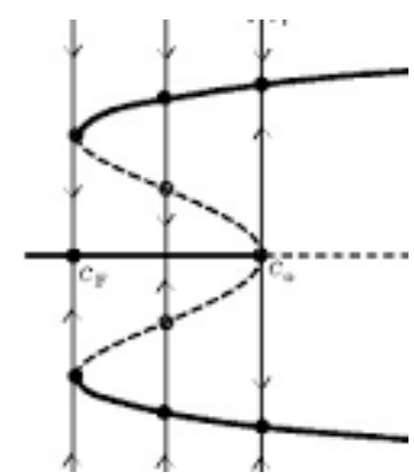
$Ro^{-1} > 1$: the first nonlinearity saturates the instability.

➡ **Supercritical** bifurcation.



$Ro^{-1} < 1$: the first nonlinearity does not saturate the instability.

➡ **Subcritical** bifurcation.



Full dynamo branch

Look for steady solutions. After integrating in z:

$$\frac{d\mathcal{B}_x}{dz} = \frac{-\mathrm{Rm}^2 R^2 \mathcal{B}_y}{\frac{\mathcal{B}_y^4}{(Ro^{-1}-1)^2} + 2R^2 \frac{\mathcal{B}_y^2}{Ro^{-1}-1} + R^2(1+R^2)}$$

$$\frac{d\mathcal{B}_y}{dz} = \frac{\mathrm{Rm}^2 R^2 \mathcal{B}_x}{\frac{\mathcal{B}_x^4}{(Ro^{-1}-1)^2} + 2R^2 \frac{\mathcal{B}_x^2}{Ro^{-1}-1} + R^2(1+R^2)}$$

Multiply the two equations leads to a conserved quantity relating the two components of B. Substituting into one equation and separating variables:

$$\mathrm{Rm}^2 \frac{\lambda}{\ell} - \frac{8i\sqrt{R^4+R^2}(M^4+R^2-2M^2R^2+R^4)M^2}{R^2(M^4-2M^2R^2)^{3/2}} \times \Bigg[M^2\mathcal{E}\left(i\sqrt{\frac{M^4-2M^2R^2}{R^2+R^4}}; i\sqrt{\frac{R^2+R^4}{M^4-2M^2R^2}}\right) + (R^2-M^2)\mathcal{F}\left(i\sqrt{\frac{M^4-2M^2R^2}{R^2+R^4}}; i\sqrt{\frac{R^2+R^4}{M^4-2M^2R^2}}\right)\Bigg]$$

dynamo branch

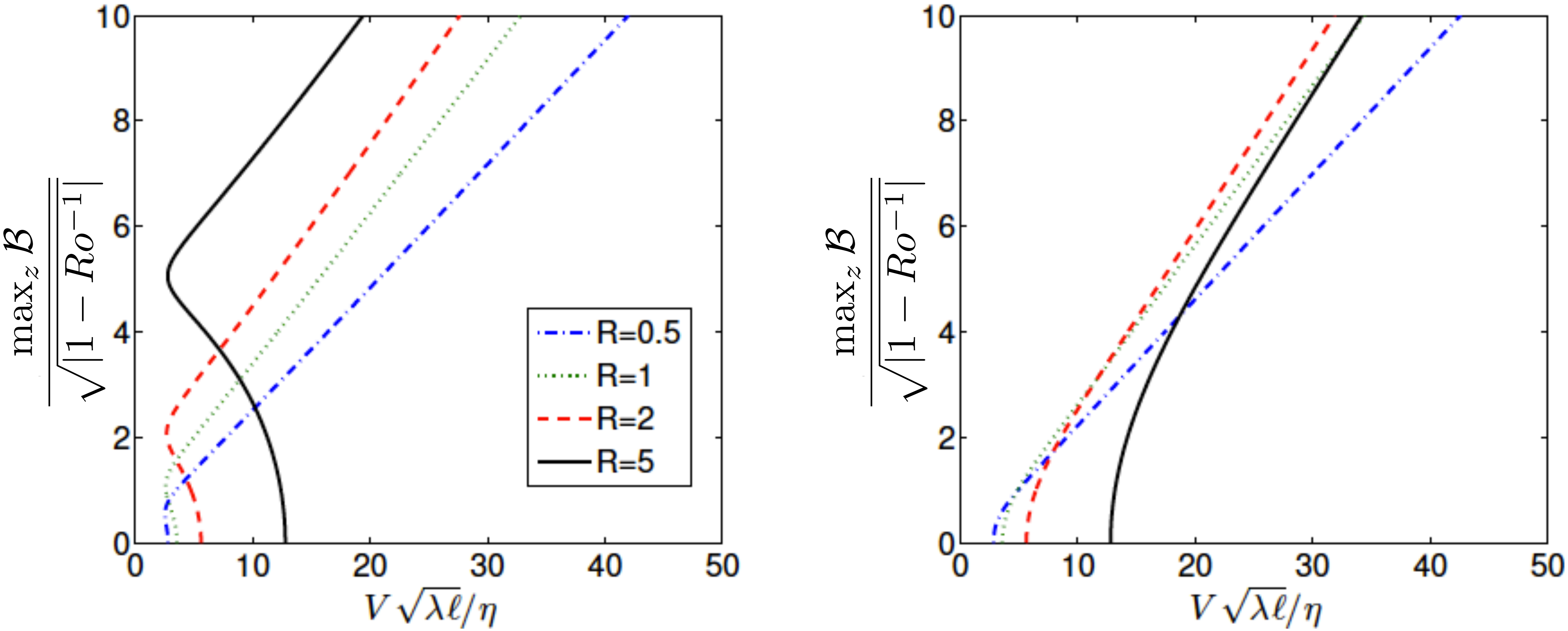
where:

$$M = \frac{\max_z \mathcal{B}_x}{\sqrt{|1-Ro^{-1}|}}$$

strong-field scaling at low Ro.

Dynamo branches

$Ro^{-1} < 1$ $Ro^{-1} > 1$



Inertial scaling regime without rotation.

Strong-field scaling regime at low Ro.

Scaling at large distance from threshold?

Precise limit compatible with the scale-separation approach:

$$\frac{\lambda}{\ell} \gg 1 \qquad Rm \ll 1$$

$$Rm\sqrt{\frac{\lambda}{\ell}} \gg 1 \qquad \frac{B\ell}{\sqrt{\rho\mu_0}\eta} \gg 1 \qquad \frac{\mathcal{U}}{V\sqrt{\lambda/\ell}} = \text{const.}$$

Taking the limiting expression of the dynamo branch we get:

- Without rotation:

$$\boxed{\frac{B^2}{\rho\mu_0} \sim \mathcal{U}^2 \sim V^2\frac{\lambda}{\ell}}$$

Equipartition scaling.

- Rapid rotation $|Ro| \ll 1$:

$$\boxed{\frac{B^2}{\rho\mu_0} \sim \Omega\,\ell\mathcal{U}}$$

Strong-field scaling, with η replaced by an eddy diffusivity.

Summary: magnetic field saturation

- The magnetic field **in state-of-the-art geodynamo DNS crucially depends on viscosity.**
- By contrast, experiments exhibit a **viscosity-independent scaling.**
- Analytically, scalings at low viscosity exemplified by the G.O. Roberts forcing together with **sweeping** $\mathcal{U}$ and **global rotation** Ω.

Close to threshold:

$$\frac{B^2\ell^2}{\rho\mu_0\eta^2} \sim \begin{cases} Rm - Rm_c & \text{turbulent scaling without rotation} \\ \dfrac{\Omega\ell^2}{\eta}(Rm - Rm_c) & \text{strong-field scaling for rapid rotation} \end{cases}$$

Far from threshold:

$$\frac{B^2}{\rho\mu_0} \sim \begin{cases} U^2 & \text{equipartition scaling without rotation} \\ U\ell\Omega & \text{strong-field scaling for rapid rotation} \end{cases}$$

Perspectives and challenges

- We have got the right force balance, how to get the right power balance?
- Realistic mechanical forcing?
- Convective forcing?

→ Could be addressed using state-of-the-art DNS of asymptotically reduced set of equations.

Thanks for your attention !

- Experiments: the VKS collaboration webpage
  http://perso.ens-lyon.fr/nicolas.plihon/VKS/index.php
- Saturation at low viscosity:
  [Seshasayanan, Gallet, Alexakis, « Transition to turbulent dynamo saturation », Phys. Rev. Lett., **119**, 204503 (2017)]
  [Seshasayanan & Gallet, « Dynamo saturation down to vanishing viscosity: strong-field and inertial scaling regimes. », JFM (2019).]