Wave–flow interactions in geophysical fluids
Part I

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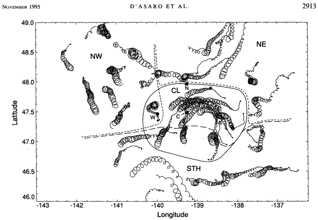
Outline

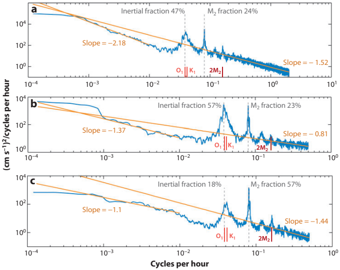
Geophysical fluids as two-time-scale systems

Observations. Mid-latitude atmosphere and ocean:

- time scales $\gg 1$ day,
- advective time scales, $T_\mathrm{a} = L/U$,
- large-scale motion near geostrophic and hydrostatic balance.

But fast oscillations, with $T \lesssim 1$ day, abound. (movie)





d'Asaro et al 1995

E_K vs ω at 27°N

Phillips & Rintoul 2000; Ferrari & Wunsch 2009

Geophysical fluids as two-time-scale systems

Fast oscillations: inertia-gravity waves, IGWs.

Two restoring mechanisms:

- density stratification, $\mathrm{d}\bar{\rho}/\mathrm{d}z < 0$,
- earth rotation.

Two key parameters:

- buoyancy (Brunt–Väisälä) frequency

$$\boxed{N = \left(-g\rho_0^{-1}\mathrm{d}\bar{\rho}/\mathrm{d}z\right)^{1/2}},$$

- inertial (Coriolis) frequency

$$\boxed{f = 2\Omega\sin(\text{latitude})}.$$

Take f and N constants with $f < N$.

Geophysical fluids as two-time-scale systems

IGWs frequencies,

$$f \le |\pm\sqrt{f^2\cos^2\theta + N^2\sin^2\theta}| \le N\,,$$

i.e. minutes $\lesssim T \lesssim$ hours,

The time-scale separation estimated by the Rossby number

$$\epsilon = \frac{U}{fL} \ll 1.$$

Ocean: $\epsilon \sim 0.01 - 0.1$; atmosphere: $\epsilon \sim 0.1 - 1$.
At small scales, near boundaries: $\epsilon \sim 1$.

Balanced flow and IGWs

Boussinesq equations for stratified–rotating fluid:

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f\boldsymbol{z} \times \boldsymbol{u} = -\nabla \phi + b\boldsymbol{z},$$

$$\partial_t b + \boldsymbol{u} \cdot \nabla b + N^2 w = 0,$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

with

- density $\rho(\boldsymbol{x}, t) = \rho_0 + \bar{\rho}(z) + \tilde{\rho}(\boldsymbol{x}, t)$,
- buoyancy acceleration $b = -g\tilde{\rho}/\rho_0$.

Potential-vorticity conservation:

$$(\partial_t + \boldsymbol{u} \cdot \nabla)q = 0 \quad \text{with} \quad q = (f\boldsymbol{z} + \nabla \times \boldsymbol{u}) \cdot (N^2\boldsymbol{z} + \partial_z b).$$

Balanced flow and IGWs

Linear modes: $(\boldsymbol{u}, b, \phi) \propto \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$.

Dispersion relation:

- balanced mode (= vortical mode): $\omega = 0$,
- IGWs, inertia-gravity waves:

$$\begin{aligned}\omega &= \pm\sqrt{f^2 k_\mathrm{v}^2 + N^2 k_\mathrm{h}^2}/|\boldsymbol{k}| \\ &= \pm\sqrt{f^2\cos^2\theta + N^2\sin^2\theta}\end{aligned}$$

Polarisation relations:

- $\boldsymbol{u}\cdot\boldsymbol{k} = 0$,
- balanced mode in geostrophic and hydrostatic equilibrium:

$$f\boldsymbol{z}\times\boldsymbol{u}_\mathrm{h} = \nabla_\mathrm{h}\phi, \quad w = 0, \quad \partial_z\phi = b,$$

- $q = 0$ for IGWs.

Balanced dynamics

Balanced view of geophysical fluids:

- balanced flows and IGWs are weakly coupled because $\epsilon \ll 1$,
- forcing is mainly large-scale and balanced (baroclinic instability).

Simplest approximation of ocean/atmosphere dynamics: quasigeostrophic approximation:

- impose geostrophic and hydrostatic balance,

$$u_x + v_y = 0, \quad w = 0 \;\Rightarrow\; \boldsymbol{u} = (-\psi_y, \psi_x, 0).$$

- no IGWs (to leading order),

$$(\partial_t + \boldsymbol{u} \cdot \nabla) q = 0,$$
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z} \left(\frac{f^2}{N^2} \frac{\partial}{\partial z} \right) \right) \psi = q.$$

Balanced dynamics

Geostrophic turbulence

2 quadratic invariants,

- energy $E = \frac{1}{2} \int \left(\psi_x^2 + \psi_y^2 + f^2 \psi_z^2 / N^2 \right) \mathrm{d}\boldsymbol{x}$,
- enstrophy $Z = \int q^2 \, \mathrm{d}\boldsymbol{x}$.

2 inertial ranges,

- forward enstrophy cascade, $E(k) \propto k^{-3}$ for $k > k_{\text{forcing}}$,
- backward energy cascade $E(k) \propto k^{-5/3}$ for $k < k_{\text{forcing}}$.

Isotropy in $(x, y, Nz/f)$-coordinates, i.e. isotropic $(\boldsymbol{k}_{\mathrm{h}}, f\, k_{\mathrm{v}}/N)$-coordinates.

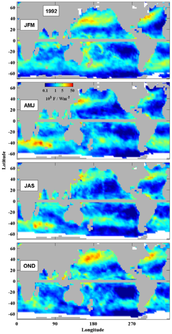
movie, J Weiss

Atmosphere/ocean turbulence:

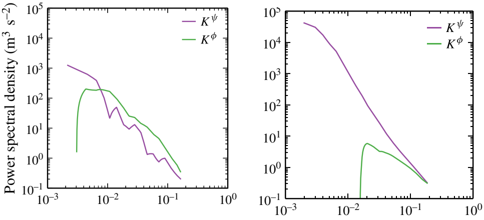
in forward-enstrophy regime, $E(k) \propto k^{-3}$.

Inertia-gravity waves

- about 10% of ocean kinetic energy is in IGWs,
- 50% kinetic energy is near inertial: $\omega \approx f$,
- forced by winds and tides, topography…
- broad range of spatial scales.



Alford et al 2003



E_K vs $k/2\pi$ in Subtropical North Pacific and Gulf Stream

Bühler, Callies & Ferrari 2014

Inertia-gravity waves

IGWs matter:

- vertical motion $\Rightarrow$ biology,
- vertical shear, instability, turbulence $\Rightarrow$ mixing,
- mixing $\Rightarrow$ pollutant dispersion,
- stratification, hence large-scale circulation.

IGWs impact the balanced motion:

- diapycnal mixing, Munk & Wunsch 1998
- energy sink for balanced turbulence.

Topics of lectures

I. Scattering of IGWs in geostrophic turbulence:
- wave propagation in random flows,
- internal tides,
- atmosphere/ocean energy spectra.

II. IGW feedback:
- ocean energy-dissipation puzzle,
- spontaneous wave generation,
- stimulated wave generation.

IGW scattering

with M Savva & H Kafiabad

What is the generic effect of a turbulent flow on NIWs?

Motivation:

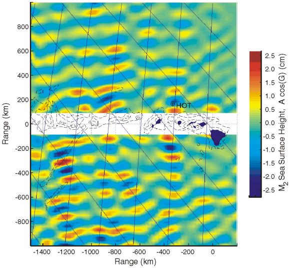
- impact of IGW on large-scale circulation,
- IGWs cascade to small scales and dissipation, Bartello 1995

IGW scattering

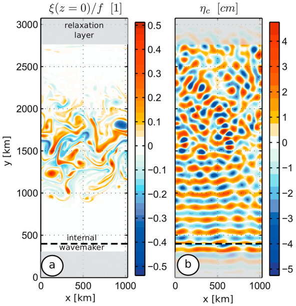
with M Savva & H Kafiabad

- impact of flow on internal tides, relevant to satellite altimetry,

Dushaw 2001



Ponte & Klein 2015



IGW scattering

No spatial scale separation: $kL_{\text{flow}} = O(1)$: scattering.

Assume:

- random flow, with homogenous statistics,
- weak flow, $\text{Ro} \ll 1$:
  IGW dispersion $\gg$ advection, refraction,
- IGWs modulated over scale $\ell \gg k^{-1} \sim L_{\text{flow}}$.

Apply theory of wave scattering in random media to obtain an equation governing the wavenumber-resolving energy density $\boxed{a(\boldsymbol{x}, \boldsymbol{k}, t)}$.

Rhyzhik, Keller & Papanicolaou 1996

IGW scattering

Starting point: Boussinesq system linearised about a fixed QG flow: $\boldsymbol{U} = \epsilon^{1/2}\nabla^{\perp}\psi$, $\epsilon \ll 1$.

Write governing equations for the perturbation as

$$\partial_t \boldsymbol{\phi} + \mathcal{L}(\nabla)\boldsymbol{\phi} + \epsilon^{1/2}\mathcal{N}(\boldsymbol{x}/\epsilon, \nabla)\boldsymbol{\phi} = 0\,,$$

with spec $\mathcal{L}(\nabla) = \{\mathrm{i}\omega\}$.

Define the Wigner matrix

$$W(\boldsymbol{x}, \boldsymbol{k}, t) = \int \boldsymbol{\phi}(\boldsymbol{x} + \epsilon\boldsymbol{y}/2, t)\boldsymbol{\phi}^*(\boldsymbol{x} - \epsilon\boldsymbol{y}/2, t)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}$$

and write its evolution. Solve pertubatively using multiple scale method: $\boldsymbol{\xi} = \boldsymbol{x}/\epsilon$:

$$W(\boldsymbol{x}, \boldsymbol{k}, t) = W_0(\boldsymbol{x}, \boldsymbol{k}, t) + \epsilon^{1/2}W_1(\boldsymbol{x}, \boldsymbol{\xi}, \boldsymbol{k}, t) + \cdots$$

IGW scattering

To leading order,

$$W_0(\boldsymbol{x},\boldsymbol{k},t) = \sum_{s=\pm} a_\pm(\boldsymbol{x},\boldsymbol{k},t)\boldsymbol{b}(\boldsymbol{x},\boldsymbol{k})\boldsymbol{b}^*(\boldsymbol{x},\boldsymbol{k}),$$

where $\boldsymbol{b}$ encodes the polarisation relations.

Next order, using ergodicity: kinetic equation

$$\frac{\partial a}{\partial t} + \nabla_{\boldsymbol{k}}\omega \cdot \nabla_{\boldsymbol{x}} a - \nabla_{\boldsymbol{x}}\omega \cdot \nabla_{\boldsymbol{k}} a = \int_{\mathbb{R}^3} \sigma(\boldsymbol{k},\boldsymbol{k}')a(t,\boldsymbol{x},\boldsymbol{k}')\mathrm{d}\boldsymbol{k}' - \Sigma(\boldsymbol{k})a(t,\boldsymbol{x},\boldsymbol{k}),$$

- $\sigma(\boldsymbol{k},\boldsymbol{k}')$ is the differential scattering cross-section,
- $\Sigma(\boldsymbol{k}) = \int \sigma(\boldsymbol{k},\boldsymbol{k}')\mathrm{d}\boldsymbol{k}'$, total cross-section.

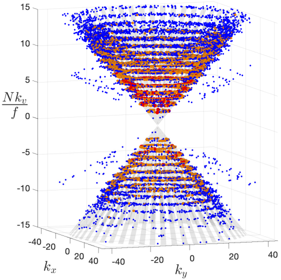
IGW scattering

The differential scattering cross section $\sigma(\boldsymbol{k}, \boldsymbol{k}')$ encodes the impact of geostrophic turbulence on IWGs:

$$\sigma(\boldsymbol{k}, \boldsymbol{k}') \propto \delta(\omega(\boldsymbol{k}) - \omega(\boldsymbol{k}'))\, E(\boldsymbol{k} - \boldsymbol{k}'),$$

with $E(\boldsymbol{k})$ the kinetic energy spectrum of the quasigeostrophic flow.

- scattering results from resonant triads: IGWs + flow = IGWs,
- strength linear in $E(\boldsymbol{k})$,
- energy transfers restricted to constant-energy surface.



IGW scattering: barotropic flow

A model of internal tide propagation assumes barotropic flow:

$$\partial_z \boldsymbol{U} = 0.$$

In a finite-depth ocean, with hydrostatic approximation, vertical mode decomposition:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_n \boldsymbol{u}_n(x, y, t) \sin(n\pi z/H).$$

IGW dispersion relation:

$$\omega = \pm\sqrt{f^2 + c_n^2 |\boldsymbol{k}_\mathrm{h}|^2}, \quad c_n = NH/(n\pi)$$

(cf. shallow-water model).

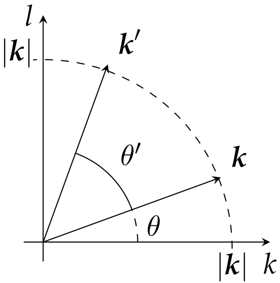
IGW scattering: barotropic flow

The scattering cross-section is given by

$$\sigma(\boldsymbol{k},\boldsymbol{k}') = \frac{2\pi}{g h_n \omega^3 |\boldsymbol{k}|^3}\Big\{|\boldsymbol{k}'\times\boldsymbol{k}|^2\big[(\omega^2+f^2)\boldsymbol{k}\cdot\boldsymbol{k}' - f^2|\boldsymbol{k}|^2\big]^2 + f^2\omega^2\big[|\boldsymbol{k}'\times\boldsymbol{k}|^2 + \boldsymbol{k}\cdot\boldsymbol{k}'(|\boldsymbol{k}|^2 - \boldsymbol{k}\cdot\boldsymbol{k}')\big]^2\Big\}\delta(|\boldsymbol{k}|-|\boldsymbol{k}'|)\hat{E}(\boldsymbol{k}'-\boldsymbol{k}),$$

using $\boldsymbol{k}=\boldsymbol{k}_{\mathrm{h}}$.

- transfers restricted to $|\boldsymbol{k}|=|\boldsymbol{k}'|$, ie $\omega(\boldsymbol{k})=\omega(\boldsymbol{k}')$,
- for isotropic flows, $\sigma=\sigma(|\boldsymbol{k}|,\theta')$,
- since $|\boldsymbol{k}|$ is fixed, scattering in angular coordinate only.



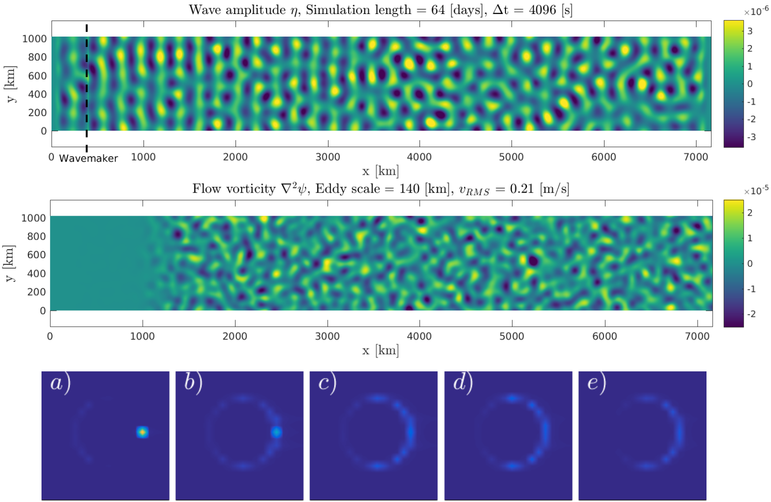
IGW scattering: barotropic flow

Ignoring spatial dependence, with $a(\boldsymbol{k}, t) = \sum_n a_n(t) \mathrm{e}^{in\theta}$, the kinetic equation reduces to

$$\partial_t a_n = (\lambda_n - \lambda_0) a_n \,,$$

with $\lambda_n = \int_0^\pi \sigma(|\boldsymbol{k}|, \theta) \cos(n\theta) \, \mathrm{d}\theta$.

- describes relaxation of IGWs towards isotropy,
- cf diffusion, $(\lambda_n - \lambda_0) \mapsto -\kappa n^2$,
- two time scales:
  - λ_0^{-1}, scattering time scale,
  - $(\lambda_1 - \lambda_0)^{-1}$, isotropisation time scale,
- equivalent spatial scales: multiply by $c_\mathrm{g} = \partial_k \omega$.

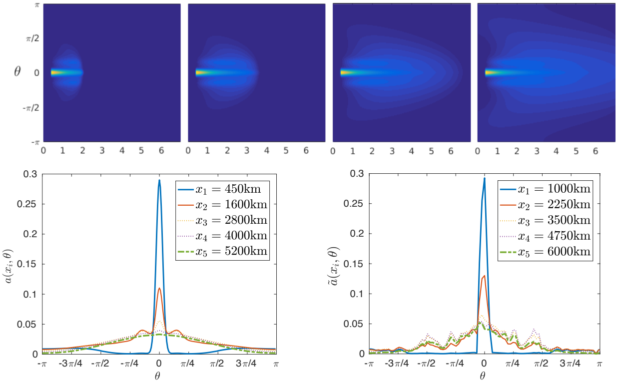


$$T_{\text{scatter}} \simeq 2.7 \text{ days}, \implies L_{\text{scatter}} \simeq 659\text{km}$$
$$T_{\text{isotropic}} \simeq 18 \text{ days}, \implies L_{\text{isotropic}} \simeq 4334\text{km}$$

IGW scattering: barotropic flow

With $\partial_y = 0$ and $|\boldsymbol{k}| = \text{const}$, $a(\boldsymbol{x}, \boldsymbol{k}, t) = a(x, \theta, t)$ solves

$$\partial_t a(x,\theta,t) + |\boldsymbol{c}_{\mathrm{g}}| \cos\theta\, \partial_x a(x,\theta,t) = \int_0^{2\pi} \sigma(\theta-\theta') a(x,\theta',t)\,\mathrm{d}\theta' - \Sigma a(x,\theta,t).$$



Wavenumber diffusion

The kinetic equation

$$\frac{\partial a}{\partial t} + \nabla_{\boldsymbol{k}}\omega \cdot \nabla_{\boldsymbol{x}} a - \nabla_{\boldsymbol{x}}\omega \cdot \nabla_{\boldsymbol{k}} a = \int_{\mathbb{R}^3} \sigma(\boldsymbol{k}, \boldsymbol{k}') a(t, \boldsymbol{x}, \boldsymbol{k}') \mathrm{d}\boldsymbol{k}' - \Sigma(\boldsymbol{k}) a(t, \boldsymbol{x}, \boldsymbol{k}),$$

can be approximated for short IGWs: $|\boldsymbol{k}| \gg |\boldsymbol{K}|$.
It reduces to the diffusion equation

$$\partial_t a + \boldsymbol{c} \cdot \nabla_{\boldsymbol{x}} a = \nabla_{\boldsymbol{k}} \cdot (\mathsf{D} \cdot \nabla_{\boldsymbol{k}} a),$$

where $\boldsymbol{c} = \nabla_{\boldsymbol{k}}\omega$ is the group velocity and

$$\mathsf{D}_{ij}(\boldsymbol{k}) = -\frac{1}{2} k_m k_n \int_{-\infty}^{\infty} \frac{\partial^2 \Pi_{mn}}{\partial x_i \partial x_j}(\boldsymbol{c}(\boldsymbol{k})s)\, \mathrm{d}s, \quad \Pi_{mn}(\cdot) = \langle U_m(\boldsymbol{x}+\cdot) U_n(\boldsymbol{x}) \rangle$$

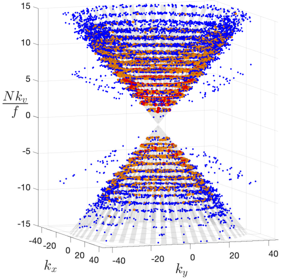
McComas & Bretherton 1977, Müller 1976, 1977, Bal et al 2010

Also directly from WKB (geometric-optics) approximation or diffusion approximation to continuous-time random walk.

Wavenumber diffusion

Energy transfers restricted to $\omega(\boldsymbol{k}) = \text{const}$:

- $\boldsymbol{D} \cdot \boldsymbol{c} = 0$:,
- results from resonant triads $\omega - \omega + 0 = 0$.



Use spherical coordinates (k, ϕ, θ) and assume $\nabla_{\boldsymbol{x}} a = \partial_\phi a = 0$.

Energy density $e(k,t) \propto k^2 a(k,t)$ satisfies

$$\partial_t e = \partial_k \left(Q k^5 \partial_k \left(k^{-2} e \right) \right),$$

where $Q = Q[N, f, \theta, E_{\text{geo}}(K_\text{h}, K_\text{v})]$.

Wavenumber diffusion

$$Q = \frac{\omega \sin^2\theta}{4\pi^3(N^2 - f^2)|\cos^5\theta|} \times \iint_{K_h^2/K_v^2 > \tan^2\theta} \frac{K_v^2}{K_h} \left(\cot^2\theta - \frac{K_v^2}{K_h^2}\right)^{1/2} E_{\mathrm{geo}}(K_h, K_v)\, \mathrm{d}K_h \mathrm{d}K_v,$$

Wavenumber diffusion

Predictions

- energy confined to one nappe of cone (e.g., upward propagating IGWs),
- energy spreading on the cone with time scale $(Qk)^{-1} \propto k^{-1}\mathrm{Ro}^{-2}$.

Initial-value problem:

- for $e(k,0) = \delta(k - k_*)$,

$$e(k,t) = \tfrac{1}{2} k_*^{-2} \int_0^\infty J_4(k^{-1/2}\lambda) J_4(k_*^{-1/2}\lambda) \mathrm{e}^{-Q\lambda^2 t/4} \lambda \, \mathrm{d}\lambda.$$

Forced problem:

- for $F(k) = \delta(k - k_*)$,

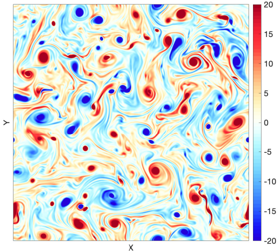
$$e(k,t) \propto \begin{cases} k^2 & \text{for } \ 0 < k < k_* \\ k^{-2} & \text{for } \ k > k_* \end{cases}.$$

- leads to dependence k_h^{-2} in horizontal wavenumber.

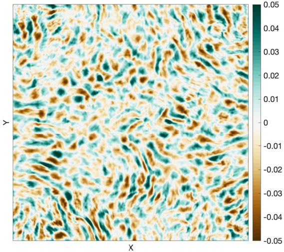
Numerical simulations

Check predictions against numerical simulations of the 3D Boussinesq equations

- pseudospectral, 768^3 resolution, Bartello 1995, Waite & Bartello 2004
- $\Delta z = f\Delta x/N$, consistent with QG scaling and IGWs with $k_\mathrm{h}/k_\mathrm{v} \sim f/N$,
- $\mathrm{Ro} = 0.01$ and $N/f = 32$,
- IGWs added to well-developed QG turbulence.



Geostrophic flow ζ, $\mathrm{Ro} = \zeta_{\mathrm{rms}} = 0.1f$

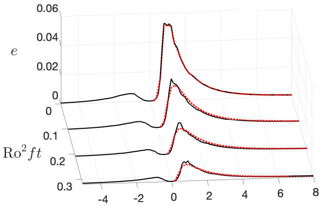


IGW w

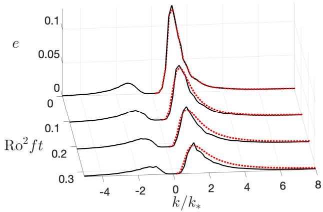
Numerical simulations

Initial-value problem

With peak $k_{\text{flow,h}} \approx 4$, take $k_{*\text{h}} = 16$.
Match spectra after a short transient during which diffusion approximation breaks down.



$\omega = 2f$

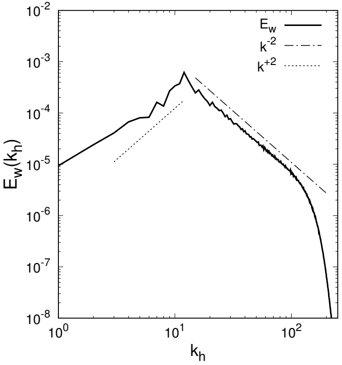


$\omega = 3f$

Numerical simulations

Forced problem

Same flow, $k_{*h} = 12$.



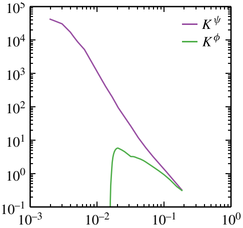
$$\omega = 2f$$

Relation to observations

Oceanic energy spectra

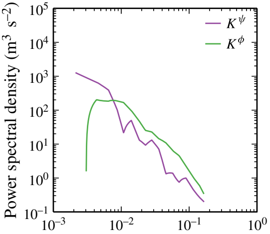
Separation between IGWs and geostrophic parts indicate:

- IGWs dominate the submesoscale range (< 20 km); this includes the Garret–Munk spectrum + larger scales,
- IGWs dominate the entire spectrum in low-energy regions,
- spectrum $\propto k_{\mathrm{h}}^{-2}$.



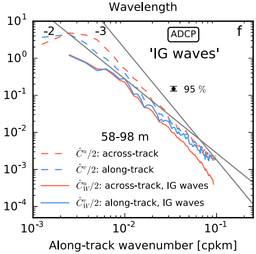
Gulf Stream

Bühler et al 2014



Subtropical North Pacific

Bühler et al 2014



ACC

Rocha et al 2016

Relation to observations

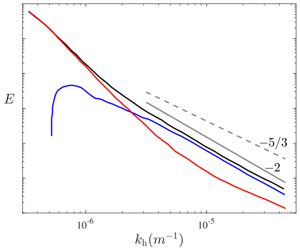
Observed atmospheric energy spectra

Revisited Nastrom–Gage (1985) spectrum:

- almost linear IGWs dominate the shallow ‘$k_\mathrm{h}^{-5/3}$’ part,

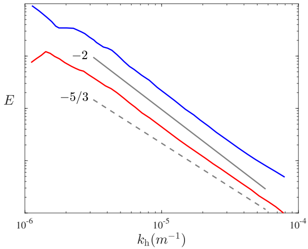
  Callies, Ferrari & Bühler 2014, 2016 (vs Li & Linborg 2018)

- the shallow part could well be k_h^{-2}.



E_tot, E_igw, E_geo

adapted from Callies et al 2014



$E_\mathrm{igw}^\mathrm{tropo}$, $E_\mathrm{igw}^\mathrm{strato}$

adapted from Callies et al 2016

Conclusions

- Statistical theory of linear IGWs in weak turbulent flow.
- Energy exchanged on constant-frequency cone through wave + wave + flow resonant interactions (catalytic interactions). Lelong & Riley 1991, Bartello 1995.
- Kinetic equation for $a(\boldsymbol{x}, \boldsymbol{k}, t)$ defined in terms of Wigner transform.
- Similar to weak turbulence.
- Diffusion approximation for $k_{\text{flow}} \ll k_{\text{igw}} \ll \omega/U$:
  - predicts k^{-2} equilibrium spectrum, consistent with oceanic and atmospheric observations,
  - predicts transient scale cascade (to dissipation),
- For $k_{\text{igw}} \sim \omega/U$, statistics of chaotic ray dynamics.