Wave–flow interactions in geophysical fluids Part II

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Outline

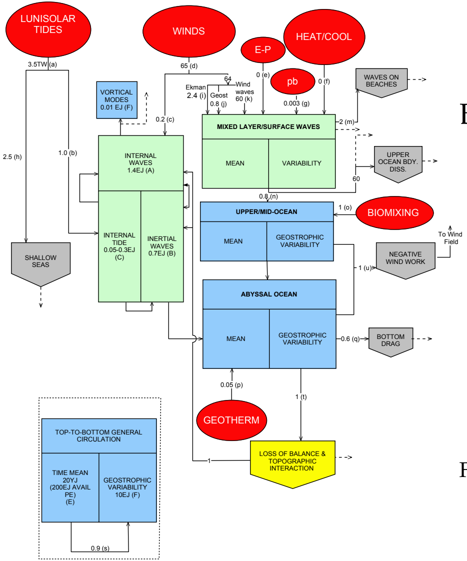
Motivation
- Ocean energy dissipation

Spontaneous IGW generation
- Lorenz 5-component model
- Numerical simulations

Stimulated IGW generation
- Mechanism
- NIW–QG model

Conclusion

The energy dissipation puzzle



Balanced dynamics:

- energy input at scales $\gtrsim$ 100 km,
- inverse energy cascade (movie),
- highly ineffective viscous dissipation.

Ferrari & Wunsch 2009

Ocean energy dissipation puzzle

How does ≈ 1 TW get dissipated?

Consider the possible role of IGWs:

1. spontaneous IGW generation, balanced motion loses energy by exciting IGWs weak except for $\epsilon = O(1)$,
2. stimulated IGW generation, externally forced IGWs extract energy from balanced motion.

Spontaneous IGW generation

Recall the Boussinesq equations,

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + f \boldsymbol{z} \times \boldsymbol{u} = -\nabla \phi + b \boldsymbol{z} ,$$

$$\partial_t b + \boldsymbol{u} \cdot \nabla b + N^2 w = 0 ,$$

$$\nabla \cdot \boldsymbol{u} = 0 ,$$

and potential-vorticity conservation:

$$(\partial_t + \boldsymbol{u} \cdot \nabla) q = 0 \quad \text{with} \quad q = (f \boldsymbol{z} + \nabla \times \boldsymbol{u}) \cdot (N^2 \boldsymbol{z} + \nabla b) .$$

Time-scale separation: Rossby number

$$\epsilon = \frac{U}{fL} \ll 1 .$$

Spontaneous IGW generation

Question: how much IGWs are generated by balanced motion?

Linear level, $O(\epsilon^0)$:

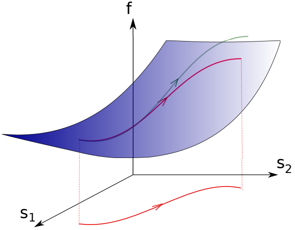
- balanced motion satisfies geostrophic + hydrostatic equilibria,
$$f\,\boldsymbol{z} \times \boldsymbol{u}_{\mathrm{h}} \approx -\nabla_{\mathrm{h}}\phi, \quad \phi_z \approx b,$$
- IGWs satisfy $q = 0$,
- balanced motion + IGWs are uncoupled.

Nonlinear level, $O(\epsilon^n)$:

- balanced motion and IGWs are coupled (presumably),
- but how are balanced motion and IGWs separated?

Dynamical-systems view: for a two-time-scale system,

- define a subspace of the state space in which the evolution is slow, slow manifold,
- ask whether trajectories remain in this subspace: invariance of the slow manifold.



Slow manifold

Write the Boussinesq equations as

$$\boxed{\frac{\partial s}{\partial t} = N_s(s,f;\epsilon)}\,, \qquad \boxed{\frac{\partial f}{\partial t} + \frac{1}{\epsilon}\mathcal{L}f = N_f(s,f;\epsilon)}\,,$$

where s is the slow (balanced) variable, f are the fast variables.

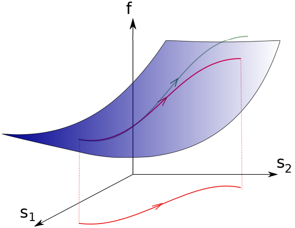
Time-scale separation: spec $\mathcal{L} = \{\mathrm{i}\omega : \omega \in \mathbf{R}, |\omega| > 1\}$.

- For $\epsilon = 0$, slow and fast dynamics split:
  - $\boxed{f \equiv 0}$ defines a slow manifold free from IGWs and exactly invariant: geostrophic and hydrostatic equilibrium,
  - $\boxed{\partial_t s = N_s(s, 0; \epsilon)}$ defines the balanced dynamics.

Slow manifold

- For $\epsilon \neq 0$, can construct approximately invariant slow manifolds perturbatively

$$f = F^N(s, \epsilon) = \sum_{n=1}^{N} \epsilon^n F_n(s) \; .$$



For a given slow manifold F^N,

- balanced models, for s only: $\partial_t s = N_s(s, F^N(s, \epsilon); \epsilon)$.
- initialisation, projection of initial data on the slow manifold $f = F^N(s_{\text{obs}}, \epsilon)$.

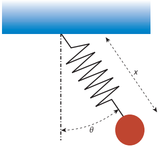
Question: does an exactly invariant slow manifold exist?

Lorenz's 5-component model

Lorenz introduced an ODE model to ask:

- how can the slow manifold be defined,
- is it invariant?

Equivalent to a simple mechanical system:



$$\dot{\phi} = w - \epsilon y, \quad \dot{w} = -\sin(2\phi)/2$$
$$\dot{x} = -\epsilon^{-1} y, \quad \dot{y} = \epsilon^{-1} x + \sin(2\phi)/2.$$

- slow $s = (\phi, w)$ and fast $f = (x, y)$,
- pendulum motion $\approx$ slow atmospheric motion,
- spring oscillation $\approx$ IGWs.

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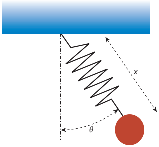

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On the Existence of a Slow Manifold

EDWARD N. LORENZ

Department of Earth, Atmospheric, and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, MA 02139

(Manuscript received and in final form 28 October 1985)

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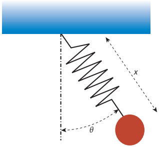

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On the Nonexistence of a Slow Manifold

E. N. LORENZ AND V. KRISHNAMURTHY

Center for Meteorology and Physical Oceanography, Massachusetts Institute of Technology, Cambridge, MA 02139

(Manuscript received 29 September 1986, in final form 13 April 1987)

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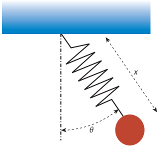

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The Slow Manifold—What Is It?

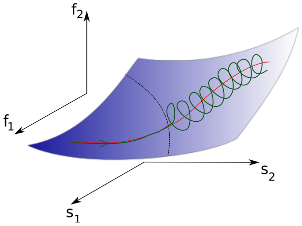
EDWARD N. LORENZ

Center for Meteorology and Physical Oceanography, Massachusetts Institute of Technology, Cambridge, Massachusetts

(Manuscript received 17 June 1991, in final form 17 March 1992)

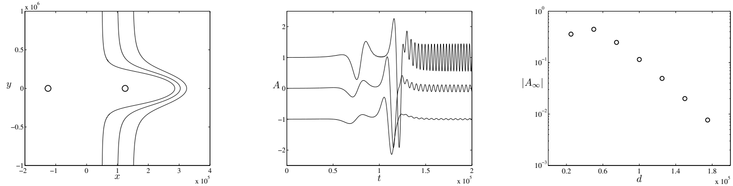
Detailed study of the Lorenz model:

- balanced motion can be defined by computing a slow manifold order-by-order in ϵ,
- the corresponding series diverge but can be truncated optimally,
- the divergence reflects the non-existence of an invariant slow manifold,
- this results from a Stokes phenomenon: generation of exponentially small fast oscillations.

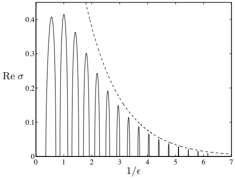


Fluid models

- Transient generation: wavepacket swept by a dipole.



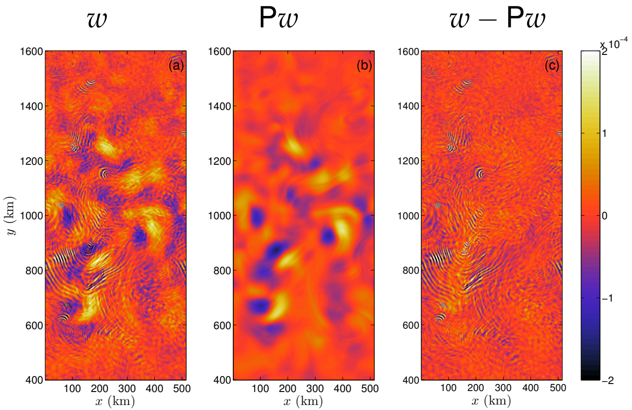
- Parametric instability: repeated Stokes phenomenon in an elliptical vortex Aspden & V 10



Numerical simulations

Ocean turbulence

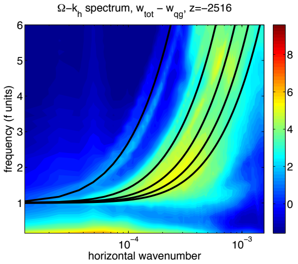
Generation by surface-intensified turbulence Danioux et al 2013.



Diagnose IGWs at $z = -2.5$ km by projecting on slow manifold.

Numerical simulations

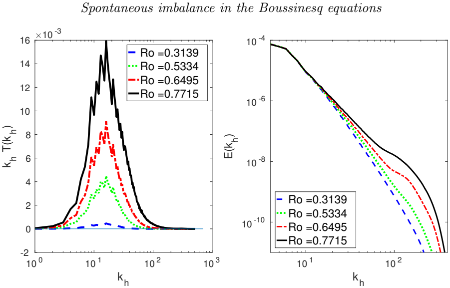
Ocean turbulence

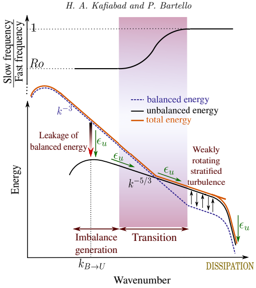


Large ϵ at surface but very weak IGWs: $E \approx 10^{-5} E_{\mathrm{GM}}$.

Numerical simulations

Turbulence





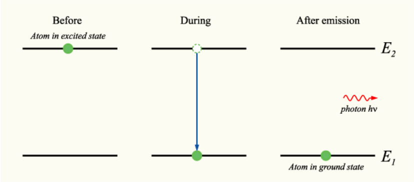
Kafiabad & Bartello 2018

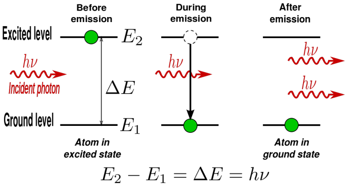
Conclusions:

- spontaneous generation only for strong flows, $\epsilon = O(1)$,
- unlikely to contribute significantly to energy dissipation.

Stimulated IGW generation:

- in the ocean, IGWs forced by winds and tides,
- effect of these forced waves on the balanced flow?





Stimulated IGW generation

More generally, stimulated loss of balance: forced IGWs provide a sink:

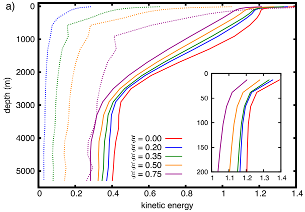
Gertz & Straub 1997

- by inducing transfers from balanced motion to IGWs,

  Whitt & Thomas 2015, Xie & V 2015

- by increasing transfers to submesoscale, unbalanced motion.

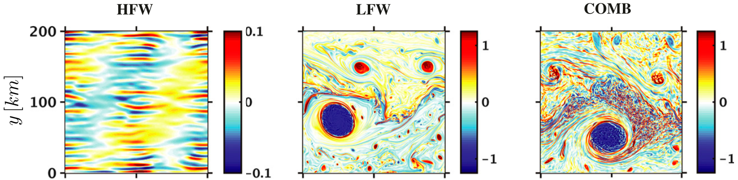
  Barkan, Winters & McWilliams 2017

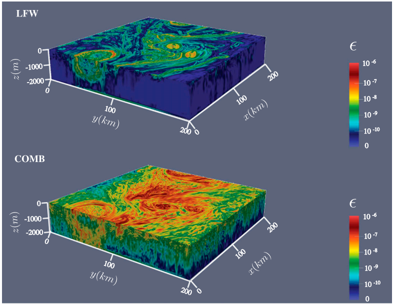


Kinetic energy of low frequency motion decreases with increasing high-frequency wind forcing

Taylor & Straub 2016

Stimulated loss of balance





Barkan, Winters & McWilliams 2017:

- simulations with and without high-frequency winds,
- increased interior dissipation of low-frequency motion,
- added diss./wind work ≈ 1.3.

Stimulated IGW generation

Expect finite-amplitude IGWs to change the dynamics of the balanced motion.

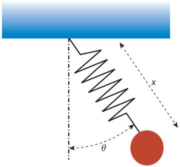
Toy models:

Kapitza pendulum: pendulum with oscillating attachment point (movie):

- lower equilibrium becomes unstable,
- upper position becomes stable.

Lorenz 5-component model:

- pendulum coupled with stiff spring,
- study the effect of fast oscillations by averaging,
- convenient to use Lagrangian formulation.



Stimulated IGW generation

Evolution of $\theta(t)$ and $x(t)$: minimisers of the action

$$\mathcal{A}[\theta, x] = \frac{1}{2} \int_0^T \left(\dot\theta^2 + \cos(2\theta + 2\epsilon x) + \epsilon^{-2}\dot x^2 - x^2 \right) \, \mathrm{d}t.$$

Fast oscillations, slowly modulated: $x(t) = \epsilon^{-1} \mathrm{Re}\, A(t) \mathrm{e}^{\mathrm{i}t/\epsilon}$.

Introduce into $\mathcal{A}$ and average over the fast time to obtain

$$\mathcal{A}[\theta, A, A^*] = \frac{1}{2} \int_0^T \left(\dot\theta^2 + J_0(2|A|) \cos(2\theta) + \mathrm{i}(A\dot A^* - \dot A A^*) \right) \, \mathrm{d}t.$$

Variations give

- $\dot A = \mathrm{i} J_0'(|A|) A / |A| \;\Rightarrow\; |A| = \text{const}$,
- $\ddot\theta + 2 J_0(2|A|) \sin(2\theta) = 0$: pendulum with frequency $2\sqrt{J_0(|A|)}$,
- oscillations $|A| \neq 0$ lead to pendulum softening and instability.

Stimulated IGW generation

Wave–mean flow interaction:

- average dynamics over fast IGW frequency,
- capture the feedback of IGWs on the balanced flow,

Generalised Lagrangian mean, GLM:

Andrews & McIntyre 1978, Bühler 2006

- mean flow is balanced $\to$ controlled by PV dynamics,
- material invariance of PV preserved by Lagrangian averaging,
- GLM emerges from multiscale treatment. Wagner & Young 2015

$$\bar{\boldsymbol{u}}^{\mathrm{L}}(\boldsymbol{x}, t) = \langle \boldsymbol{u}(\boldsymbol{X}(\boldsymbol{a}, t), t) \rangle, \qquad \boldsymbol{x} = \langle \boldsymbol{X}(\boldsymbol{a}, t) \rangle .$$

GLM PV conservation

$$(\partial_t + \bar{\boldsymbol{u}}^{\mathrm{L}} \cdot \nabla) q^{\mathrm{L}} = 0,$$

Stimulated NIW generation

with J-H Xie

NIW–QG model

- a mechanism of stimulated wave generation,
- focus on near-inertial waves (NIWs), $\omega \approx f$,
- derive a simplified theoretical model
  - by averaging over oscillation frequency f,
  - making no assumption of spatial-scale separation, wavenumber k assumed to satisfy $kL_{\text{flow}} = O(1)$ (vs WKB's $kL_{\text{flow}} \gg 1$).

Assumptions:

1. near-inertial waves, $\boxed{\delta = Nk_{\text{h}}/(fk_{\text{v}}) \ll 1}$ $\Rightarrow \omega = f + O(\delta^2)$,
2. small Rossby number $\boxed{\epsilon = U_{\text{flow}}/(fL) = O(\delta^2)}$,
3. strong waves $\boxed{\alpha = U_{\text{waves}}/(fL) = O(\delta) = O(\epsilon^2)}$.

Stimulated NIW generation

To leading order: NIWs in resting basic state,

$$\partial_t u - fv = 0, \quad \partial_t v + fu = 0,$$

hence $u + \mathrm{i}v = M_z(\boldsymbol{x}, \epsilon t)\exp(-\mathrm{i}ft)$. At the next order, evolution of wave amplitude $M(\boldsymbol{x}, \epsilon t)$ and flow $\boldsymbol{U}(\boldsymbol{x}, \epsilon t)$.

To derive this, use

- variational (Lagrangian) formulation and averaging,
- GLM averaging.

Salmon 2015

Coupled model

Start with hydrostatic–Boussinesq Lagrangian

$$\mathcal{L}[\boldsymbol{x}, p] = \int \left(\frac{1}{2} \left(\dot{x}^2 + \dot{y}^2 \right) - \left(fy + \frac{\beta y^2}{2} \right) \dot{x} + bz + p \left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{a}} - 1 \right) \right) \mathrm{d}\boldsymbol{a}$$

and introduce $\boldsymbol{x}(\boldsymbol{a}, t) = \boldsymbol{X}(\boldsymbol{a}, t) + \boldsymbol{\xi}(\boldsymbol{X}(\boldsymbol{a}, t), t)$.

To leading order, $\boldsymbol{\xi}$ describes NIWs:
$\partial_t \xi^{(1)} - f\eta^{(1)} = 0, \quad \partial_t \eta^{(1)} + f\xi^{(1)} = 0, \quad \xi_x^{(1)} + \eta_y^{(1)} + \zeta_z^{(1)} = 0.$

Solve in terms of the NIW amplitude: $M(x, y, z, t)$, with

$$\xi^{(1)} + \mathrm{i}\eta^{(1)} = M_z \mathrm{e}^{-\mathrm{i}ft}, \quad \zeta^{(1)} = -\tfrac{1}{2}(\partial_x - \mathrm{i}\partial_y) M \mathrm{e}^{-\mathrm{i}ft} + \text{c.c.}.$$

Whitham average, using $\overline{\boldsymbol{\xi}^{(2)}} = \frac{1}{2}\overline{\boldsymbol{\xi}^{(1)} \cdot \nabla \boldsymbol{\xi}^{(1)}}$ (glm) to obtain $\bar{\mathcal{L}}[\boldsymbol{X}, M, P]$.

Coupled model

Take variations of

$$\int_0^T \bar{\mathcal{L}}[\boldsymbol{X}, M, P] \, \mathrm{d}t$$

with respect to $\boldsymbol{X}$, M and P to obtain:

- mean-flow equations,
- Young–Ben Jelloul (YBJ) NIW equation.

For a balanced mean flow, $\bar{\boldsymbol{u}}^{\mathrm{L}} = (\nabla^{\perp}\psi, 0)$, with

$$\psi = \bar{\psi} + \psi_{\mathrm{Stokes}},$$

we obtain the coupled YBJ/QG model for the joint evolution of M and $q = \bar{q}^{\mathrm{L}}$.

Stimulated NIW generation

$$(D_t M_z)_z + \frac{\mathrm{i}}{2}\left(\nabla^2\psi M_{zz} + (\frac{N^2}{f} + \psi_{zz})\nabla^2 M - 2\nabla\psi_z \cdot \nabla M_z\right) = 0\,,$$

$$\partial_t q + \partial(\psi, q) = 0\,,$$

with the PV-streamfunction relation

$$\left(\nabla^2 + \partial_z\left(f^2/N^2\partial_z\right)\right)\psi = q + F(M^*, M)\,.$$

This includes the 'wave' PV

$$F(M^*, M) = \mathrm{i}f\partial(M_z^*, M_z)/2 + f\left(2|\nabla M_z|^2 - M_{zz}\nabla^2 M^* - M_{zz}^*\nabla^2 M\right)/4.$$

Xie & V 2015, Wagner & Young 2016, Salmon 2016

Stimulated NIW generation

NIW equation is the YBJ equation Young & Ben Jelloul 1997

$$\partial_t M_{zz} + \underbrace{\nabla^{\perp}\psi \cdot \nabla M_{zz}}_{\text{advection}} + \underbrace{\mathrm{i}\frac{N^2}{2f}\nabla^2 M}_{\text{dispersion}} + \underbrace{i\frac{\Delta\psi}{2}M_{zz}}_{\text{refraction}} = 0.$$

- advection: by the background flow,
- dispersion: approximates frequency as

$$(f^2 + N^2k_{\mathrm{h}}^2/|\boldsymbol{k}|^2)^{1/2} \quad \approx \quad f + N^2k_{\mathrm{h}}^2/(2fk_{\mathrm{v}}^2),$$

- refraction: $\Delta\psi/2$, frequency shift Kunze 1985

YBJ model used to demonstrate:

- effect of background flow on vertical propagation

  Balmforth et al 1998, Balmforth & Young 1999, Klein et al 2004

- concentration of NIW energy in anticyclones

  Llewellyn-Smith 1999, Klein et al 2004, Danioux, V & Bühler 2015

Stimulated NIW generation

The model is Hamiltonian, conserves action and energy:

$$\boxed{\mathcal{A} = \int |M_z|^2 \, \mathrm{d}\boldsymbol{x}} = \text{NIW kinetic energy},$$

$$\boxed{\mathcal{H} = \tfrac{1}{2} \int \left(|\nabla \psi|^2 + \tfrac{f^2}{N^2} (\partial_z \psi)^2 + \tfrac{N^2}{2} |\nabla M|^2 \right) \mathrm{d}\boldsymbol{x}}$$

$$= \text{QG energy} + \text{NIW potential energy} \qquad .$$

Physical implications:

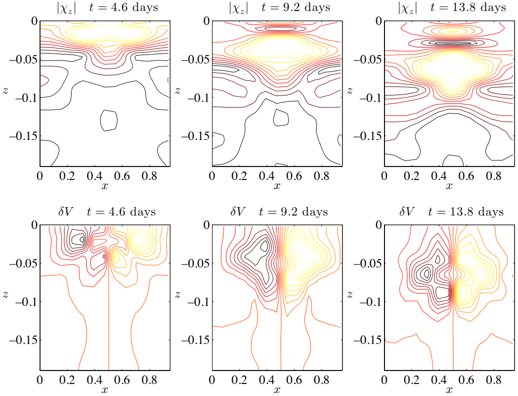
- $\mathcal{A} =$ const: no spontaneous NIW generation,
- $\mathcal{H} =$ const: mean-flow energy decays as $|\nabla M|$ increases:

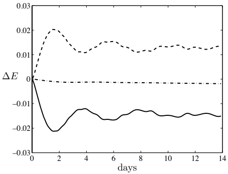
stimulated wave generation

Stimulated NIW generation

Numerical simulation

Slice model ($\partial_y = 0$): $\psi \propto \cos x$, NIWs in mixed layer at $t = 0$.





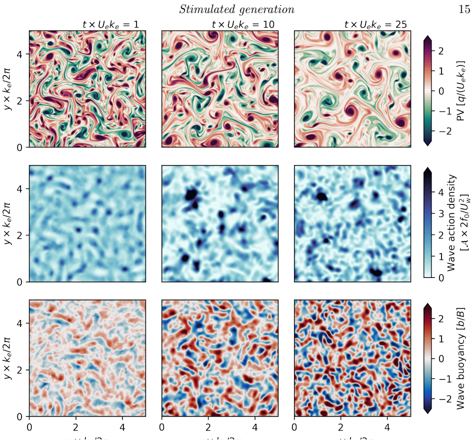
Wave propagation and flow slowdown

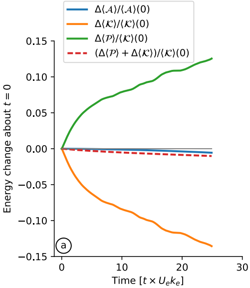
Stimulated NIW generation

Numerical simulations

Rocha, Wagner & Young 2018

Barotropic flow, vertically plane waves: $M \propto \mathrm{e}^{imz}\phi(x, y, t)$.

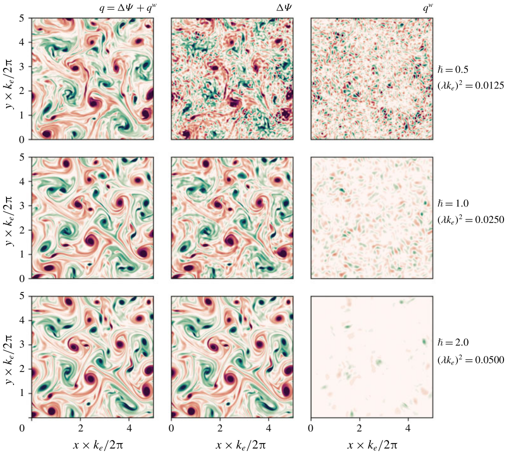




Stimulated NIW generation

Numerical simulations

Rocha, Wagner & Young 2018



Stimulated NIW generation

Impact on mesoscale

Mesoscale motion:

- 1 TW input (by baroclinic instability),
- inverse energy cascade → negligible viscous dissipation,
- dissipation mechanisms: bottom drag, side friction, loss of balance...

Can stimulated NIW generation provide an energy sink?

$$\boxed{\dot{E}_{\mathrm{QG}}} = -\frac{N^2k^2}{2f^2m^2}\dot{E}_{\mathrm{NIW}} = \boxed{-\frac{\Delta\omega}{f}\dot{E}_{\mathrm{NIW}}}$$

With $\dot{E}_{\mathrm{NIW}} = 0.6$ TW and $\Delta\omega/f = 0.2$, $\dot{E}_{\mathrm{QG}} \approx 0.1$ TW (cf. 0.1 TW for bottom drag).

Conclusion

Spontaneous IGW generation

- balanced approximation: quasigeostrophy + higher-order corrections, hold to all orders in ϵ,
- for ODE models, spontaneous wave generation is exponentially small, $\exp(-\alpha/\epsilon)$,
- significant only for $\epsilon = O(1)$.

Stimulated IGW generation

- simplified model; conservation laws constrain energy exchanges,
- naturally a Generalised Lagrangian mean model,
- near-inertial waves gain potential energy at the expanse of mean-flow energy.

Collaborators: Jonathan Aspden, Eric Danioux, Hossein Kafiabad, Miles Savva, Jin-Han Xie.