credit: S. Viboud

Experimental studies on Wave Turbulence

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Wave turbulence ?

single point frequency elevation spectrum

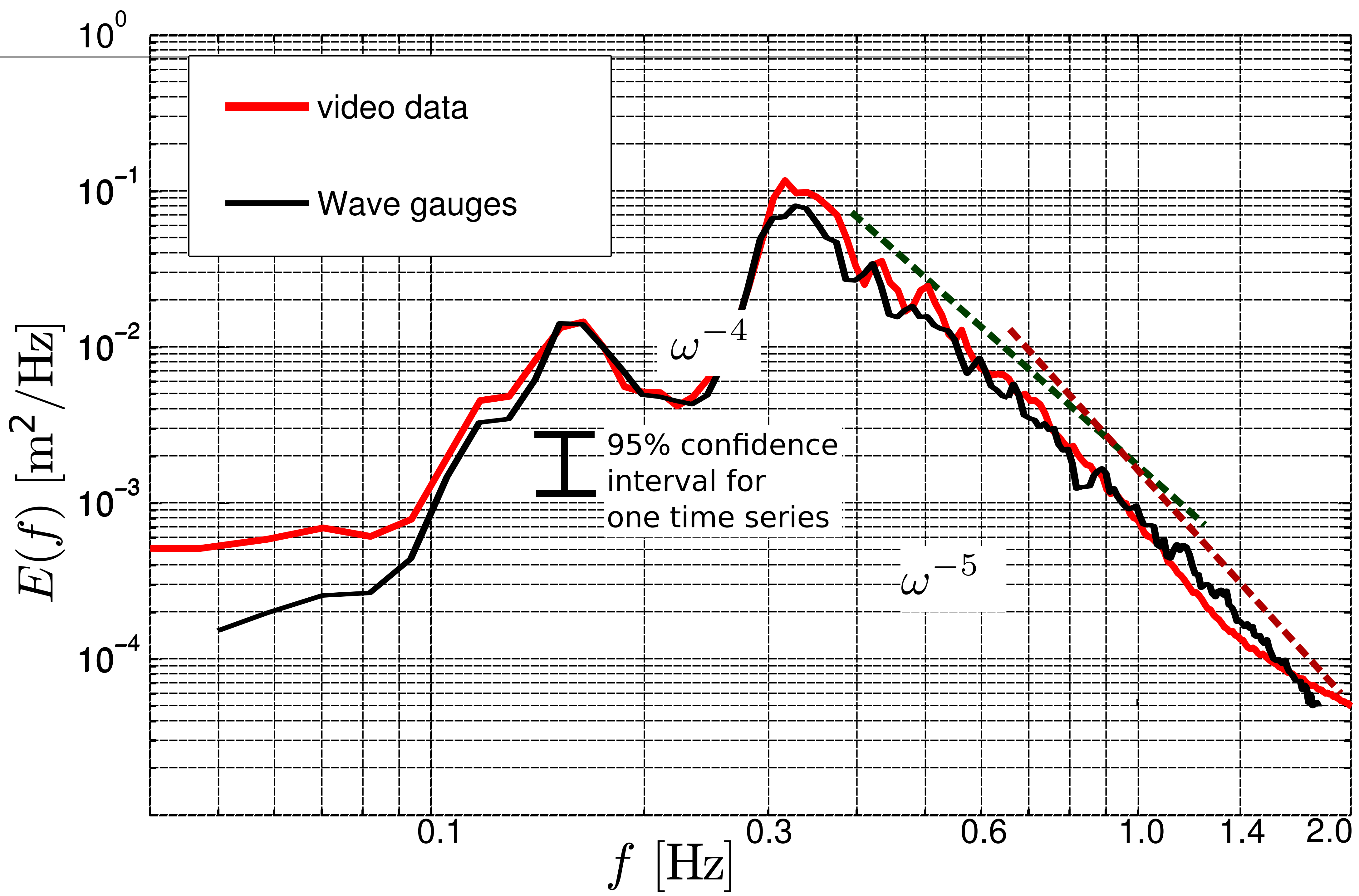


FIG. 4. Frequency spectrum $E(f)$ estimated from the stereo video system, without any smoothing, and from the wire wave gauges mounted on the platform. The dashed lines show the f^{-4} and f^{-5} asymptotes. The error bar corresponds to random sampling errors for single time series, as given by the expected χ^2 distribution with 40 degrees of freedom. The spectrum from the video is averaged over the 10.8-m square analysis window, and thus the random sampling error is actually smaller for the shorter waves with many uncorrelated waves in the field of view.

Leckler *et al. JPO 2015* :
observation of waves in the Black Sea with stereoscopic reconstruction

physical ingredients:

- many waves ➙ many degrees of freedom
- non linearity ➙ energy transfer among waves
- forcing at large scale
- dissipation at small scale ➙ energy cascade in scale (if transfer is local in scale)

Weak Turbulence Theory

there is a statistical theory of wave turbulence, starting from the deterministic wave equation !

nonlinear
wave equation

statistical equation:
evolution of statistical quantities

ex: temporal evolution
of the wave spectrum

hypotheses of weak turbulence theory:

- asymptotically large system ➙ no discrete modes
- dispersive waves
- weak non linearity

➙ scale separation $T_{linear} \ll T_{nonlinear} \quad (\ll T_{dissipation})$

multiscale analysis

see Falkovich *et al.*, *Kolmogorov spectra of turbulence* 1992
S. Nazarenko, *Wave Turbulence* 2011
AC Newell & B. Rumpf, *Ann. Rev. Fluid Mech.*, 2011

Weak turbulence

➤ evolution equation for the wave spectrum: kinetic equation

$n(k,t)$ wave action spectrum (number density of waves)

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = 4\pi\varepsilon^2 \int |V_{k,k_1,k_2}|^2 n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) \left[\left(\frac{1}{n_{\mathbf{k}}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} \right) \delta(\omega-\omega_1-\omega_2) + \left(\frac{1}{n_{\mathbf{k}}} - \frac{1}{n_{\mathbf{k}_1}} + \frac{1}{n_{\mathbf{k}_2}} \right) \delta(\omega_1-\omega-\omega_2) + \left(\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} \right) \delta(\omega_2-\omega_1-\omega) \right] d\mathbf{k}_1 d\mathbf{k}_2 \ ,$$

see Falkovich *et al.*, *Kolmogorov spectra of turbulence* 1992
S. Nazarenko, *Wave Turbulence* 2011
AC Newell & B. Rumpf, *Ann. Rev. Fluid Mech.*, 2011

energy transfers only through resonant waves (weak non linearity)

3-wave coupling: $\omega_1 = \omega_2 + \omega_3$ $\qquad \mathbf{k}_1 = \mathbf{k}_2 + \mathbf{k}_3$

4-wave coupling: $\omega_1 + \omega_2 = \omega_3 + \omega_4$ $\qquad \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$

(energy conservation) (momentum conservation)

dispersive waves
for non trivial solutions

Kolmogorov-Zakharov spectrum

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = 4\pi\varepsilon^2 \int |V_{k,k_1,k_2}|^2 n_{\mathbf{k}} n_{\mathbf{k}_1} n_{\mathbf{k}_2} \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) \left[\left(\frac{1}{n_{\mathbf{k}}} - \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} \right) \delta(\omega-\omega_1-\omega_2) + \right.$$

$$\left. \left(\frac{1}{n_{\mathbf{k}}} - \frac{1}{n_{\mathbf{k}_1}} + \frac{1}{n_{\mathbf{k}_2}} \right) \delta(\omega_1-\omega-\omega_2) + \left(\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_{\mathbf{k}_1}} - \frac{1}{n_{\mathbf{k}_2}} \right) \delta(\omega_2-\omega_1-\omega) \right] d\mathbf{k}_1 d\mathbf{k}_2 \ ,$$

stationary solutions of the kinetic equation:

- with neither forcing nor dissipation: equipartition spectra (Rayleigh-Jeans)
- with forcing and dissipation: out of equilibrium spectrum, energy cascade
  Kolmogorov-Zakharov spectrum

usually has the shape: $$n(k) = C P^{1/(N-1)} k^{-\alpha}$$

spectral exponent

Kolmogorov
constant
(can be computed)

average
energy flux

number of
interacting waves (N=3, 4…)
nonlinearity of order N-1

see Falkovich *et al.*, *Kolmogorov spectra of turbulence* 1992
S. Nazarenko, *Wave Turbulence* 2011

Experiments (and simulations) in wave turbulence

Issues with experiments:

- **finite size** of the experiment. Is it large enough ? (discrete modes vs kinetic regime)

- **scale separation:** $T_{linear} \ll T_{nonlinear} \ll T_{dissipation}$

non linearity:
weak enough ?

non linearity:
not too weak ?

dissipation:
weak enough ?

- for weak nonlinearity: long transient…

- **finite nonlinearity:** other structures than waves

  non resonant coupling, bound waves (parasitic waves)

  singularities (strongly non linear)

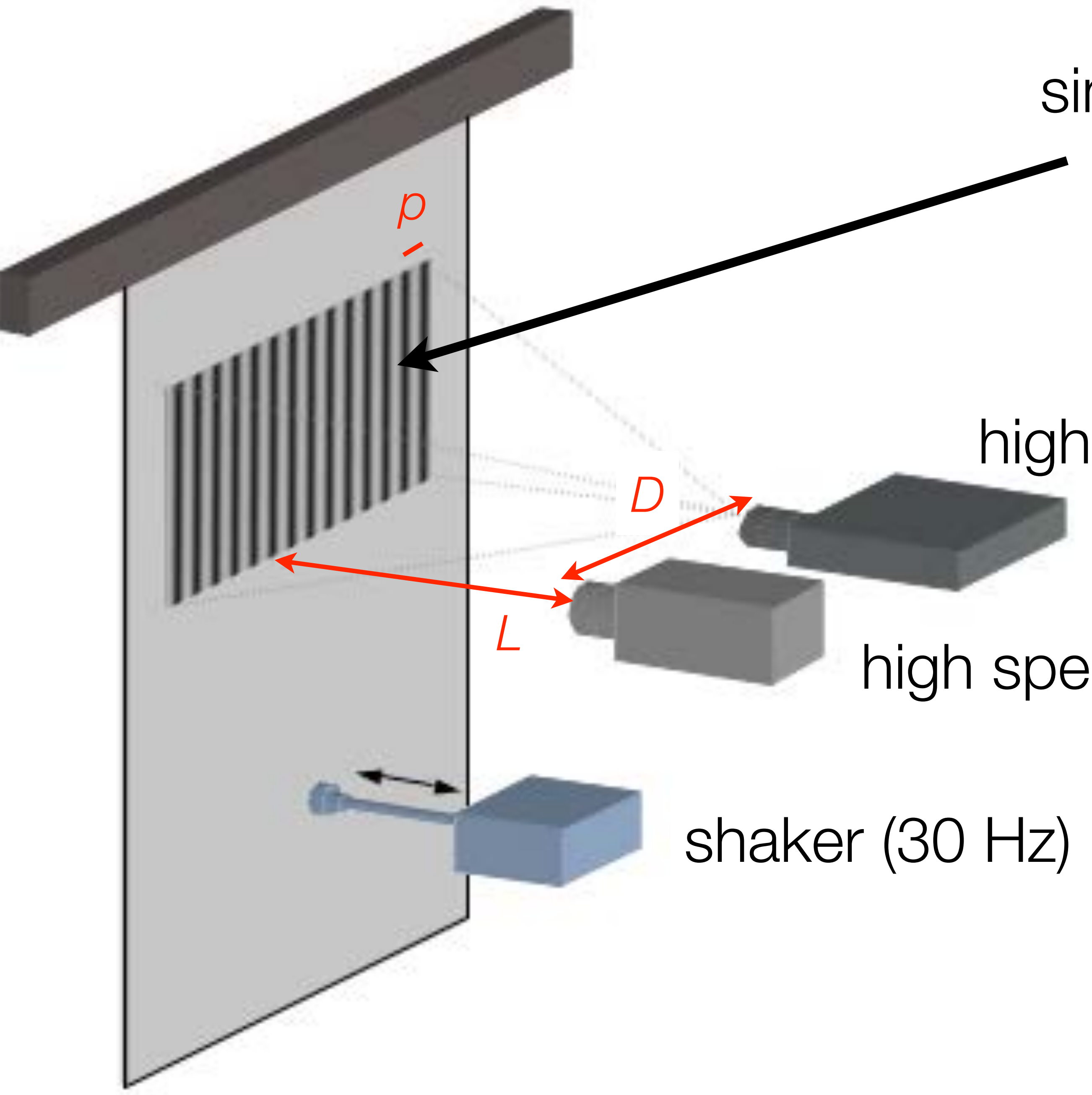
- **other degrees of freedom** that waves: vortices, jets…

Observation of weak turbulence:

- the thunder plate: vibrating a thin elastic plate
- internal waves & stratified turbulence

the thunder plate…

high speed Fourier transform profilometry:



sine grayscale pattern

high definition video projector

high speed camera

shaker (30 Hz)

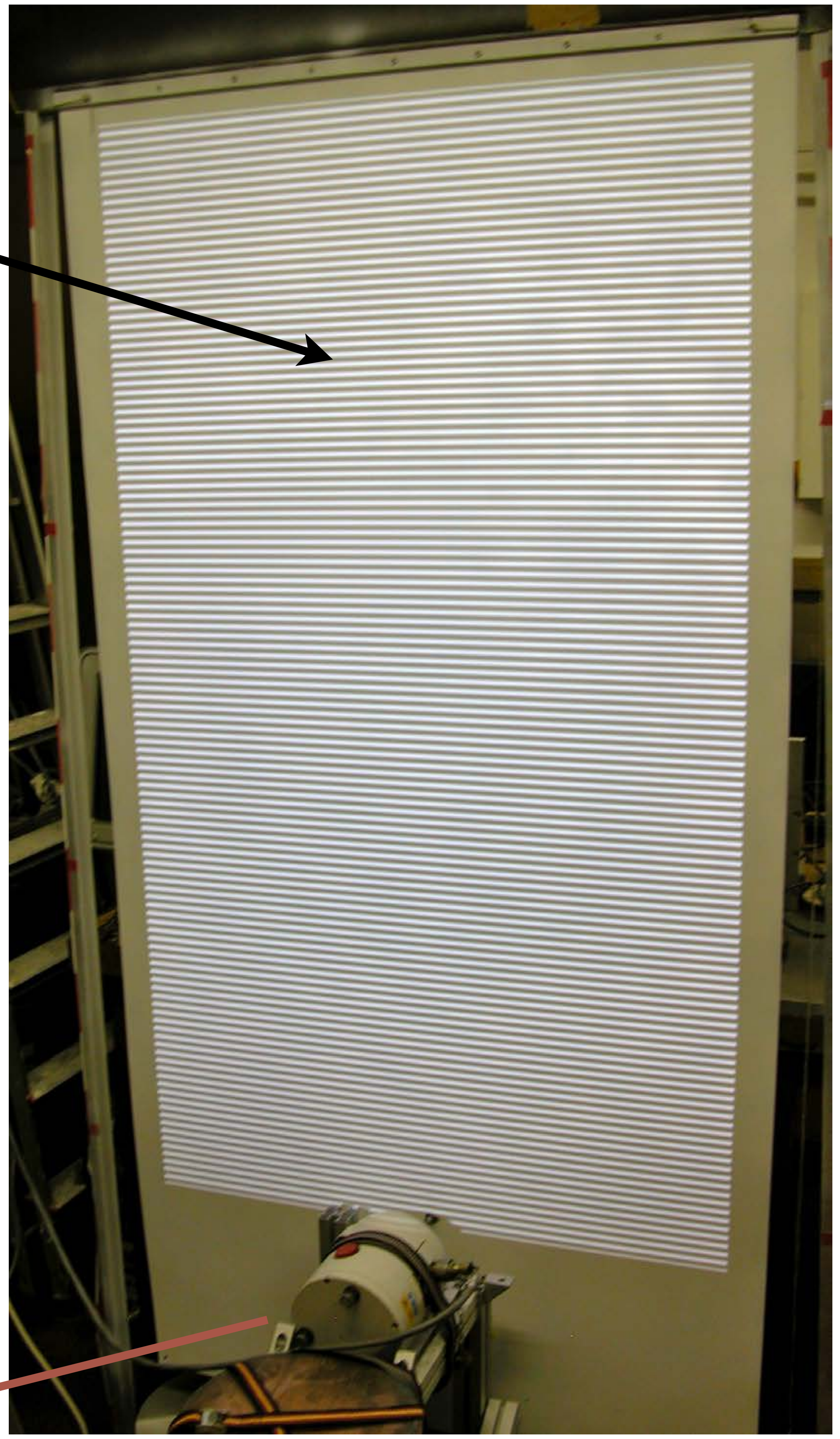
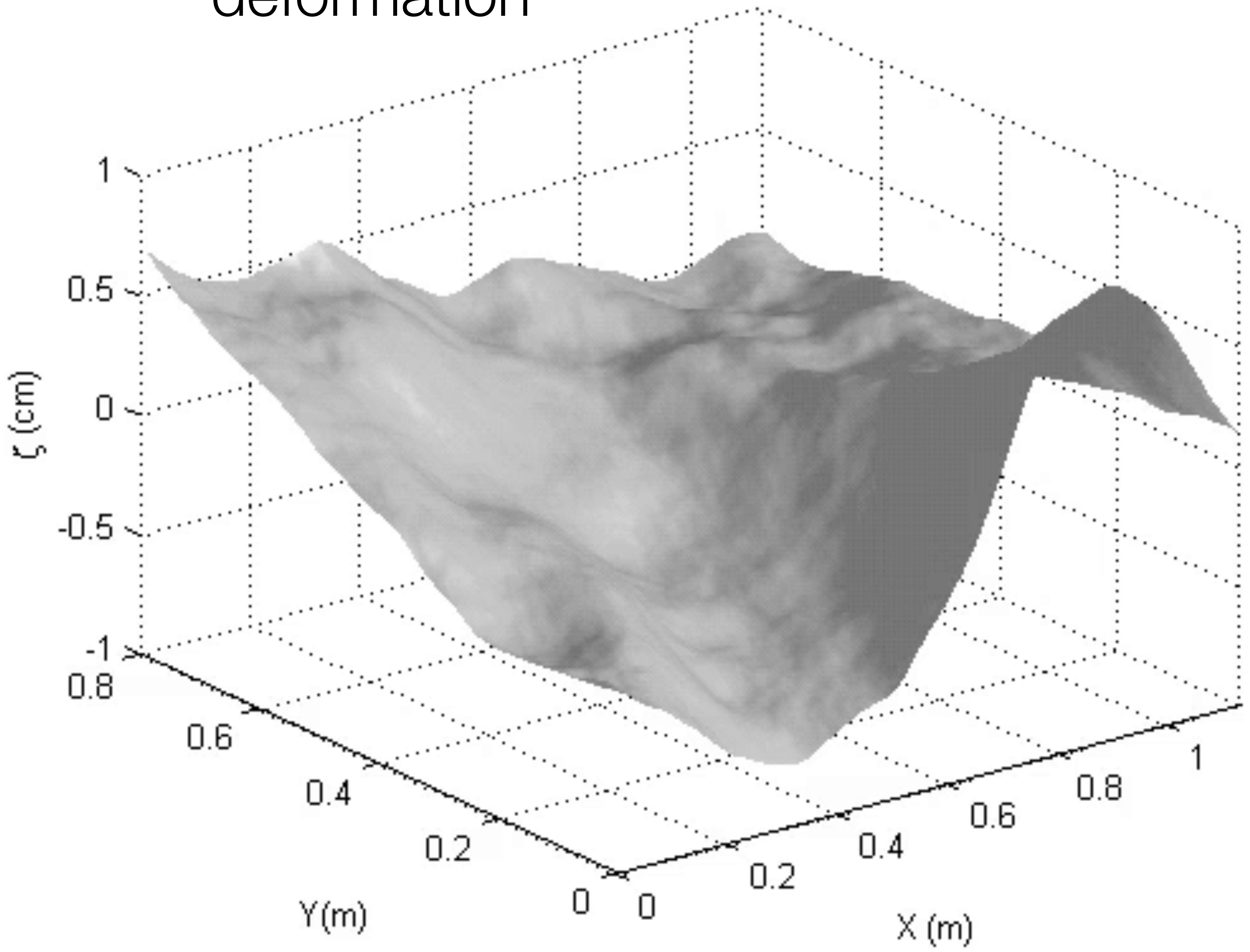


plate:
stainless steel
2m x 1m
thickness 0.4 mm

forcing
electromagnetic shaker
sine forcing at 30 Hz

➨ space and time resolved measurement

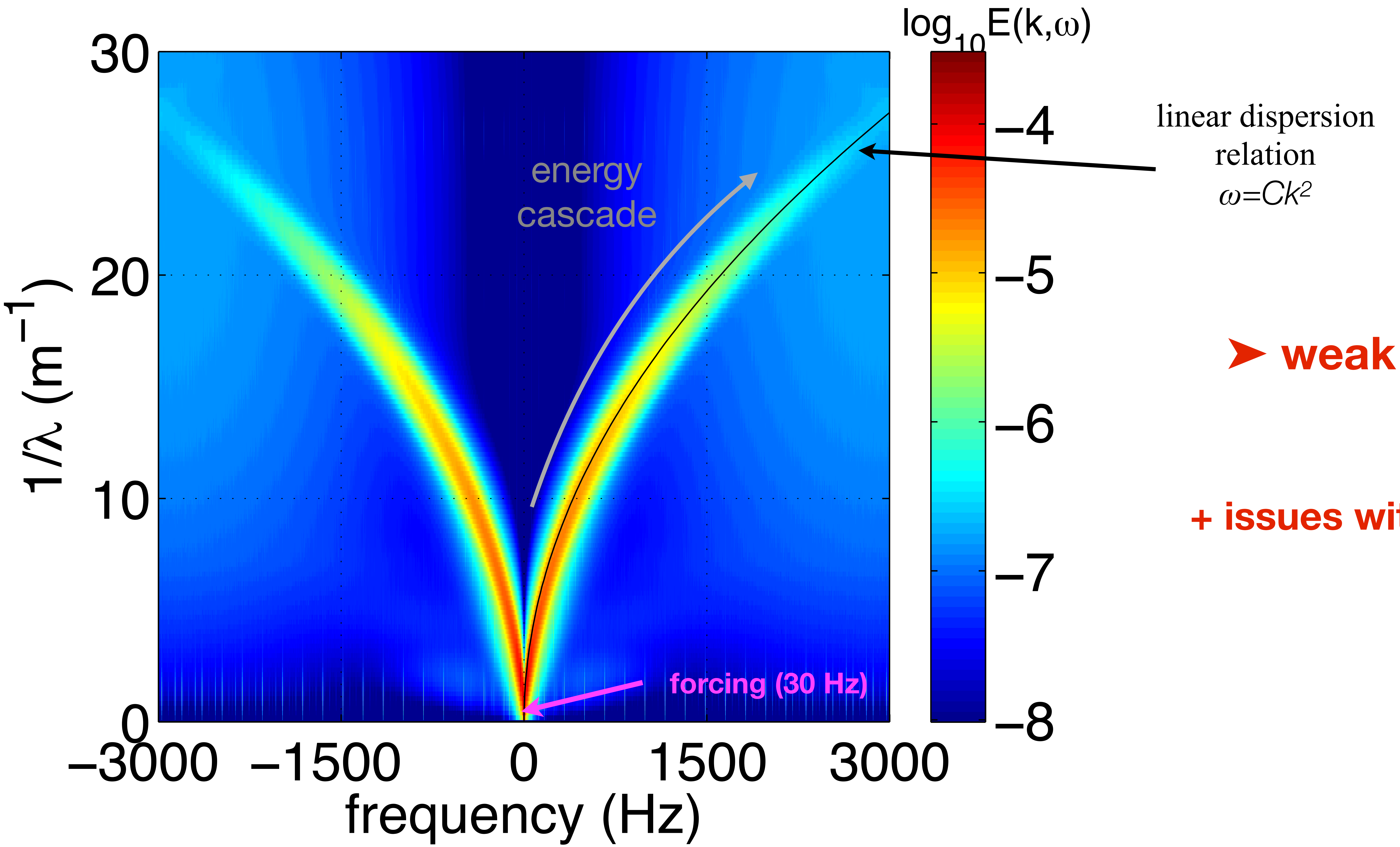
deformation



slowed down 25 times
original sampling 6000 fps

P. Cobelli, A. Maurel, V. Pagneux, P. Petitjeans
«Global measurement of water waves by Fourier transform profilometry» *Exp. in Fluids*, **46** (2009)

Fourier spectrum *E(k,ω)*



➤ **weak wave turbulence…**

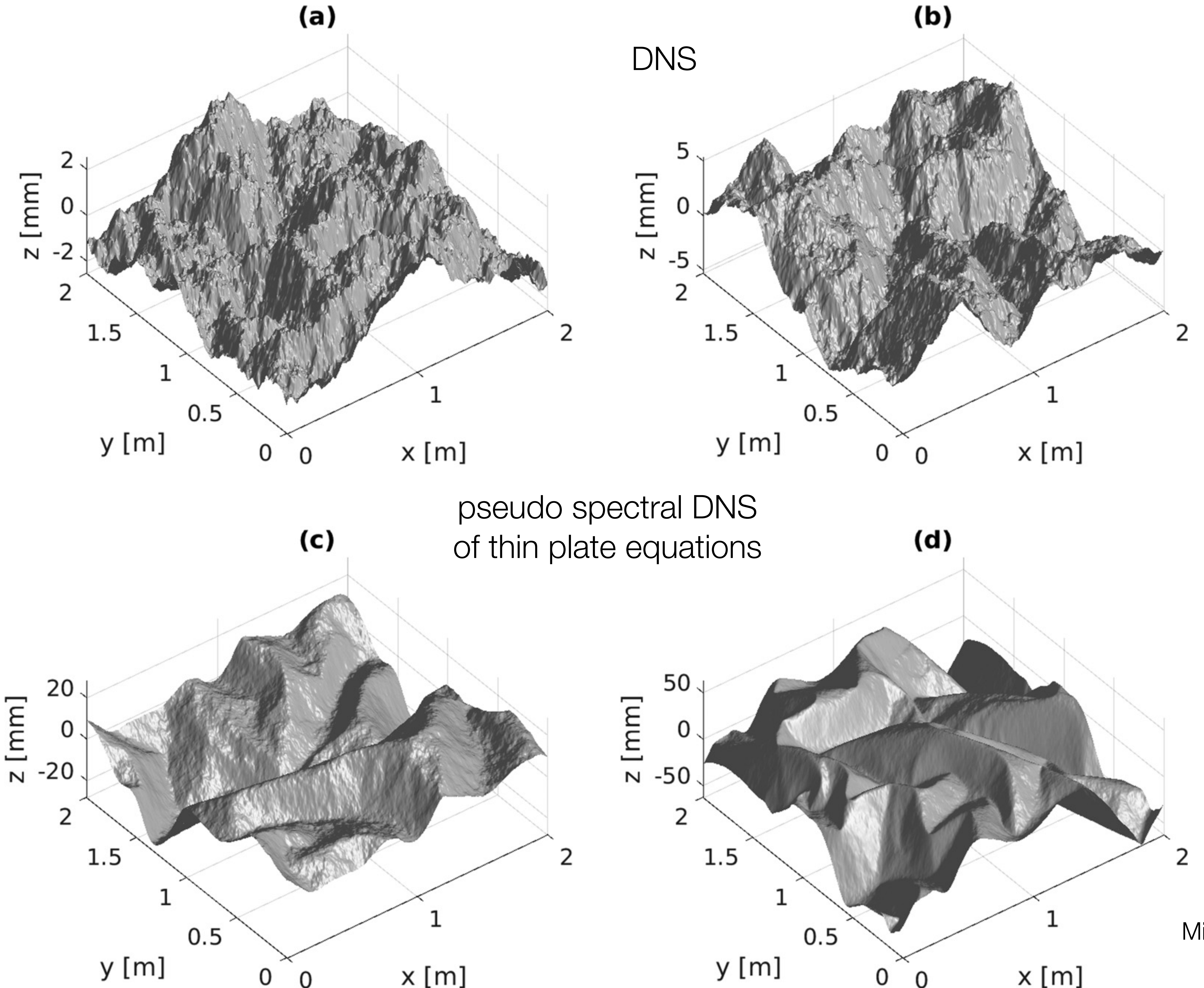
+ issues with finite size, dissipation

Cobelli *et al. PRL* 2009
Mordant *EPJB* 2010

stronger forcing: emergence of singularities

DNS

weakest forcing

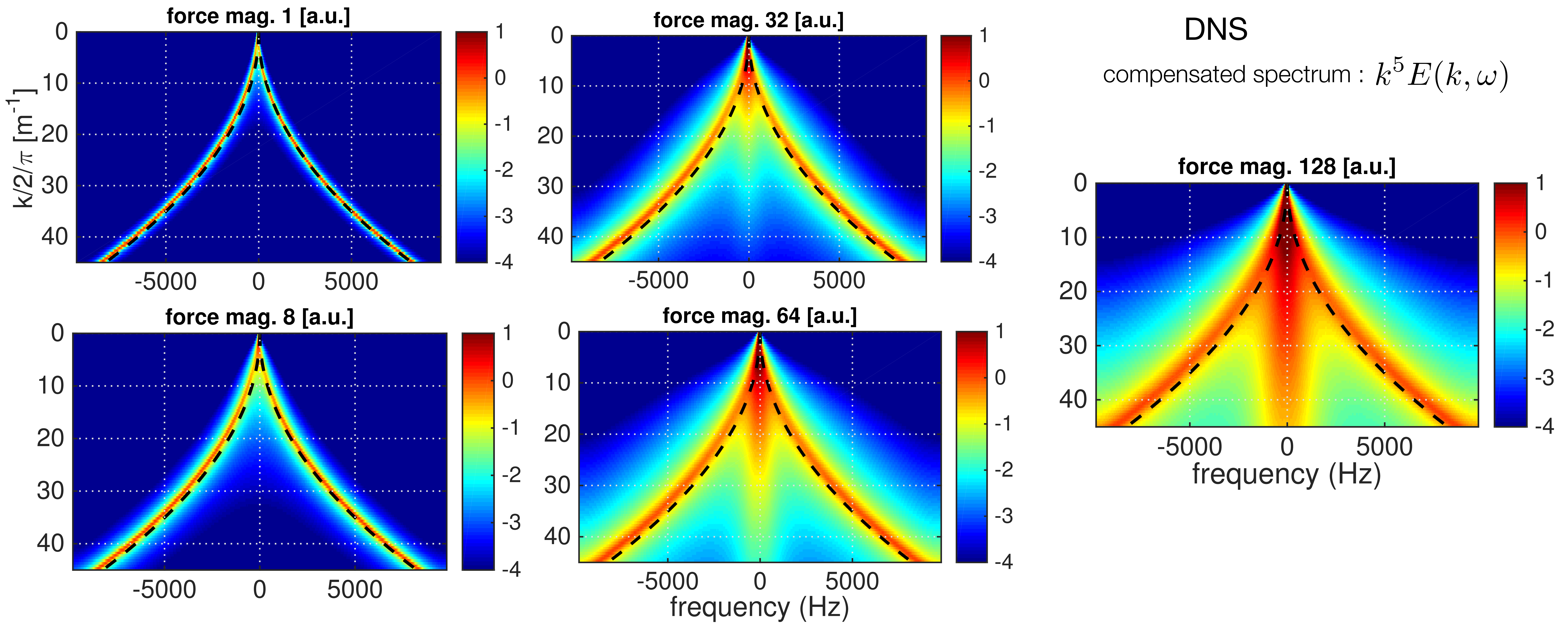


pseudo spectral DNS of thin plate equations

strongest forcing

Miquel, Alexakis, Josserand & Mordant, *Phys. Rev. Lett.* 2013

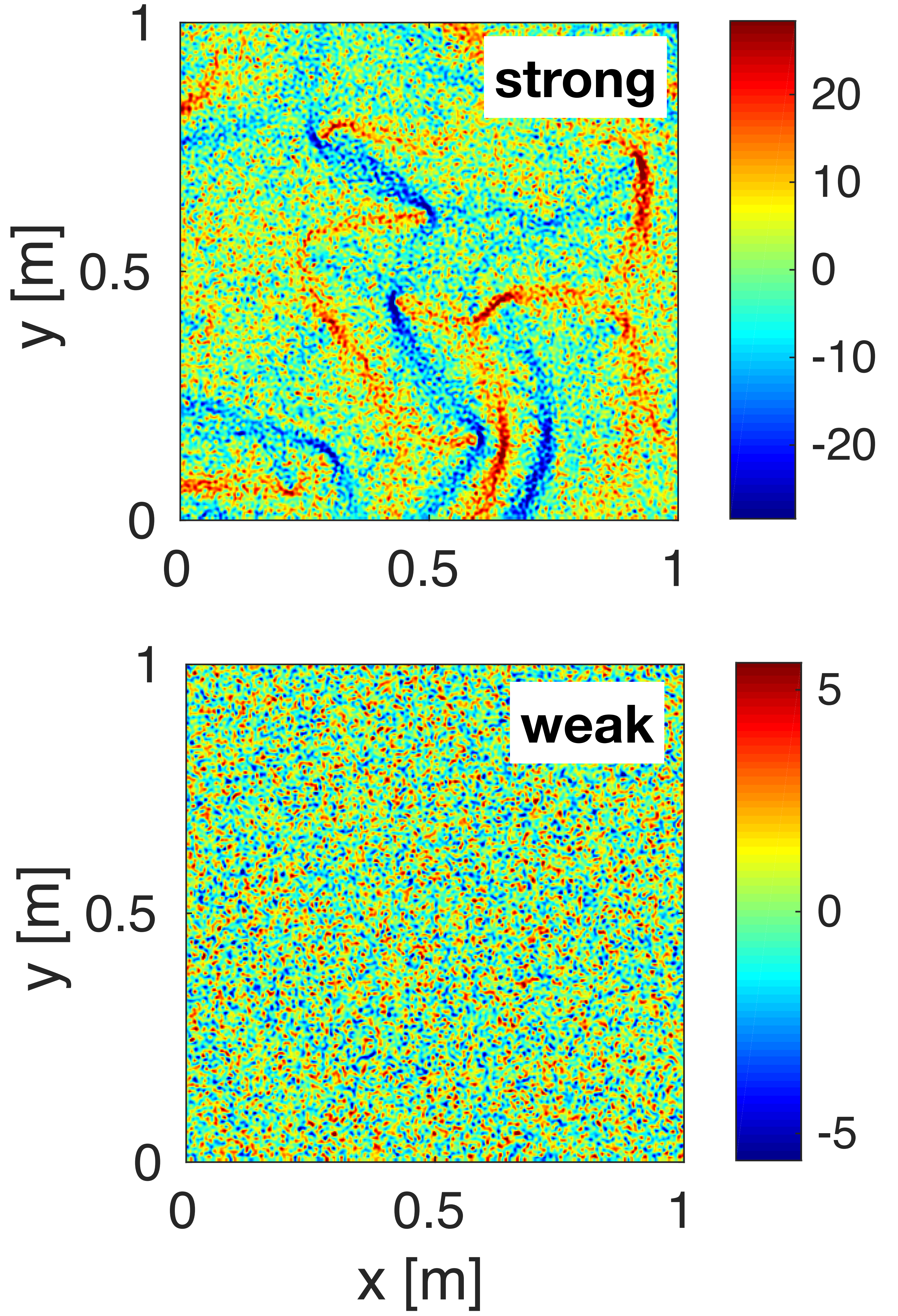
space-time spectrum

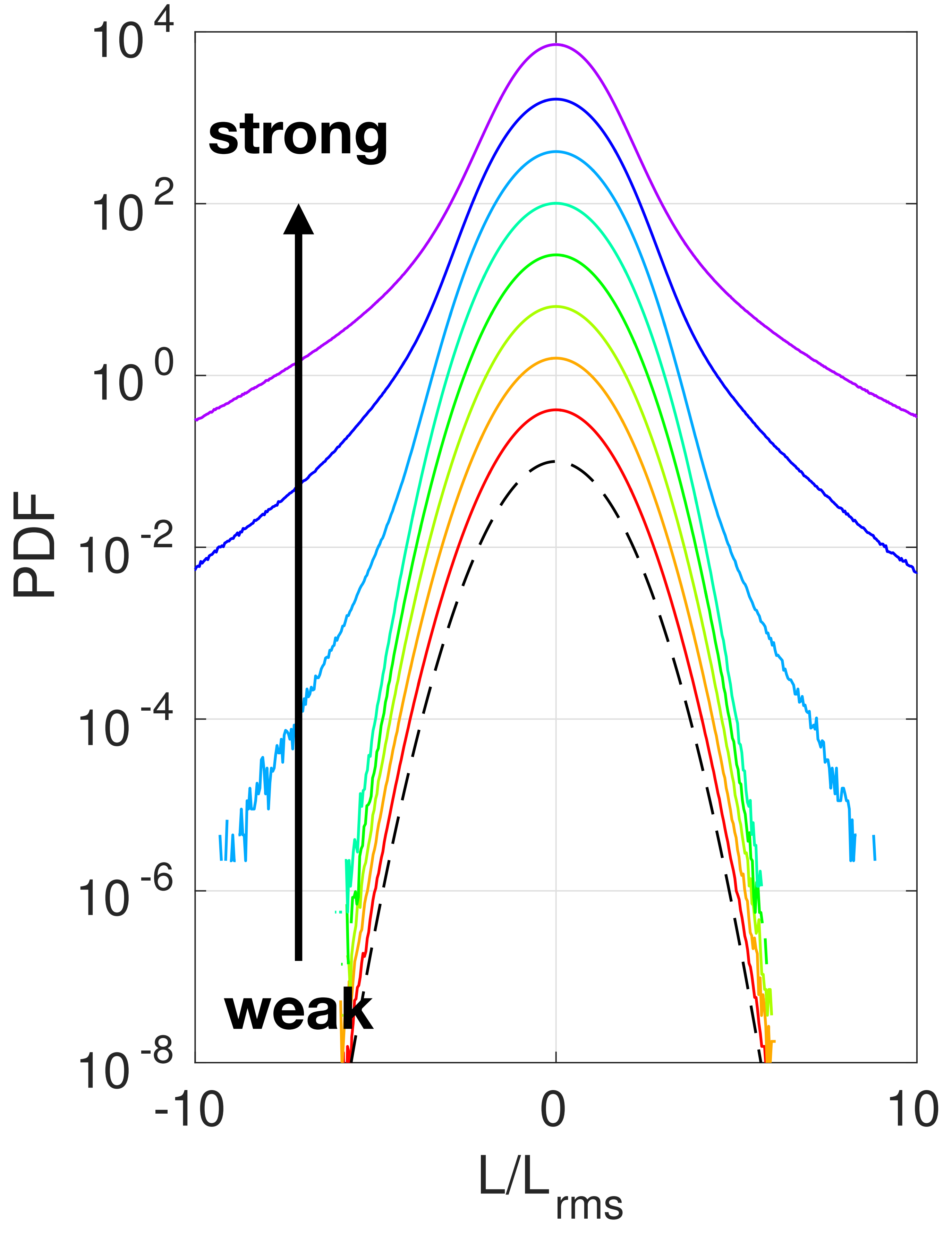


emergence of a new non-propagative component in the spectrum and intermittency

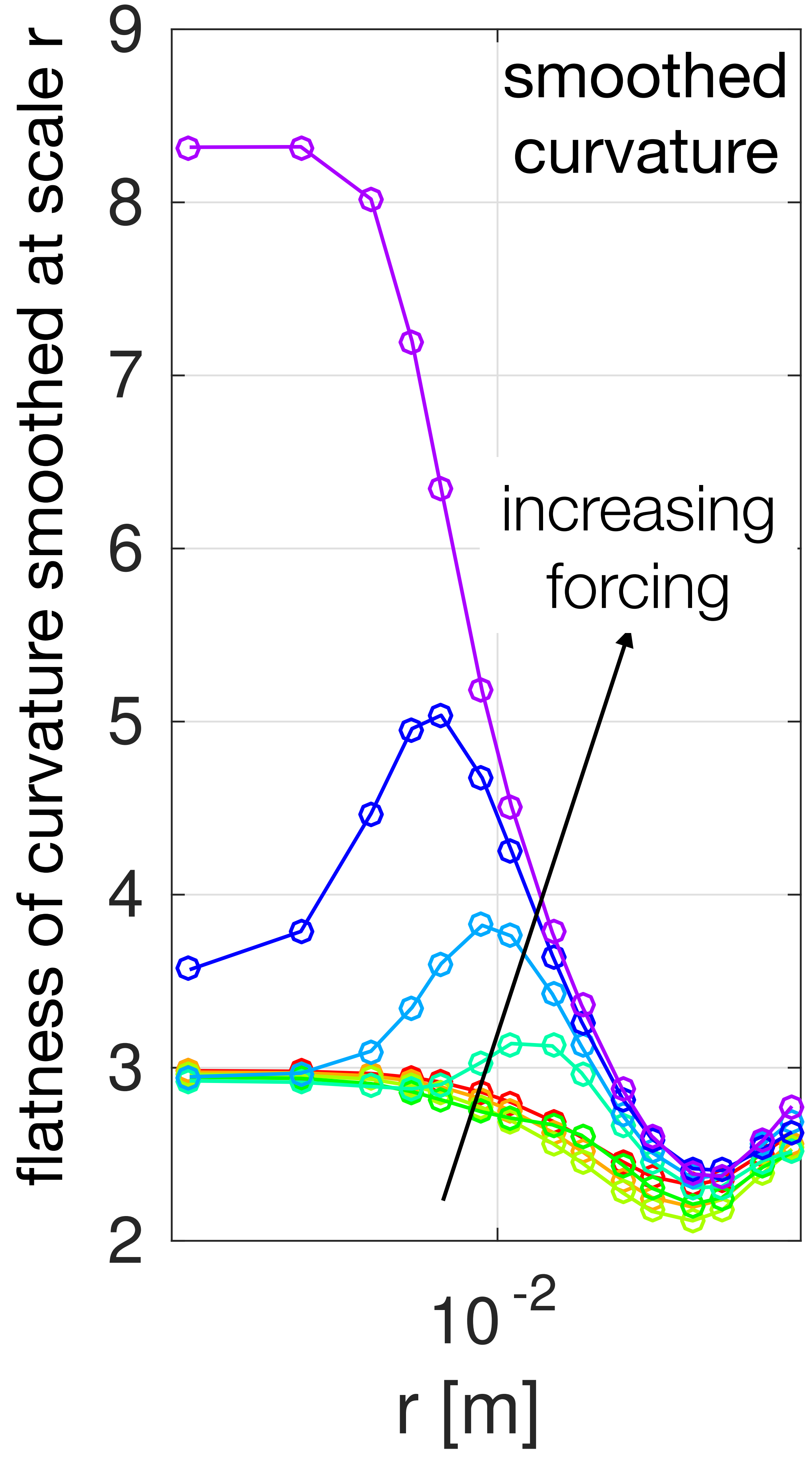
Mordant & Miquel *Phys. Rev. E* 2017

intermittency: statistics of curvature $\Delta\zeta$ (Laplacian)



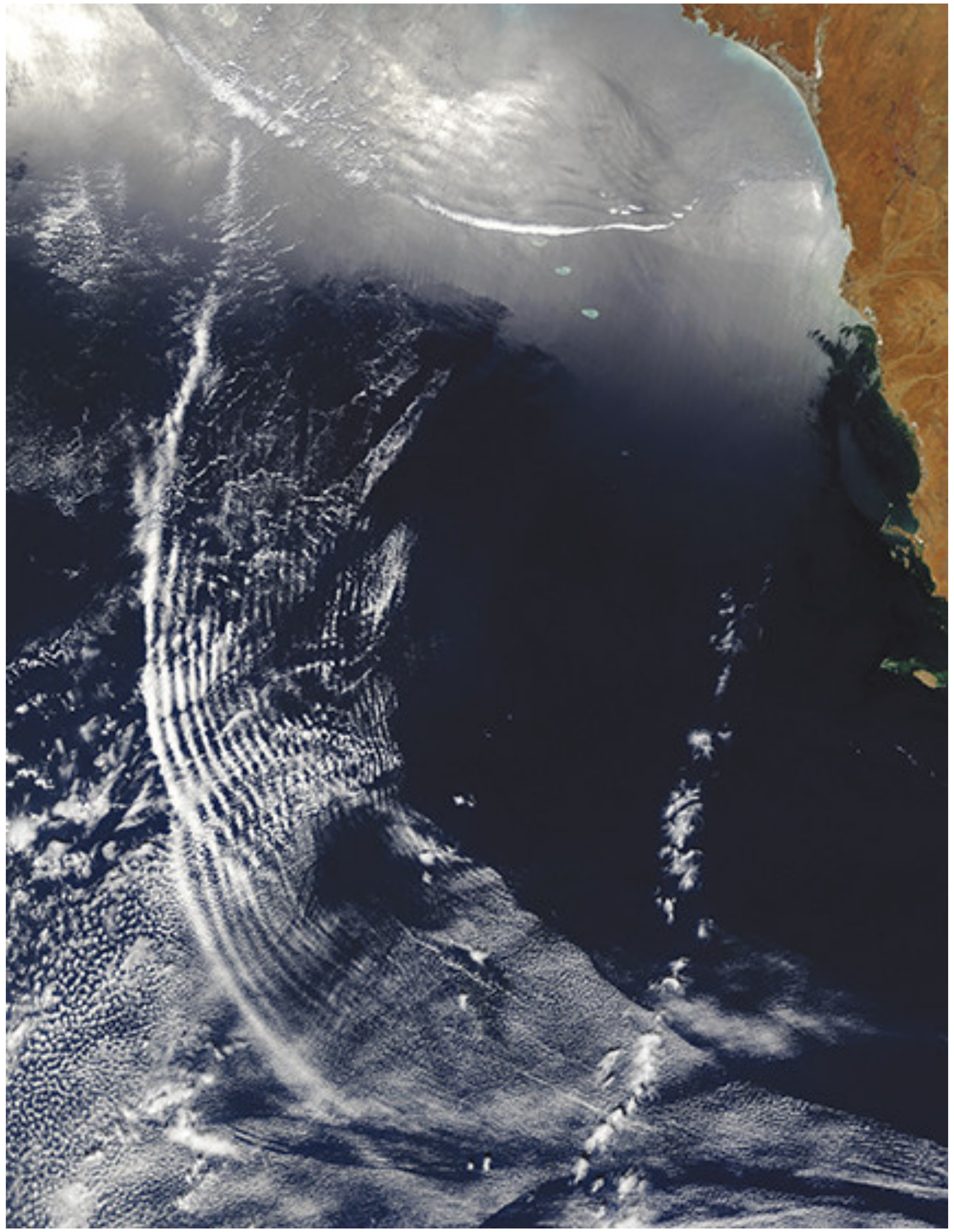


non Gaussian at strong forcing

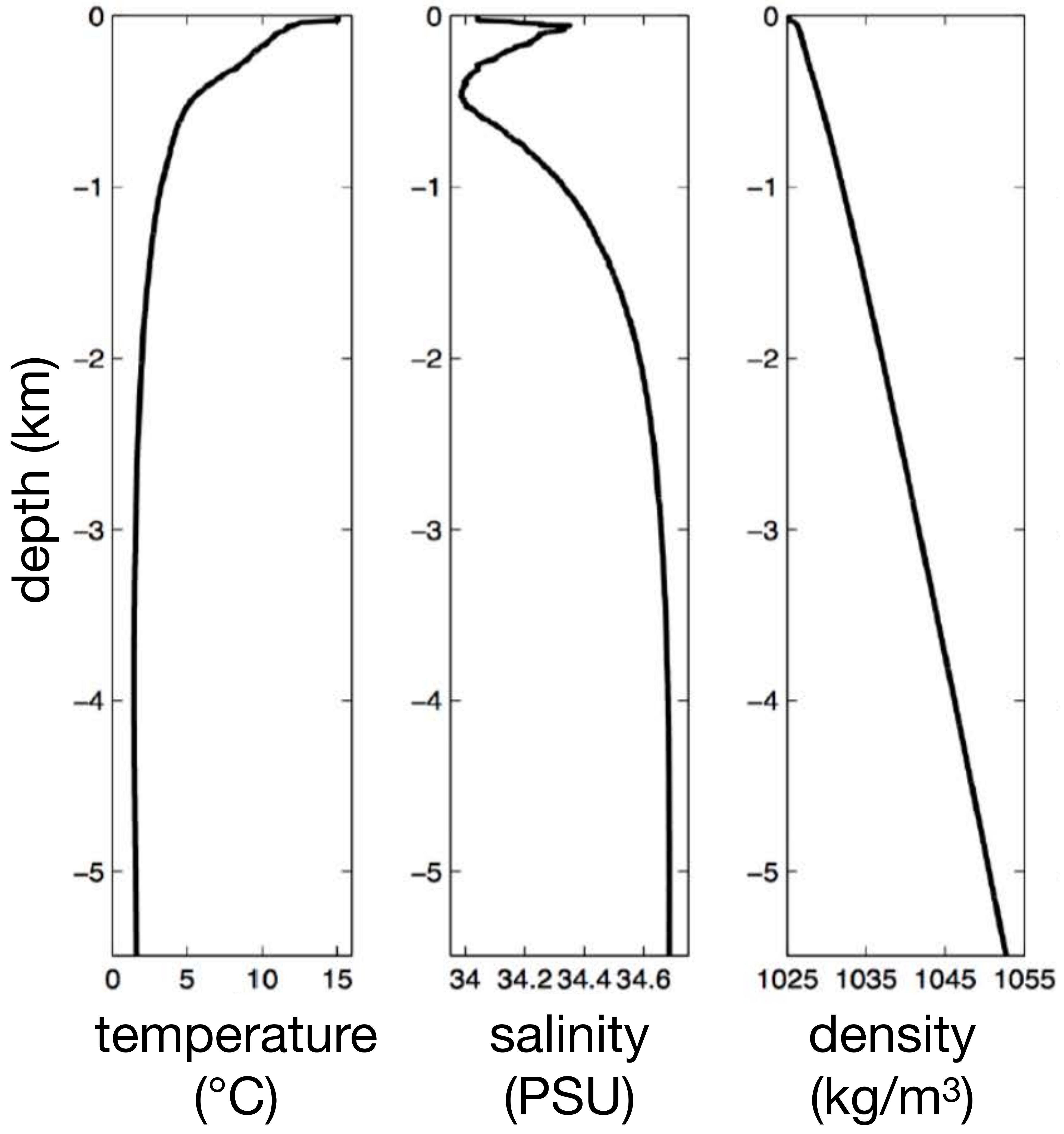


➥ intermittency

Internal waves — Stratified turbulence



Stratification



- the ocean is vertically stratified
- subject to the Earth rotation (Coriolis force)

geophysical flow

Boussinesq approximation

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{u} = -\vec{\nabla} p' + \nu \Delta \vec{u} - b\vec{e}_z - 2\vec{\Omega} \wedge \vec{u}$$

buoyancy (gravity) $b = \frac{\rho' g}{\bar{\rho}_0}$

Coriolis (rotation)

$$\frac{\partial b}{\partial t} + \vec{u} \cdot \vec{\nabla} b = -u_z N(z)^2$$

scalar conservation

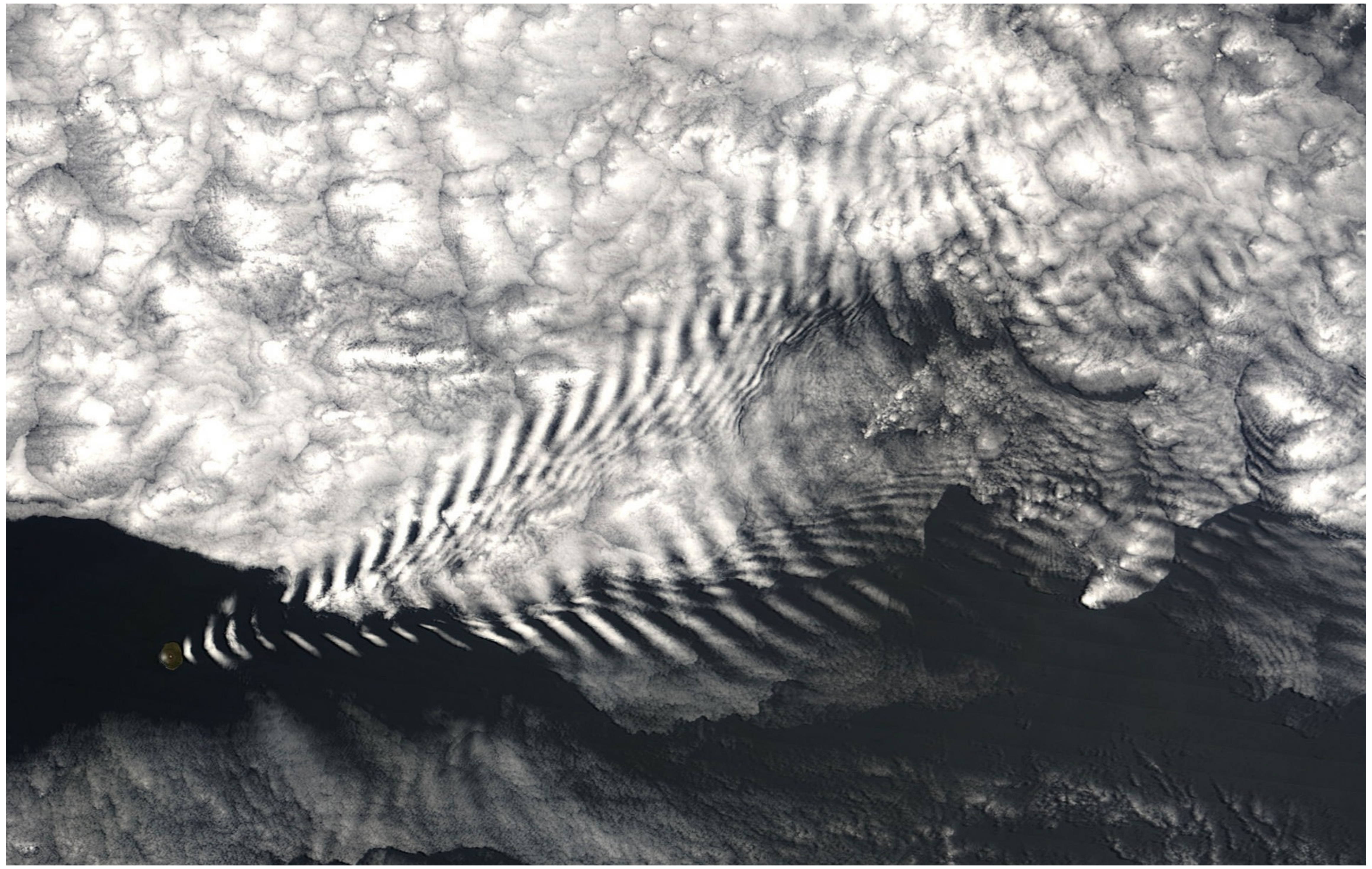
$$N(z) = \sqrt{-\frac{g}{\bar{\rho}_0}\frac{d\rho_0}{dz}}$$

Brünt-Vaissala frequency

Reynolds $Re = \frac{UL}{\nu}$ Rossby $Ro = \frac{U}{L\Omega}$ Froude $Fr = \frac{U}{NL}$

linear terms → waves

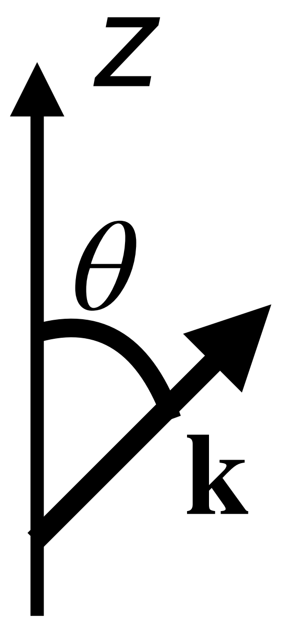
internal waves



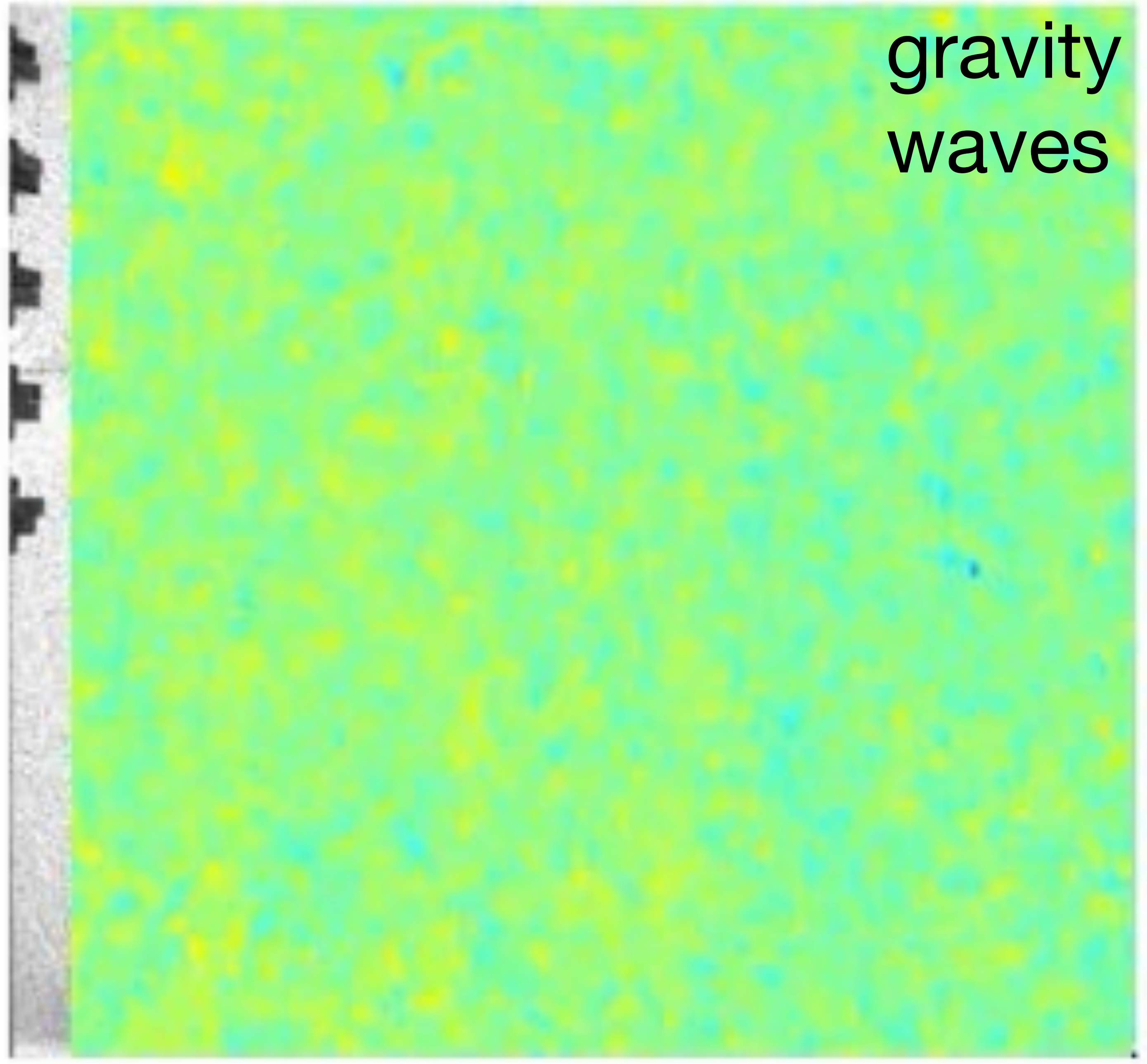
$$\omega^2 = N^2 \sin^2\theta + f^2 \cos^2\theta$$

gravity rotation

$$f = 2\Omega \le \omega \le N$$



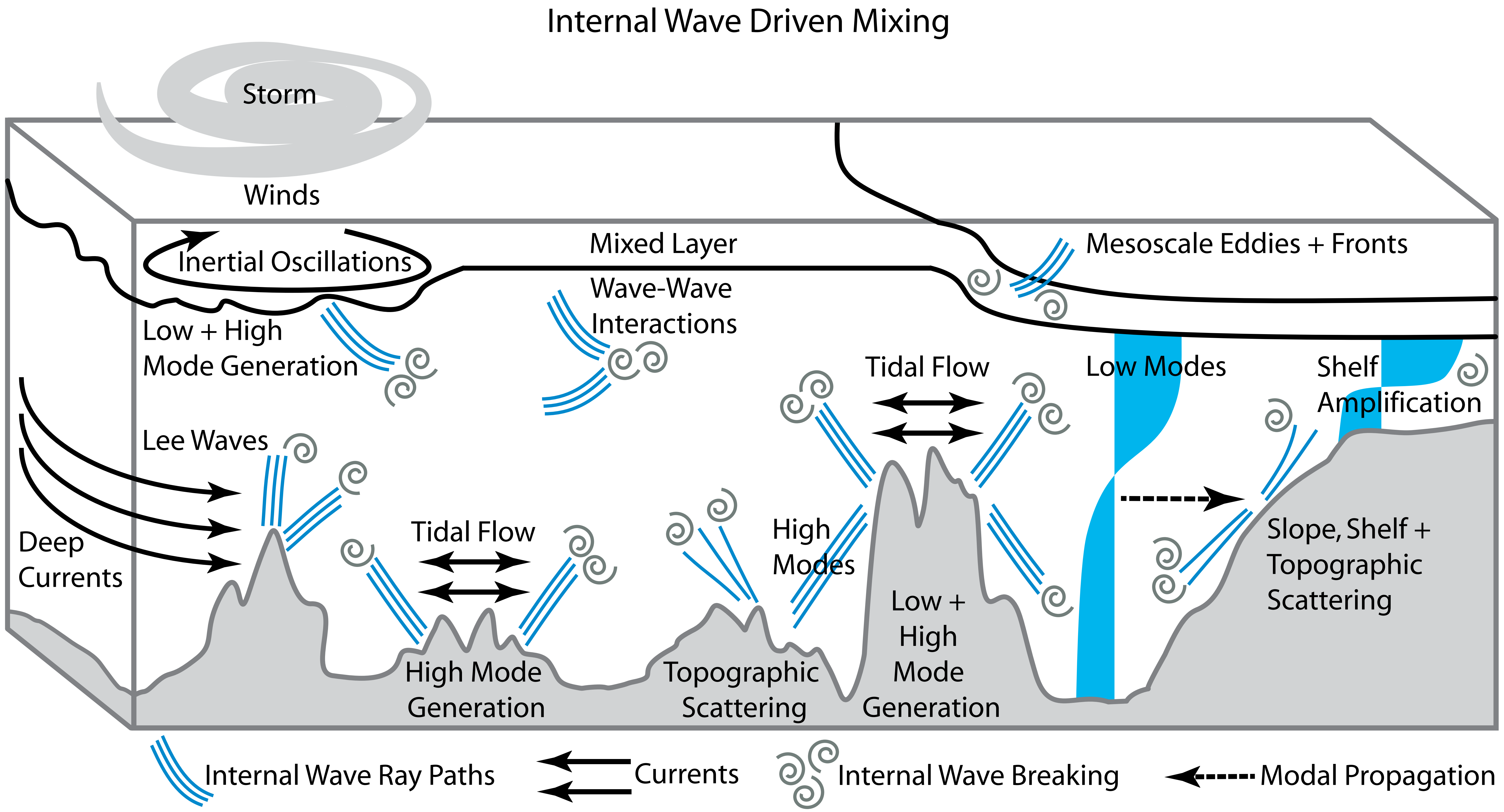
wave generator



movie
ENS Lyon

Internal waves

density stratified fluid
inertia-gravity waves

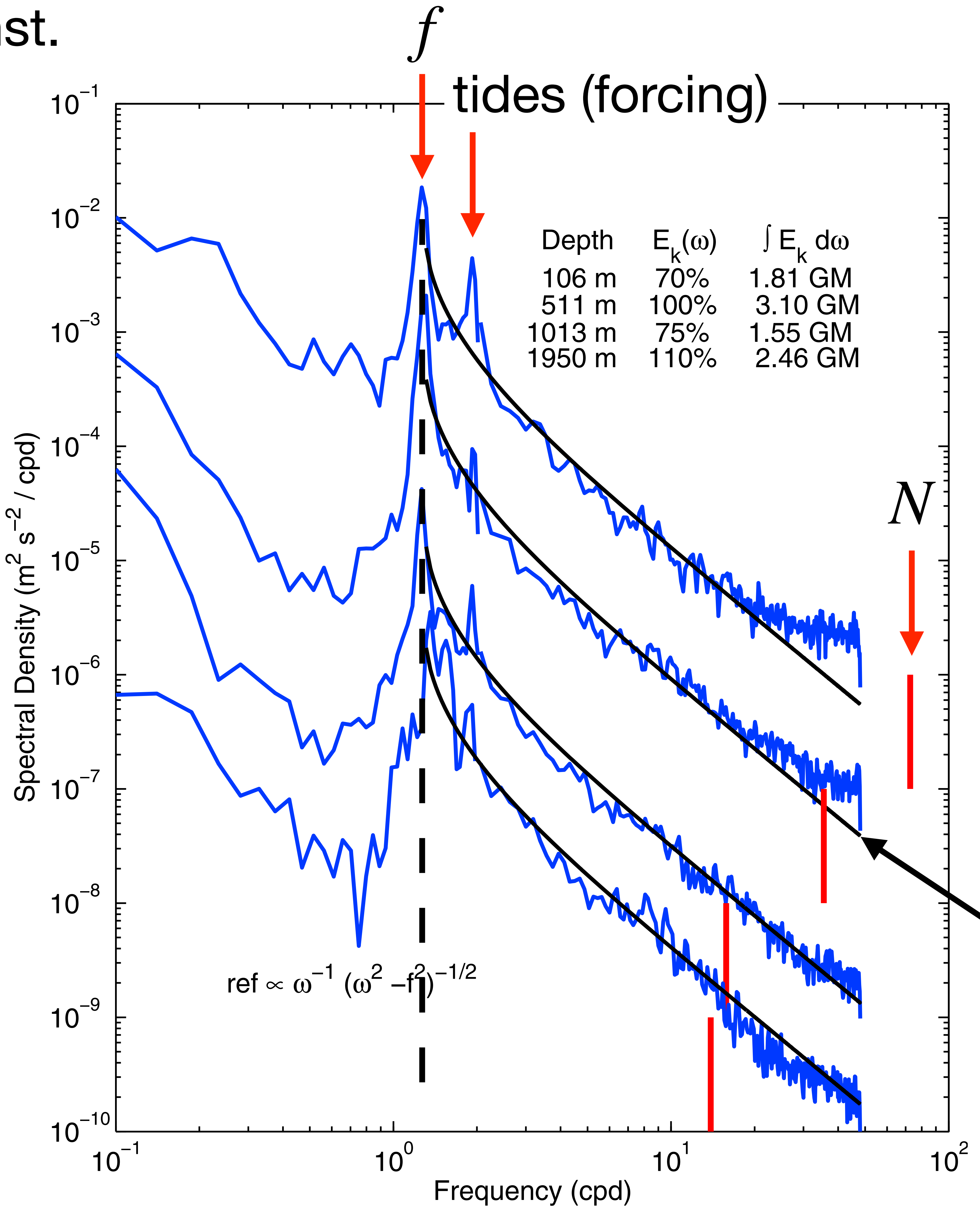


MacKinnon et al. *Bull. Am. Meteor. Soc* **98** 1997

Stratified and/or rotating turbulence

N. Mordant (Grenoble) with C. Savaro, C. Rodda (Simons post-doc)

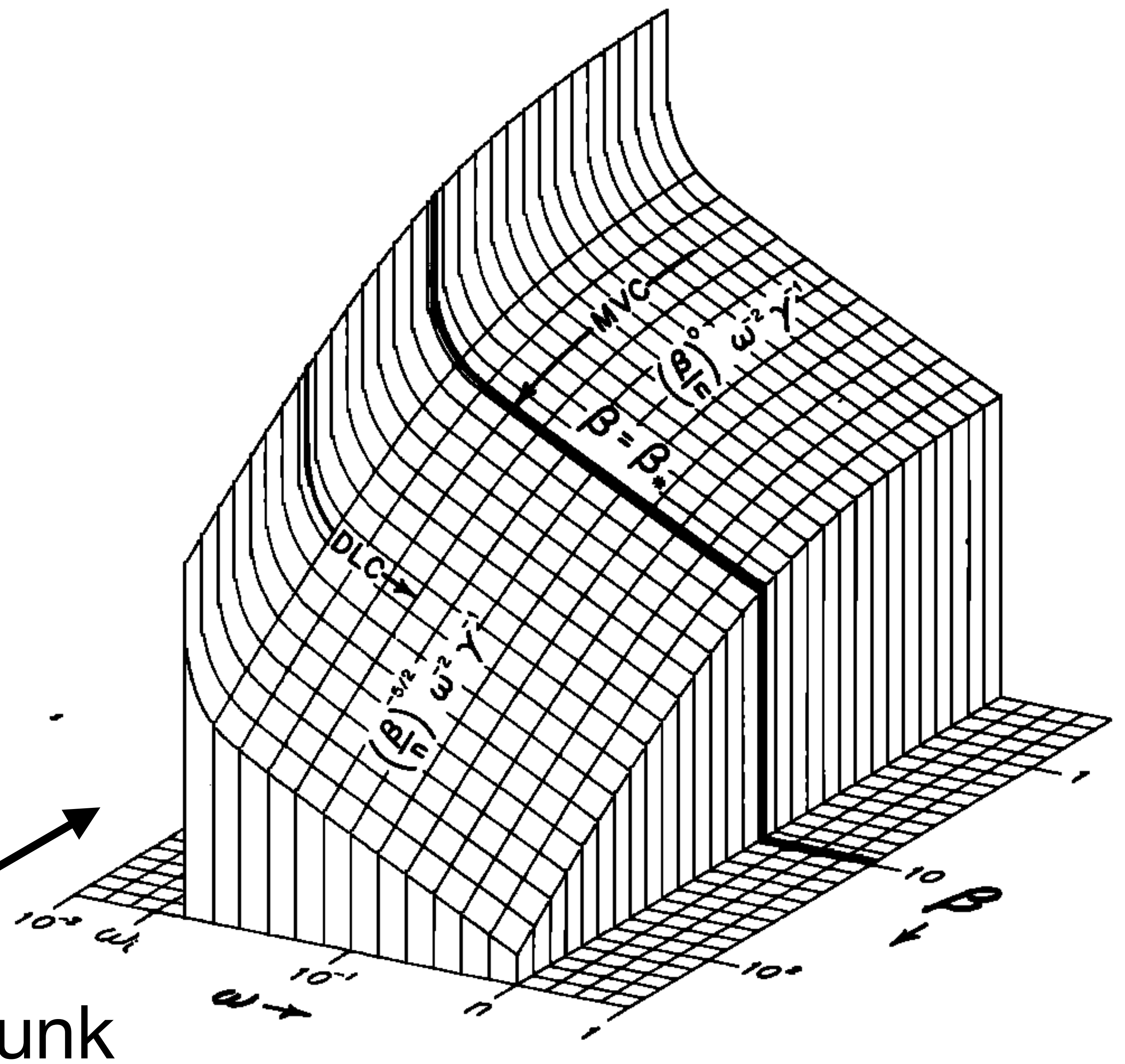
Atlantic ocean,
near Woods Hole Ocan. Inst.



Polzin & Lvov, *Rev. Geophys.* 2011

interpreted as
turbulence of internal waves

Garrett & Munk
spectrum



Garrett & Munk J. Geophys. Res. 1975

weak turbulence of internal waves ?

L'vov, Polzin & Tabak, *PRL* 2004

derivation of the kinetic equation
for internal gravity waves
$(f \ll \omega \ll N)$

$$\frac{dn_{k,m}}{dt} = \frac{1}{k}\int (R^{p}_{p_1 p_2} - R^{p_1}_{p p_2} - R^{p_2}_{p_1 p})\, dp_1 dp_2/\Delta^{k}_{k_1 k_2},$$

$$R^{p}_{p_1 p_2} = \delta_{\omega_p - \omega_{p_1} - \omega_{p_2}} f^{p}_{p_1 p_2} |V^{p}_{p_1 p_2}|^2 \delta_{m - m_1 - m_2 k k_1 k_2}, \qquad (4)$$

where $f^{p}_{p_1 p_2} = n_{p_1} n_{p_2} - n_p(n_{p_1} + n_{p_2})$ and $\Delta^{k}_{k_1 k_2} = \{2[(kk_1)^2 + (kk_2)^2 + (k_1 k_2)^2] - k^4 - k_1^4 - k_2^4\}^{1/2}/2.$

$$V^{\mathbf{p}}_{\mathbf{p}_1\mathbf{p}_2} = U^{\mathbf{p}}_{\mathbf{p}_1\mathbf{p}_2} + U^{\mathbf{p}_1}_{\mathbf{p}\mathbf{p}_2} + U^{\mathbf{p}_2}_{\mathbf{p}\mathbf{p}_1} \text{ with } U^{\mathbf{p}}_{\mathbf{p}_1\mathbf{p}_2} = -\frac{N}{4\sqrt{2g}} \frac{\mathbf{k}_2 \cdot \mathbf{k}_3}{k_2 k_3} \sqrt{\frac{\omega_{\mathbf{p}_2}\omega_{\mathbf{p}_3}}{\omega_{\mathbf{p}_1}}} k_1.$$

family of solutions

$$n(k,m) \propto k^{-a}|m|^{-b}$$

only stationary solution:

$$n(k,m) \propto k^{-3.69}|m|^{0}$$

direct cascade in scales

see Dematteis & Lvov, *ArXiv 2010.06717*

Garrett & Munk:

$$n(k,m) \propto k^{-4}|m|^{0}$$

experiments for strongly stratified turbulence

Reynolds $Re = \frac{UL}{\nu}$ (Rossby $Ro = \frac{U}{L\Omega}$) Froude $Fr = \frac{U}{NL}$

(for pure rotation: see work by P.-P. Cortet, PRL 2020)

for stratified turbulence:

$$Re = \frac{UL}{\nu} \gg 1 \qquad Fr = \frac{U}{NL} \ll 1$$

technically with water & salt $N \lesssim 1$ rad/s and ν given → U moderate and L large → **large facility**

buoyancy Reynolds number $Re_b = ReFr^2 \gg 1$

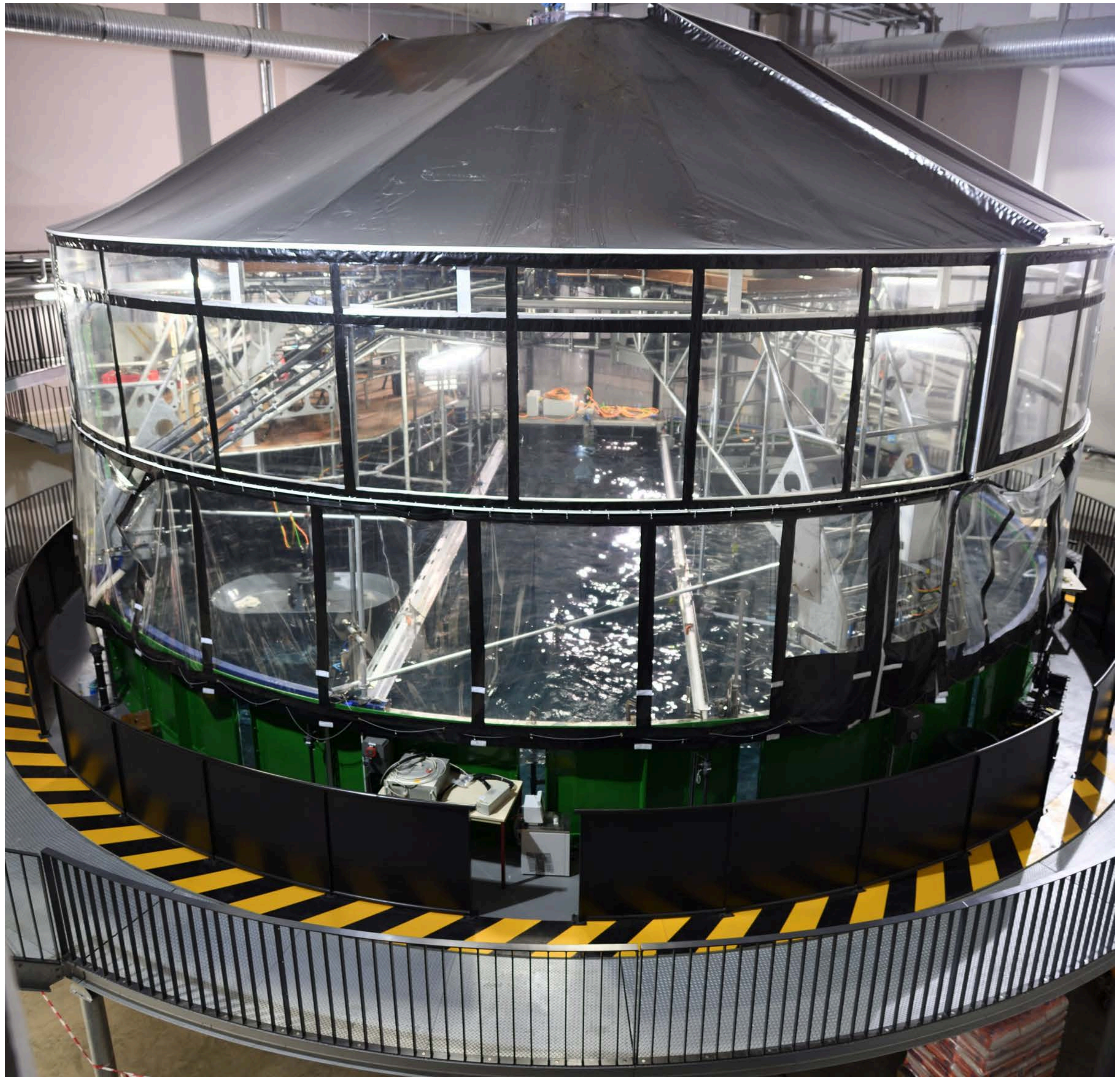
see Brethouwer, Billant, Lindborg & Chomaz *JFM* 2007

there are not only waves but also vorticity !

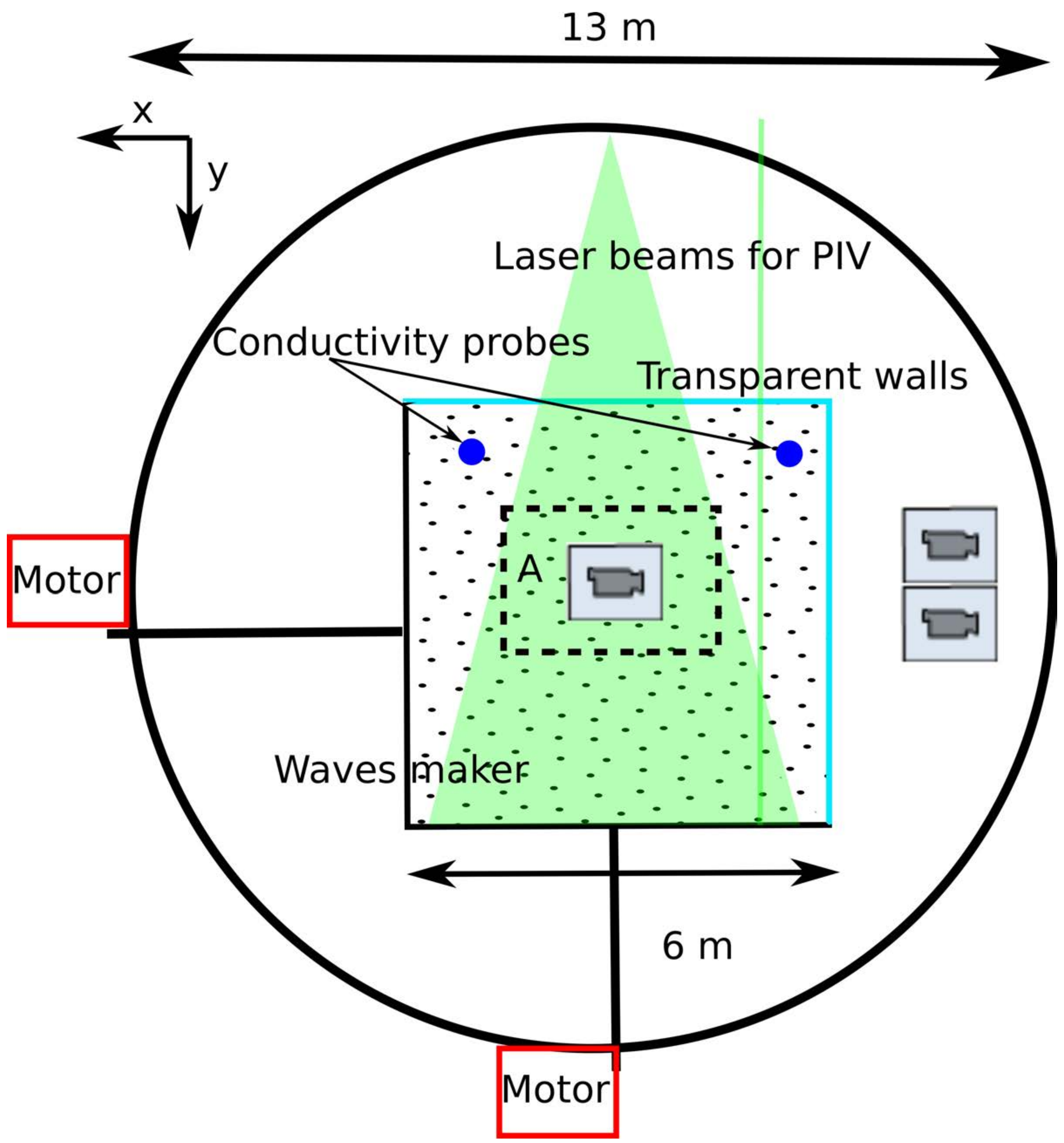
internal gravity wave turbulence in Coriolis facility

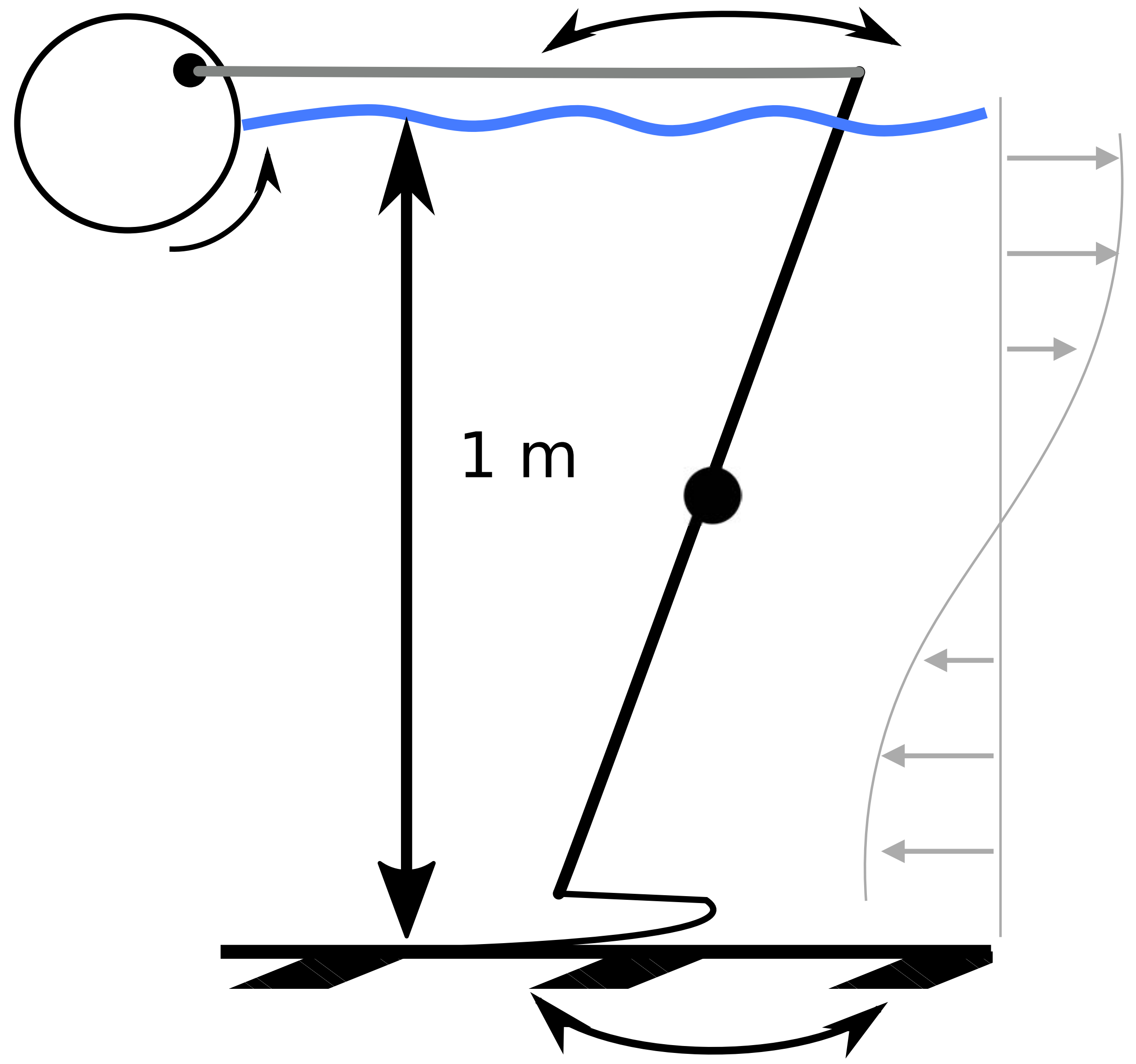
Coriolis facility in Grenoble

- 13m-diameter, 1m-deep, 130 tons water
- rotation up to 6 rpm
- stratification with salt (~1ton of salt)

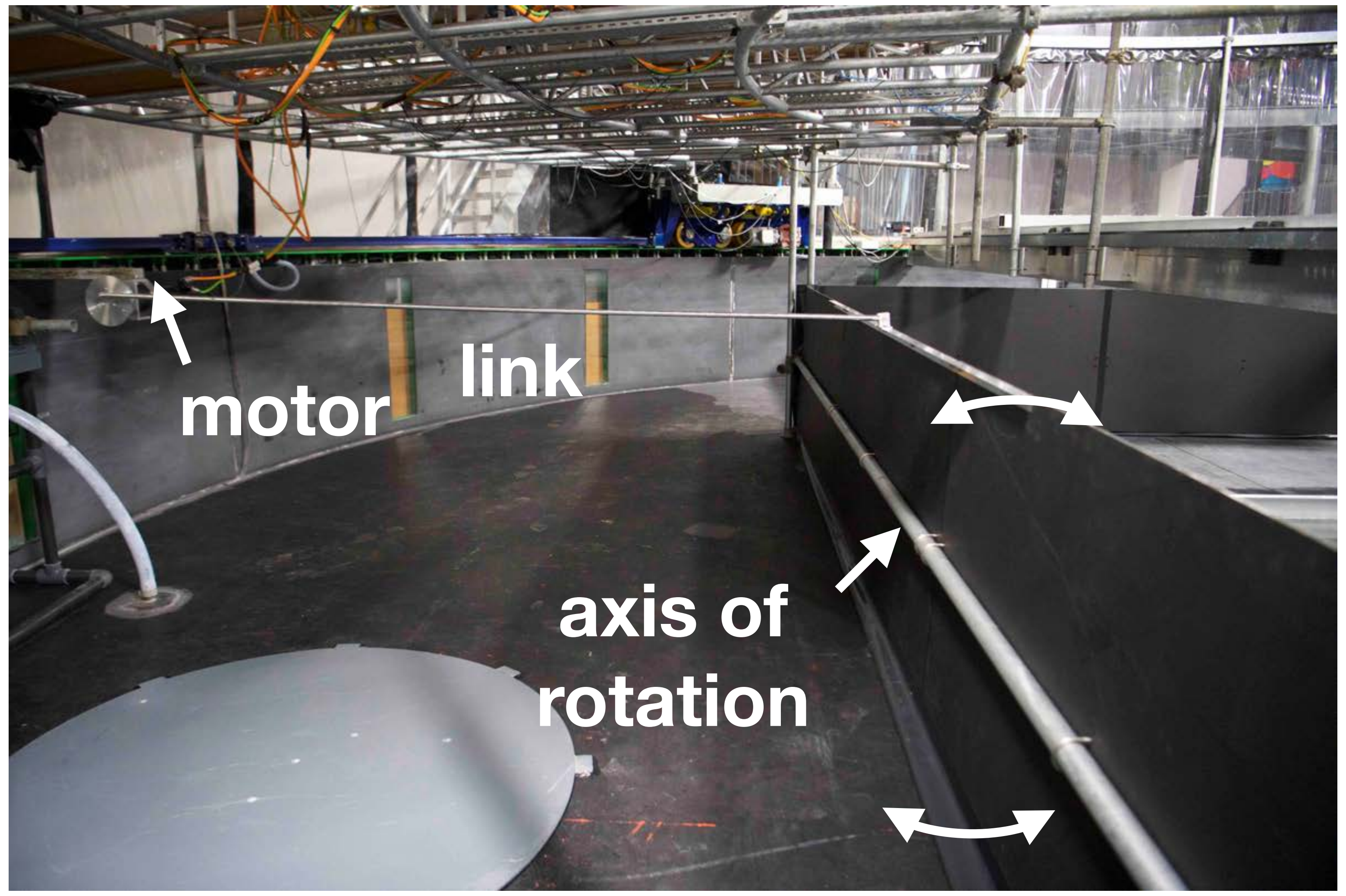


internal gravity wave turbulence in Coriolis facility

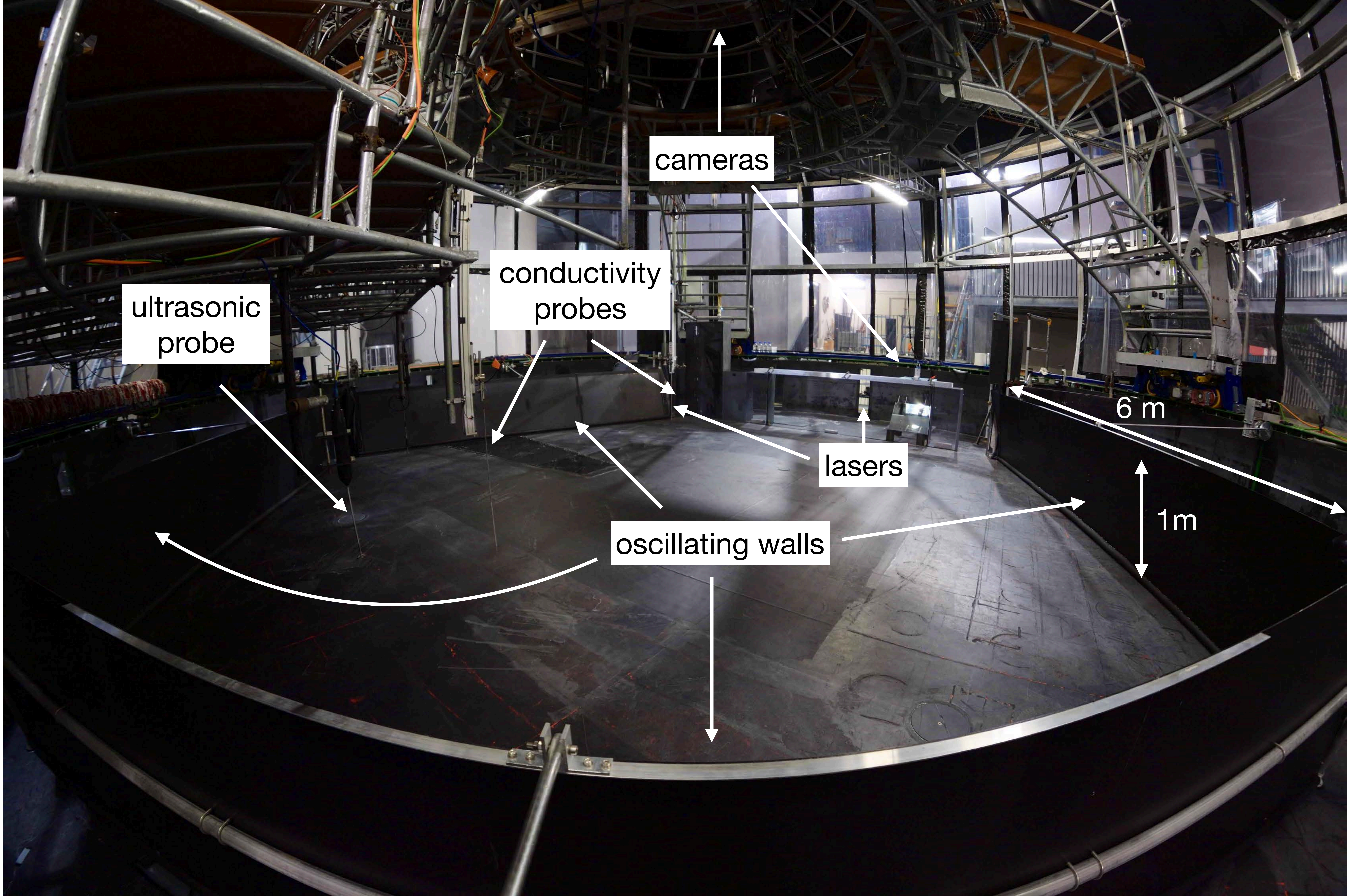


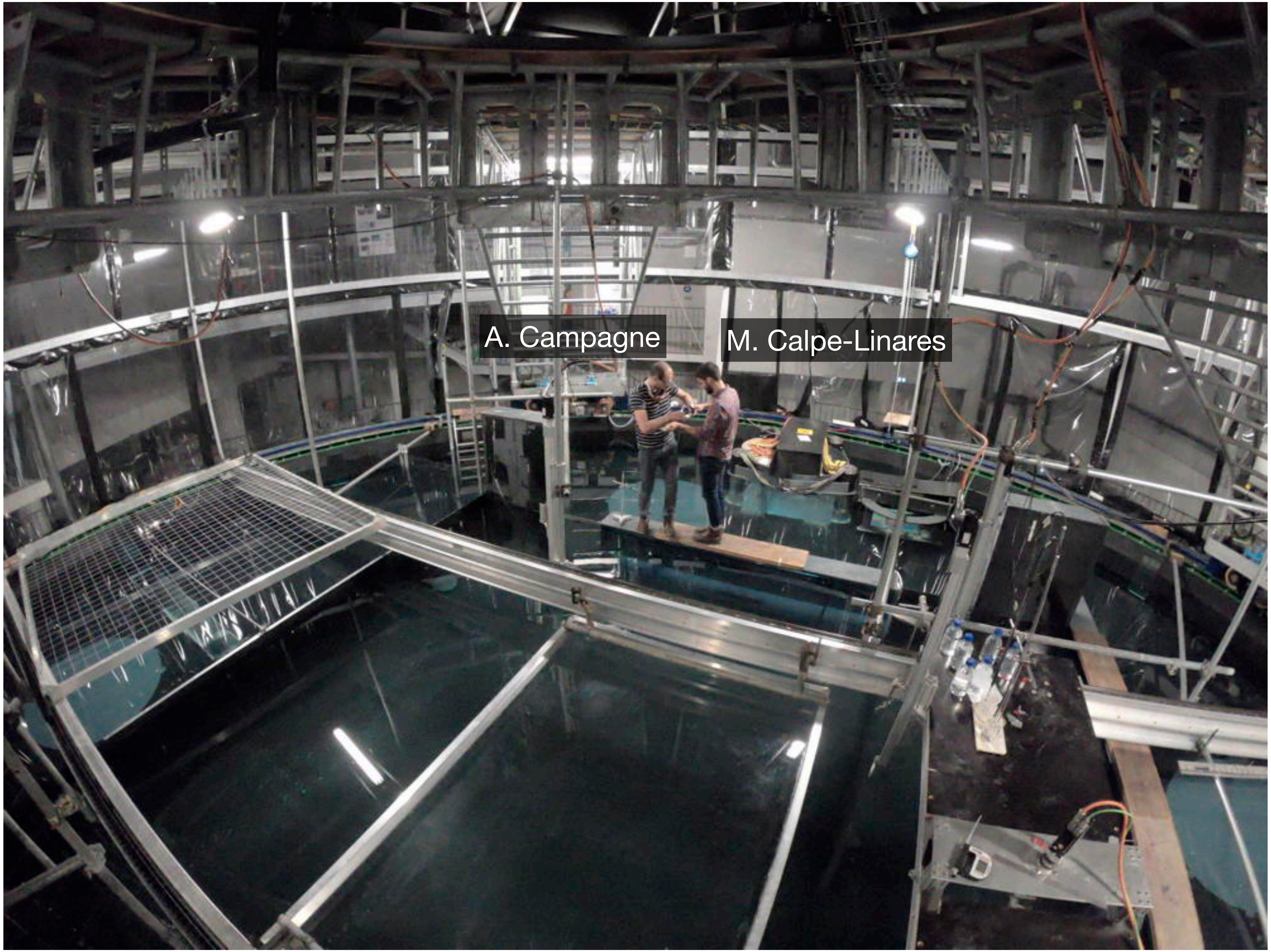


linearly stratified water+salt
(with ~1 ton of salt !)



difficulty: 3D and anisotropic !


cameras
conductivity probes
ultrasonic probe
lasers
6 m
1m
oscillating walls

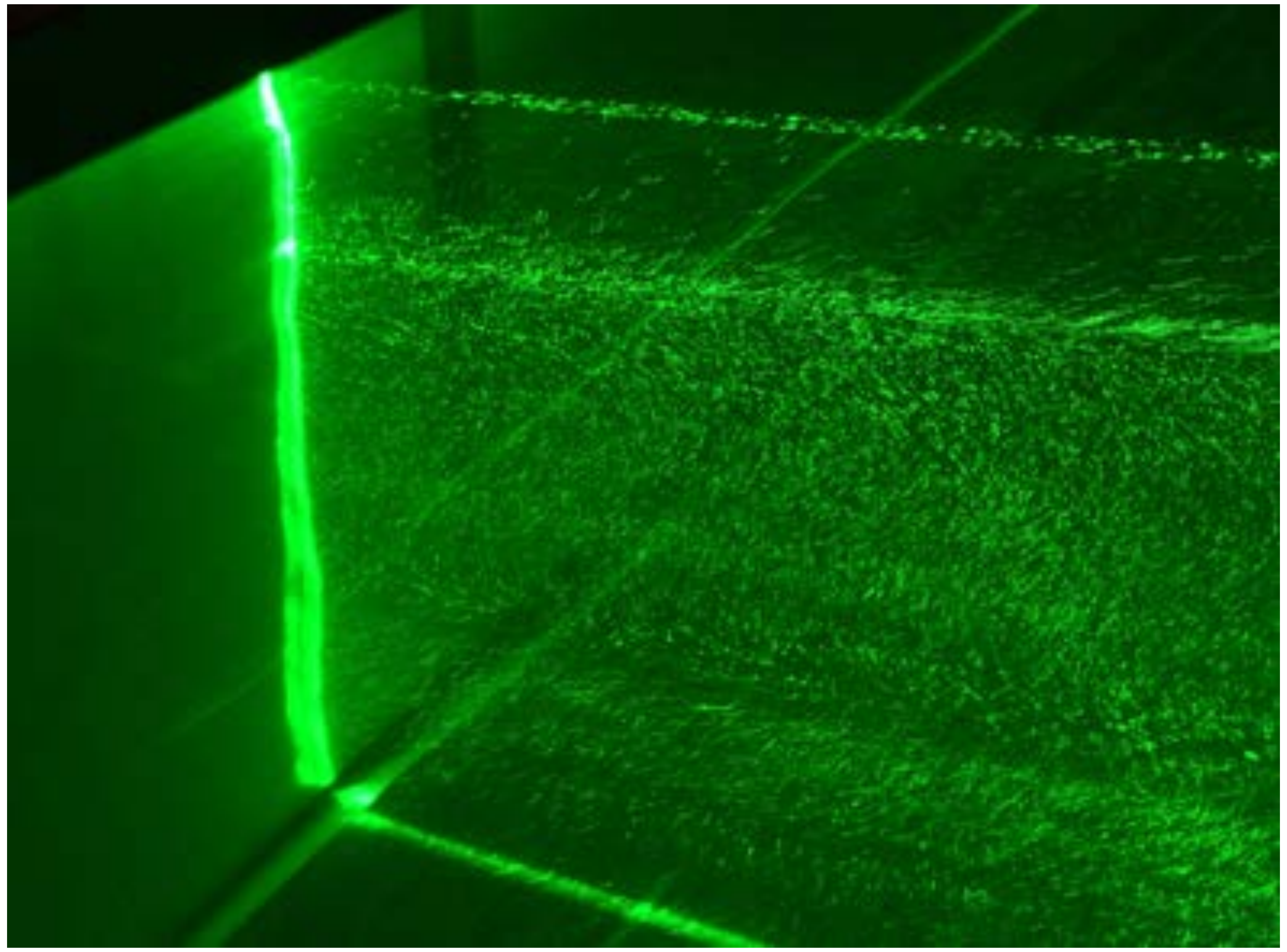

A. Campagne
M. Calpe-Linares

time-resolved Particle Image Velocimetry (PIV) measurement

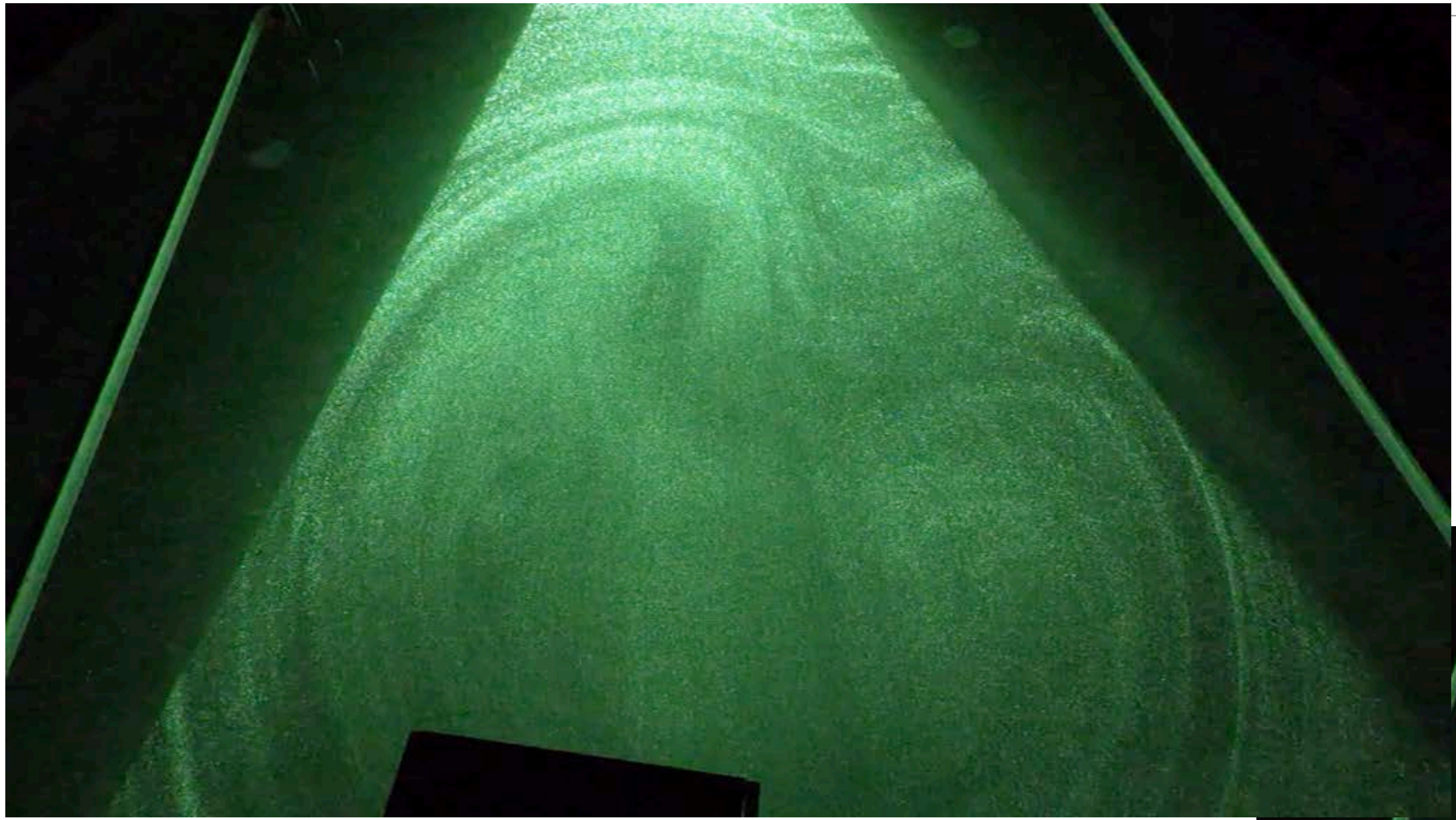
horizontal laser sheet (25W): horizontal velocity
scanned vertically for 3D measurement (3 scans/s)
12 Mpixels camera, 100 frames/s
3D-2C PIV

seeding of the fluid by
polystyrene particles matched in density
(for stratified flows)

vertical laser sheet: 2D PIV

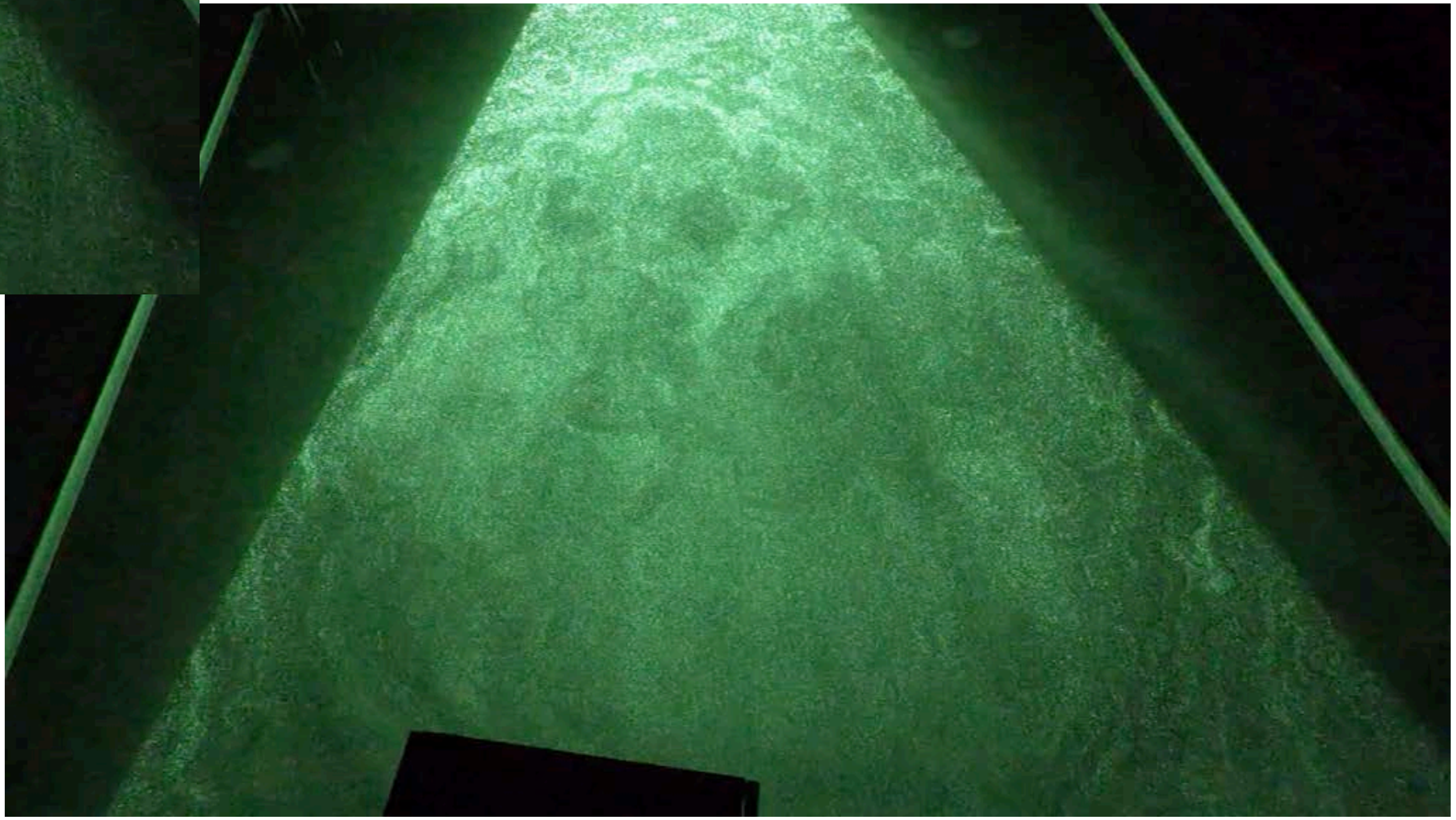


time-resolved Particle Image Velocimetry (PIV) measurement



accelerated 50 times…

after a few minutes



transitory state after starting the forcing

- development of small scales
- development of other frequencies
- mixing

frequency spectrum: parameters

A [cm]	u_f [cm/s]	Re_f	Fr_f	Re_{bf}	σ_u [cm/s]	σ_v [cm/s]	σ_w [cm/s]	k_b [m^{-1}]
2	0.8	8400	0.014	1.6	0.4	0.6	0.8	70
3	1.3	13000	0.021	5.6	0.7	0.8	1.0	50
4	1.7	17000	0.028	13	1.0	1.4	1.1	35
5	2.1	21000	0.035	26	1.5	1.4	n/a	28

$N = 0.6$ rad/s

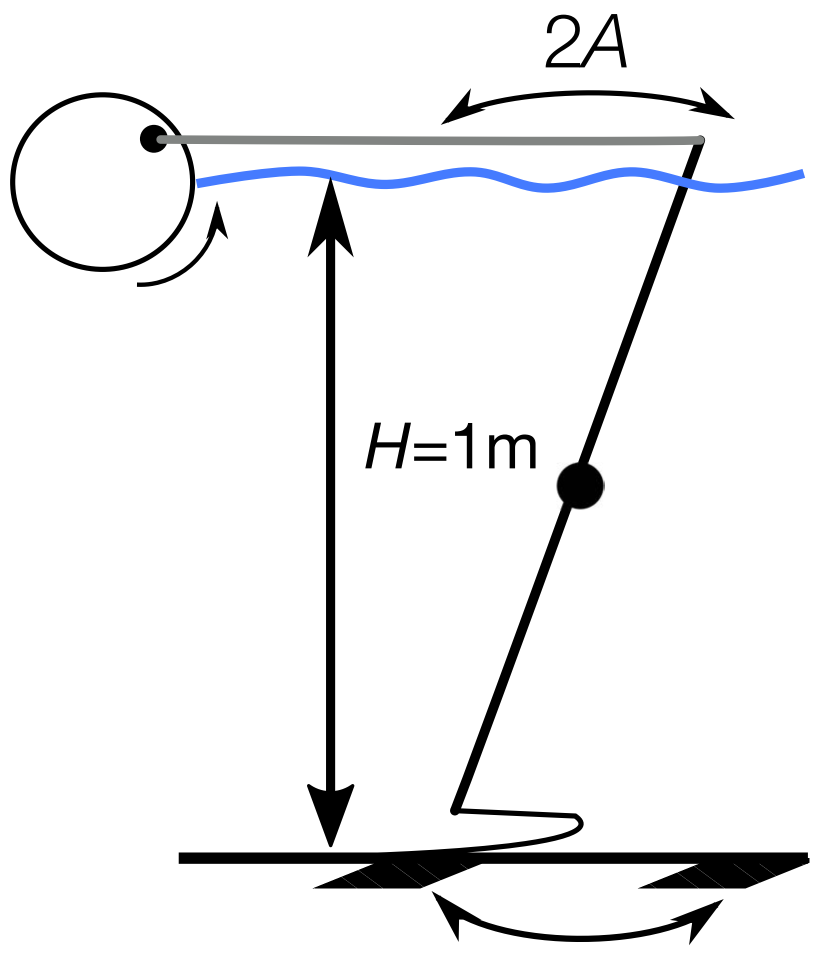
frequency of forcing $\omega_f/N = 0.7$

velocity of forcing $u_f = 0.7NA$

Reynolds number $Re_f = u_f H/\nu$

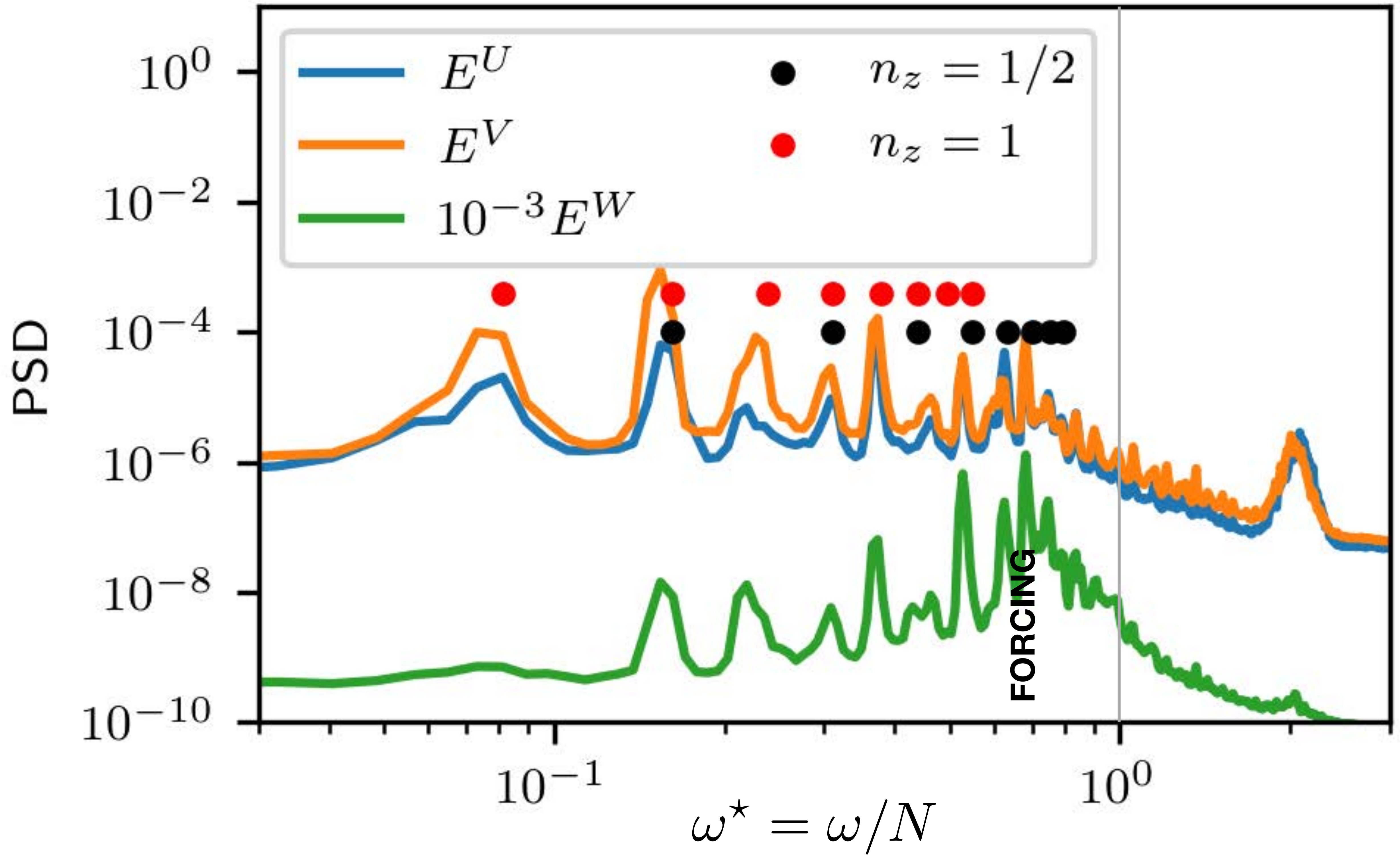
Froude number $Fr_f = u_f/NH = 0.7A/H$

buoyancy Reynolds number $Re_{bf} = Re_f Fr_f^2 = 0.34 A^3 N/H\nu$



Savaro et al. PRFluids 2020

frequency spectrum: very weak forcing



modes of the rectangular box $p(x, y, z) \propto \cos(2\pi n_x x/L)\cos(2\pi n_y y/L)\cos(2\pi n_z z/H)$

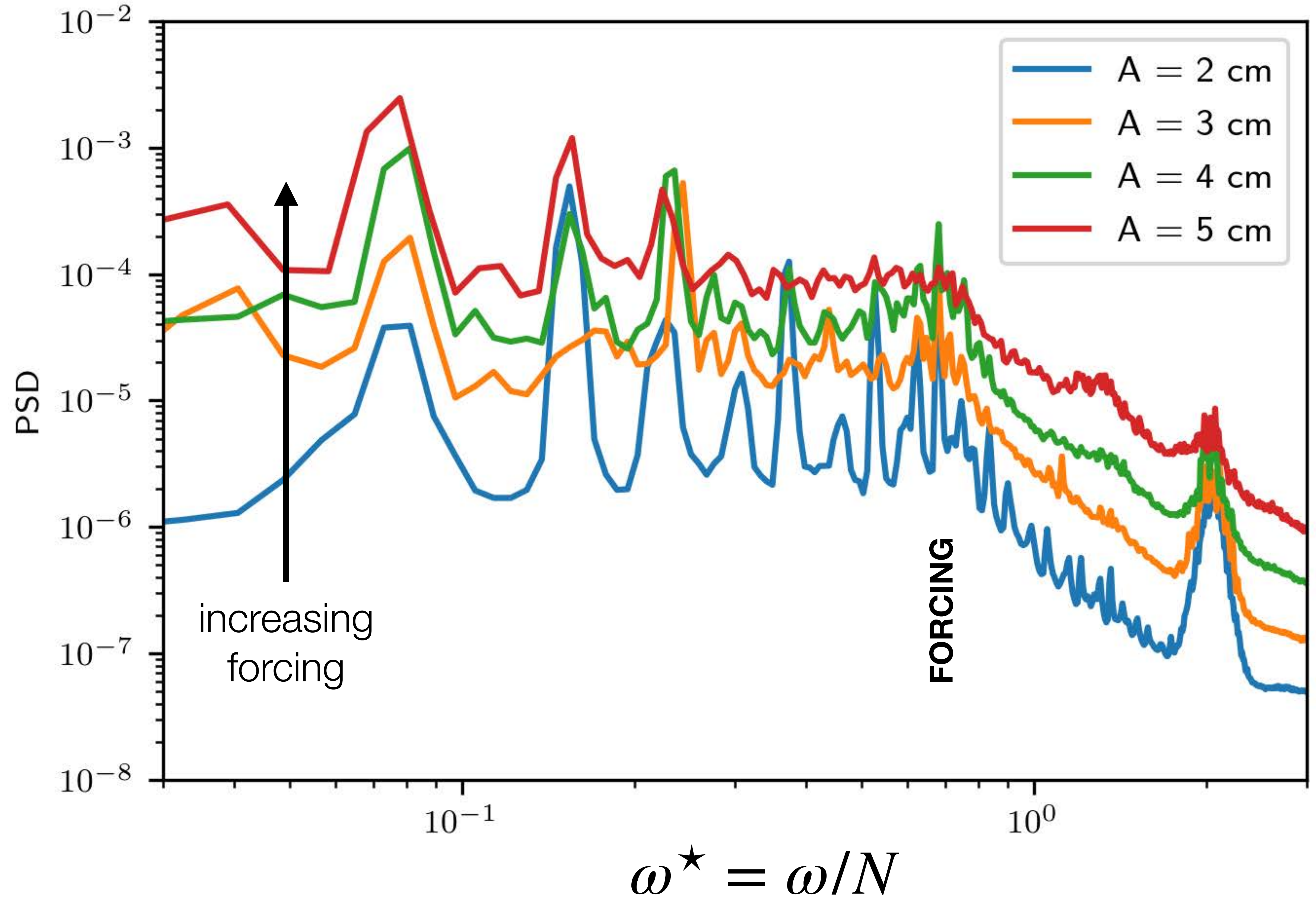
$$\omega^* = \frac{1}{\sqrt{1 + \frac{(n_z/H)^2}{(n_x^2+n_y^2)/L^2}}}$$

2D modes $n_x = 0$ or $n_y = 0$

➤ **non linearly generated waves !**

Savaro et al. PRFluids 2020

frequency spectrum



$Re_{bf} = 1.6$

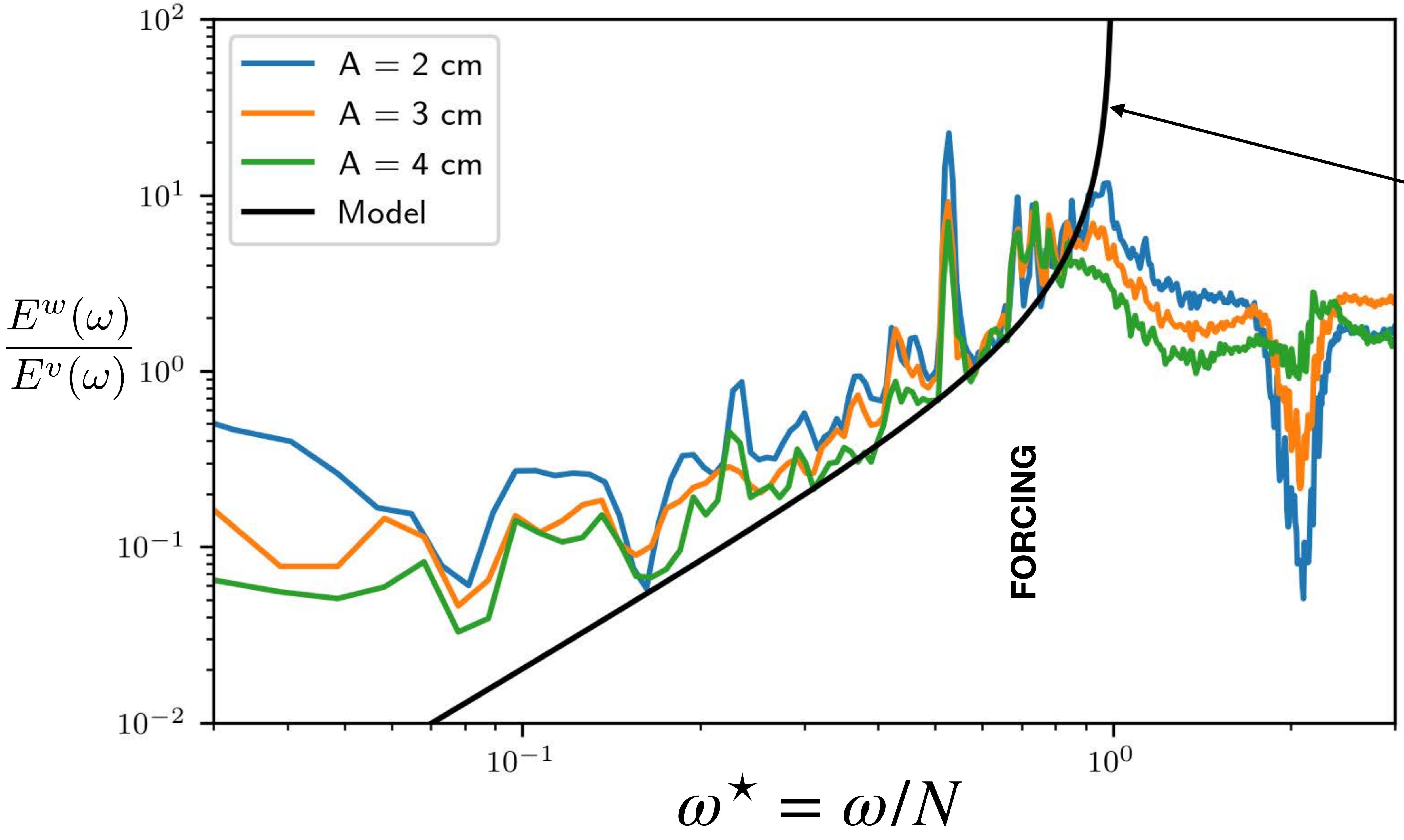
$Re_{bf} = 26$

A: amplitude of wave maker oscillation

when increasing the forcing: the spectrum becomes more continuous

Savaro et al. PRFluids 2020

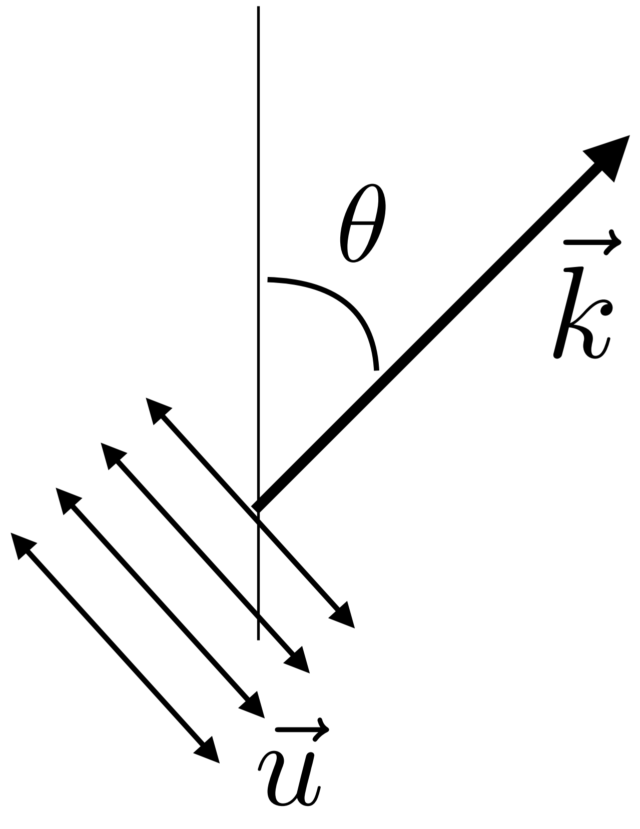
wave turbulence ?



axisymmetric superposition of random linear waves:

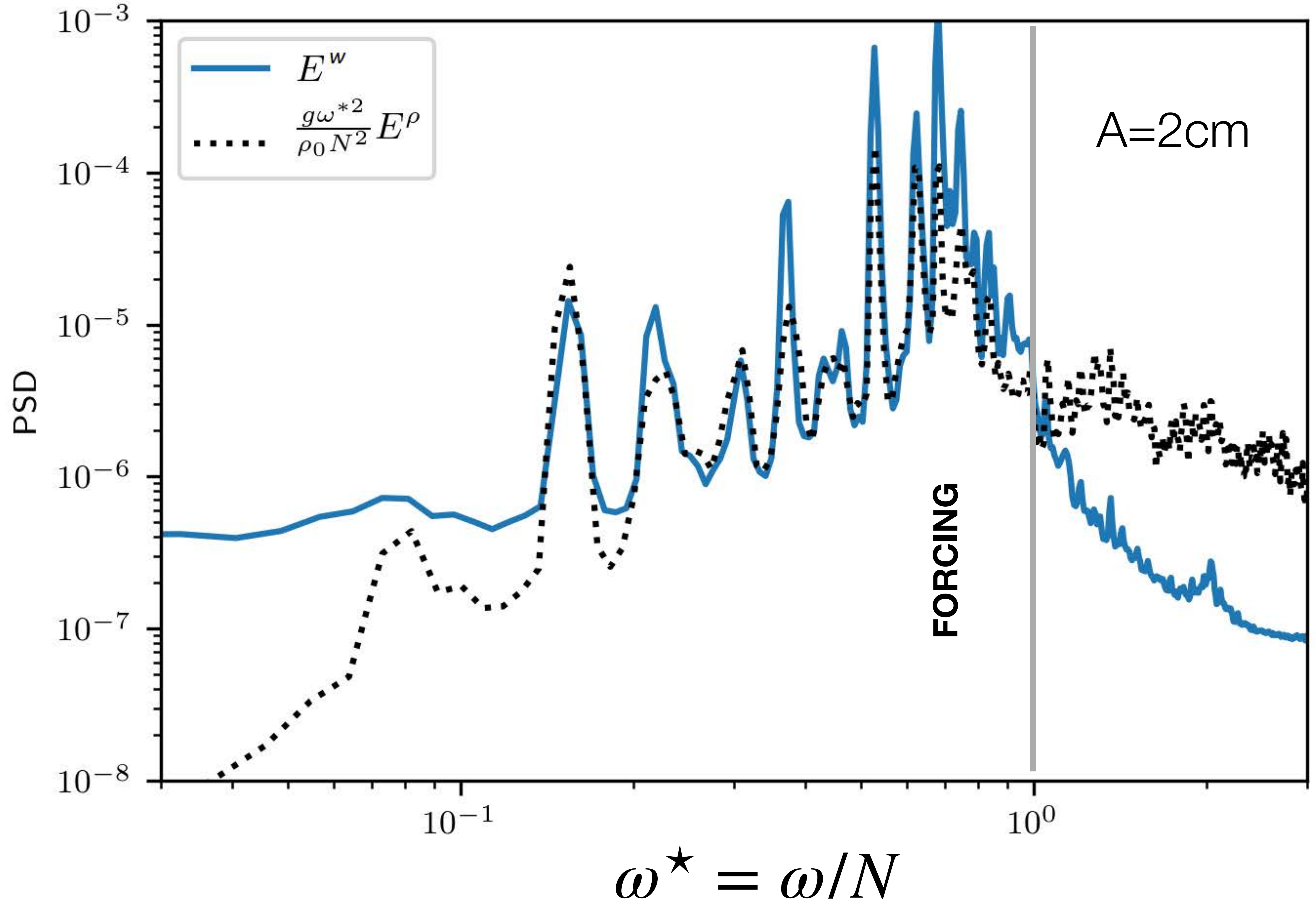
kinematic relation

$$\frac{E^w(\omega)}{E^v(\omega)} = \frac{2\sin^2\theta}{\cos^2\theta} = \frac{2(\omega^*)^2}{1-(\omega^*)^2}$$



Savaro et al. PRFluids 2020

wave turbulence ?



linearized equation for buoyancy

$$b = -\frac{g\delta\rho}{\rho_r}$$

$$\frac{\partial b}{\partial t} = -N^2 w$$

⬇

$$E^w(\omega) = E^b(\omega)\omega^2/N^4$$

Savaro et al. PRFluids 2020

looking for the dispersion relation…

space-frequency correlations $$C^{ver}(\mathbf{r},\omega) = \frac{\langle v(\mathbf{R}_0+\mathbf{r},\omega)v^\star(\mathbf{R}_0,\omega) + w(\mathbf{R}_0+\mathbf{r},\omega)w^\star(\mathbf{R}_0,\omega) + c.c.\rangle}{\langle |v(\mathbf{R}_0,\omega)|^2 + |w(\mathbf{R}_0,\omega)|^2\rangle}$$

model: axisymmetric superposition of random linear plane waves

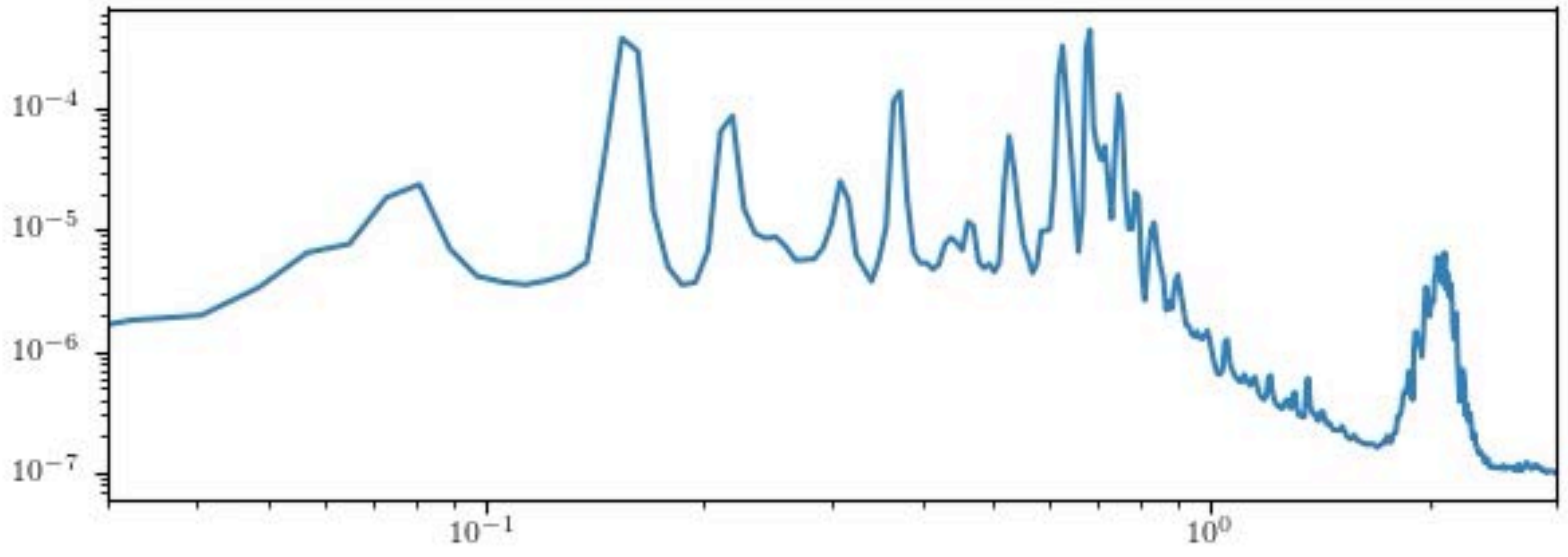
inspired by Campagne et al. *PRE* 2015 for rotating turbulence

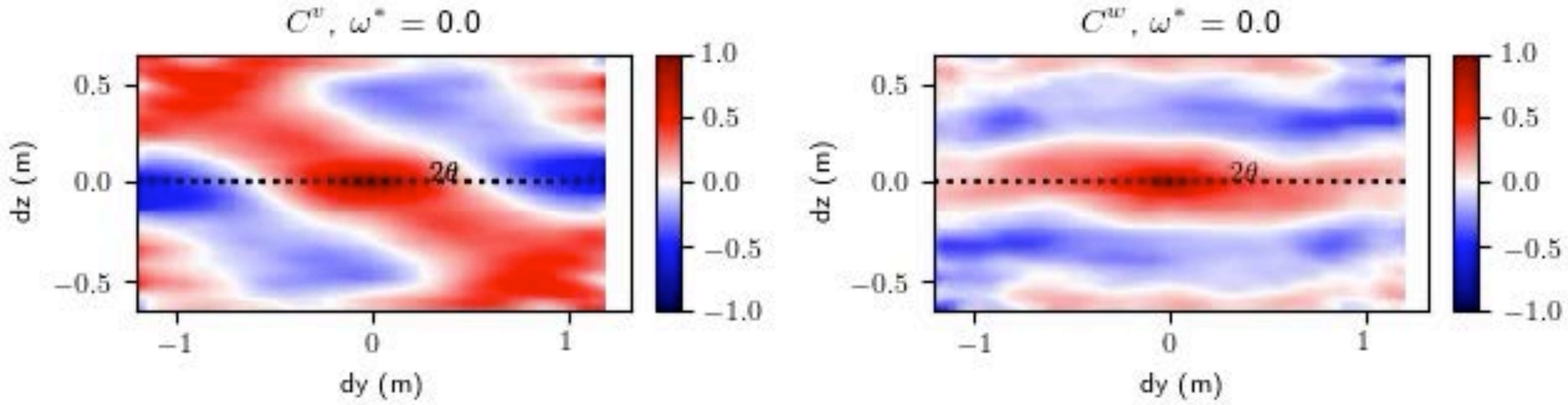
$$C^{u+w}(x,z,\omega) = 2\pi\sin\theta \int \langle |\mathbf{a}(\mathbf{k},\omega)|^2\rangle \left[J_0(kx\sin\theta) - \cos^2\theta \frac{J_1(kx\sin\theta)}{kx\sin\theta}\right]\cos(kz\cos\theta)kdk$$

Savaro et al. PRFluids 2020

looking for the dispersion relation…

space-frequency correlations $$C^{ver}(\mathbf{r},\omega) = \frac{\langle v(\mathbf{R}_0+\mathbf{r},\omega)v^\star(\mathbf{R}_0,\omega) + w(\mathbf{R}_0+\mathbf{r},\omega)w^\star(\mathbf{R}_0,\omega) + c.c.\rangle}{\langle |v(\mathbf{R}_0,\omega)|^2 + |w(\mathbf{R}_0,\omega)|^2\rangle}$$

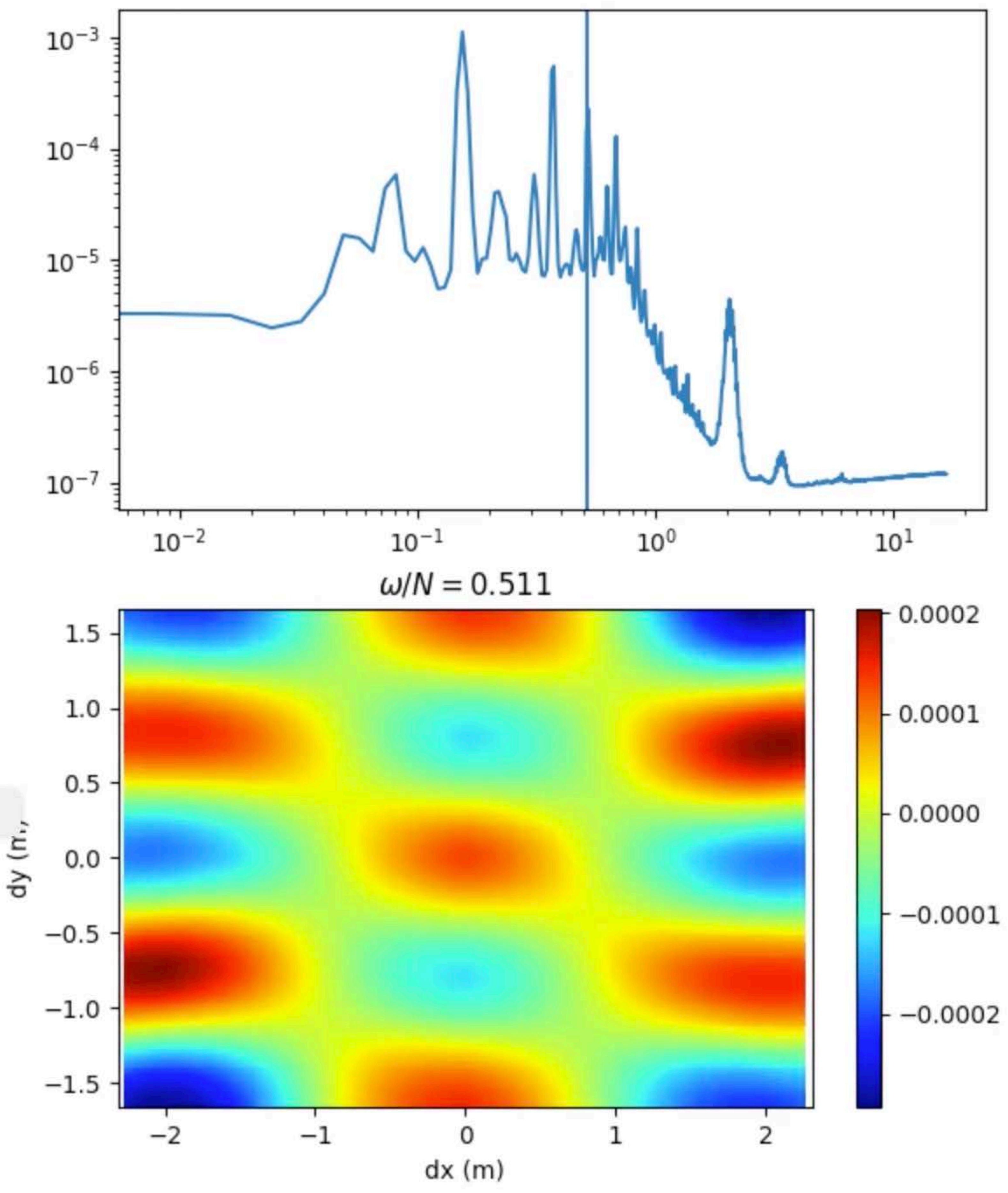




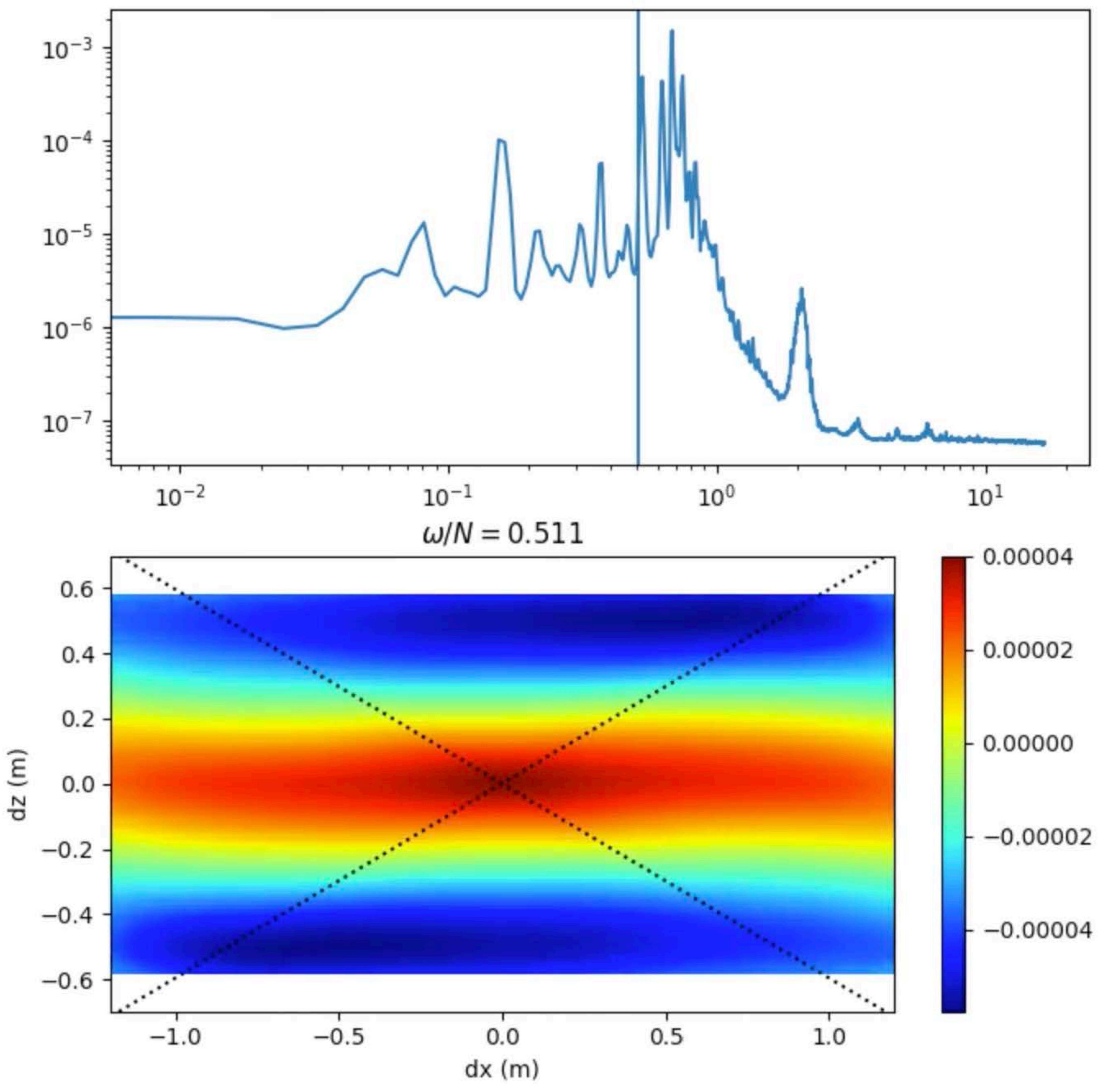
Savaro et al. PRFluids 2020

frequency less than N
on a peak of the spectrum

horizontal components correlation



vertical component correlation

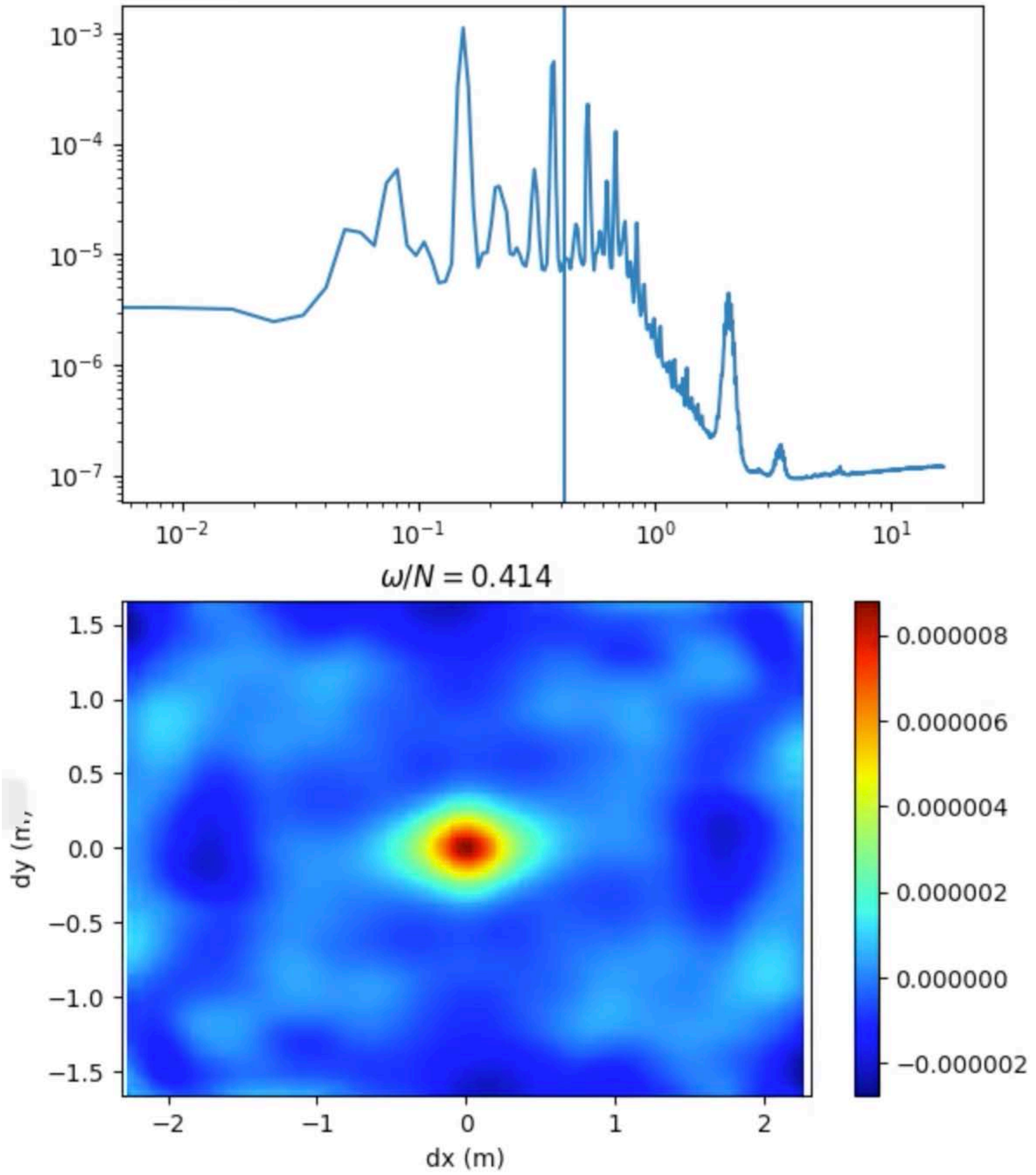


linear mode of the tank

Savaro et al. PRFluids 2020

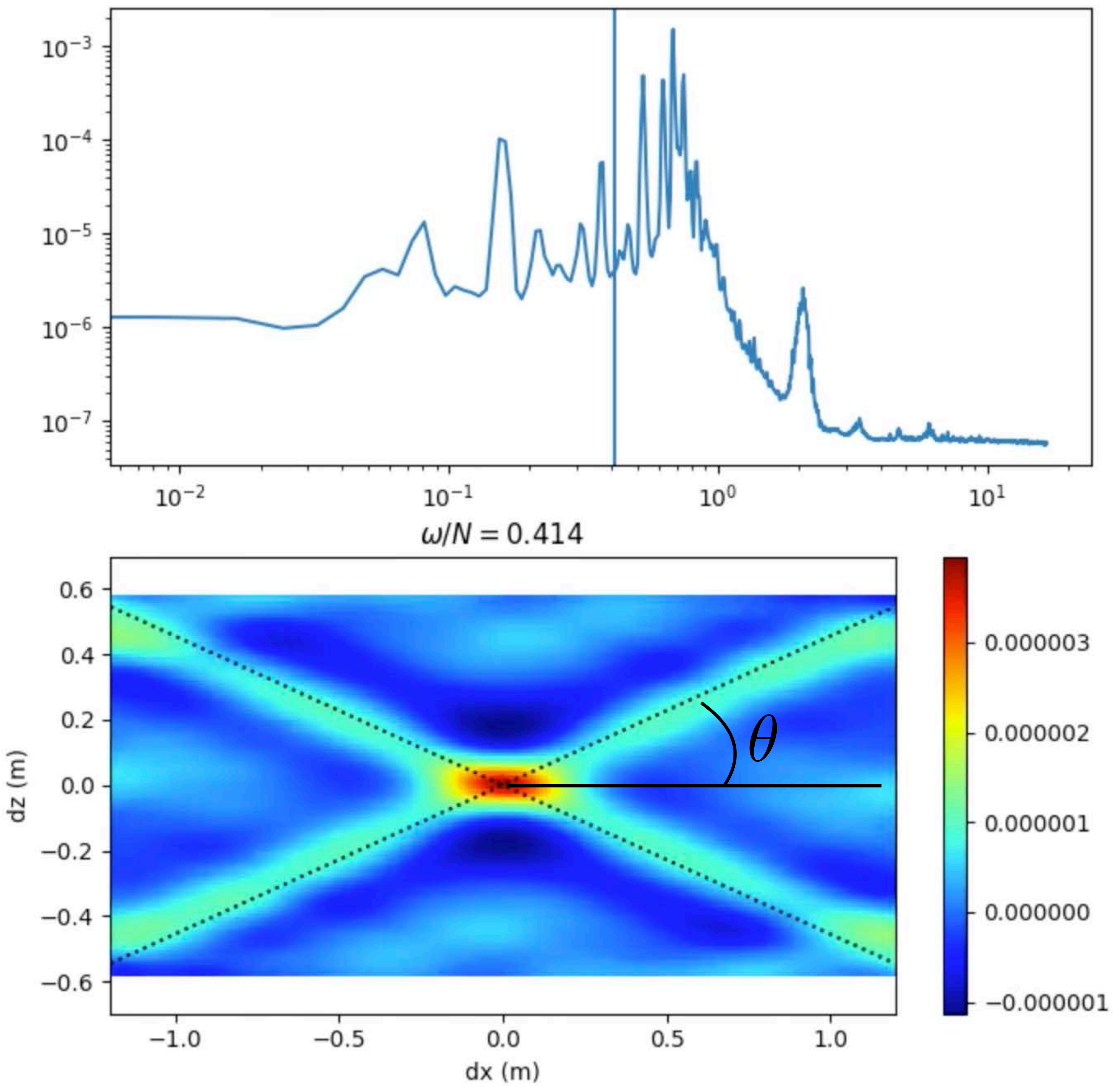
frequency less than N
between peaks of the spectrum

horizontal components correlation



axisymmetry

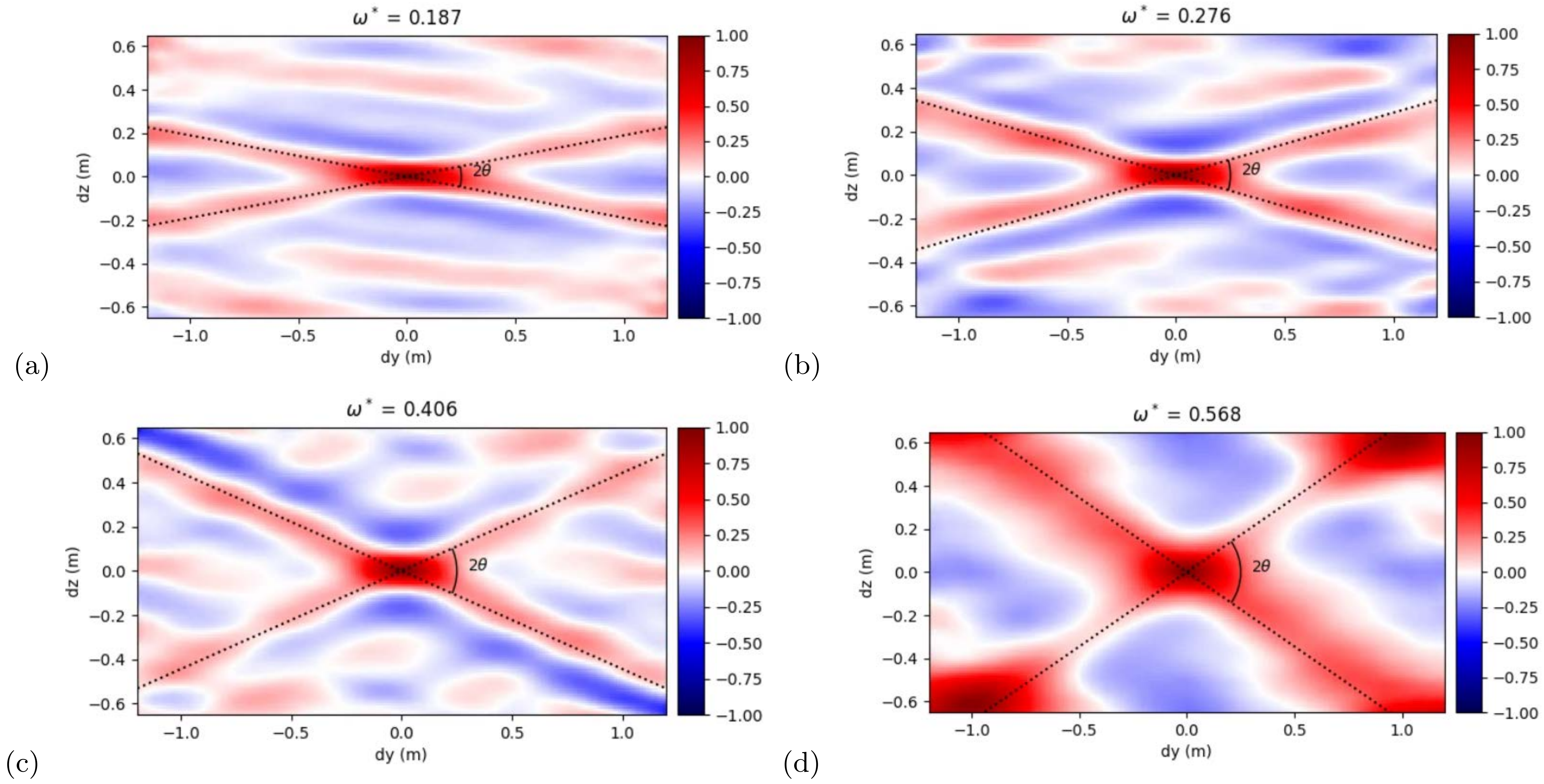
vertical component correlation



cross pattern ?

dashed line: angle of propagation at ω *Savaro et al. PRFluids 2020*

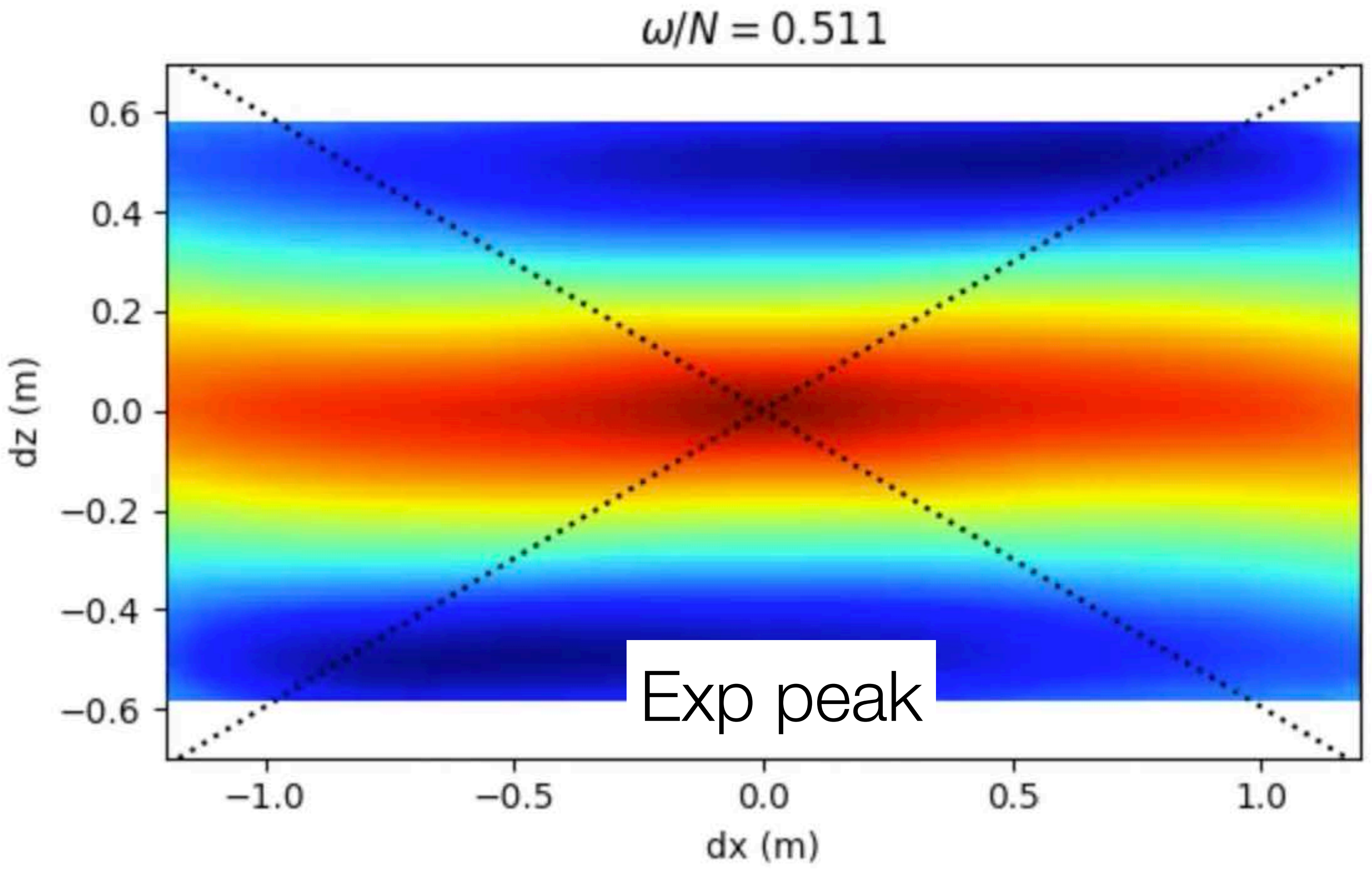
frequency less than N
between peaks of the spectrum

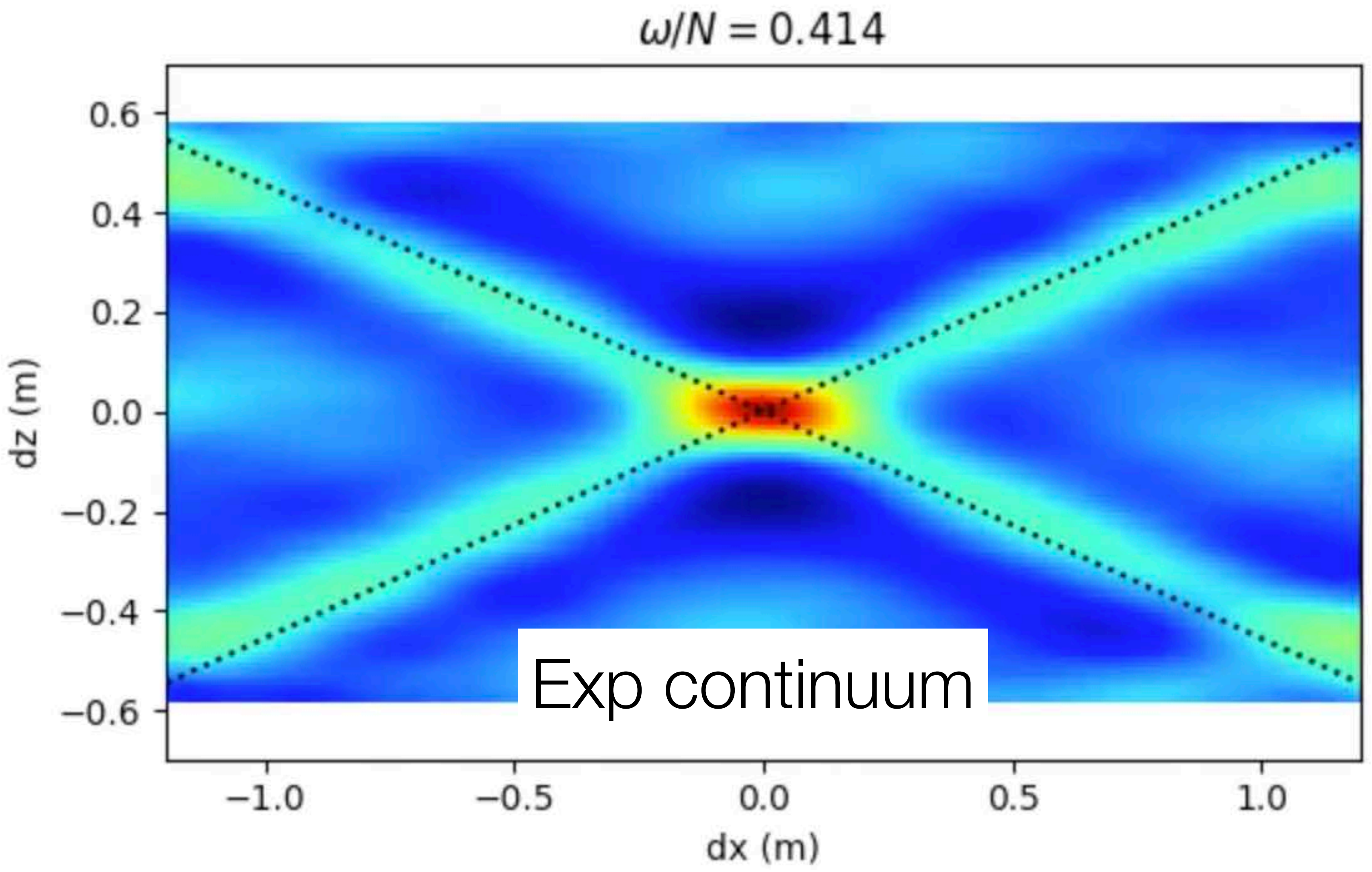


Savaro et al. PRFluids 2020

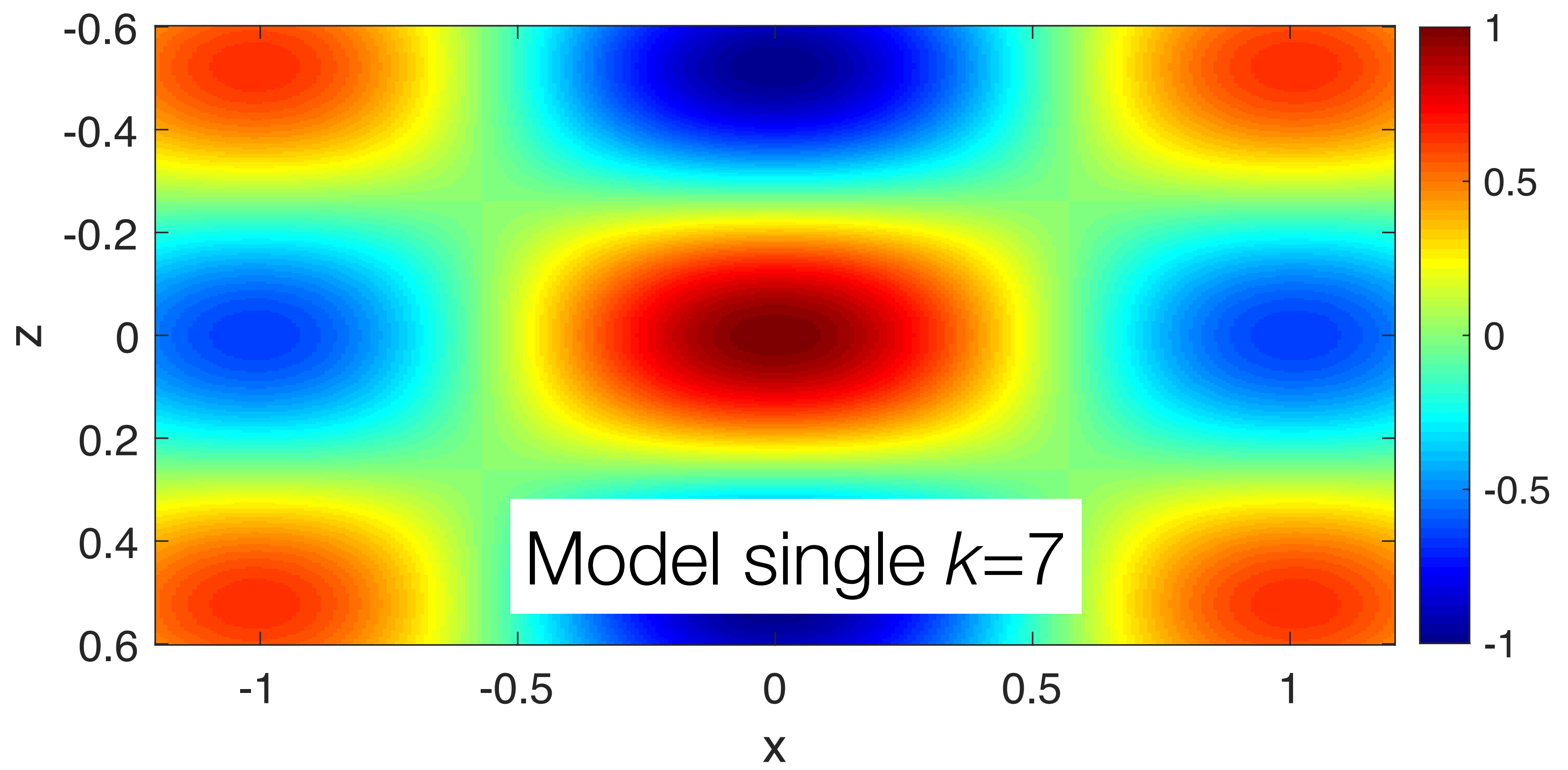
space correlations

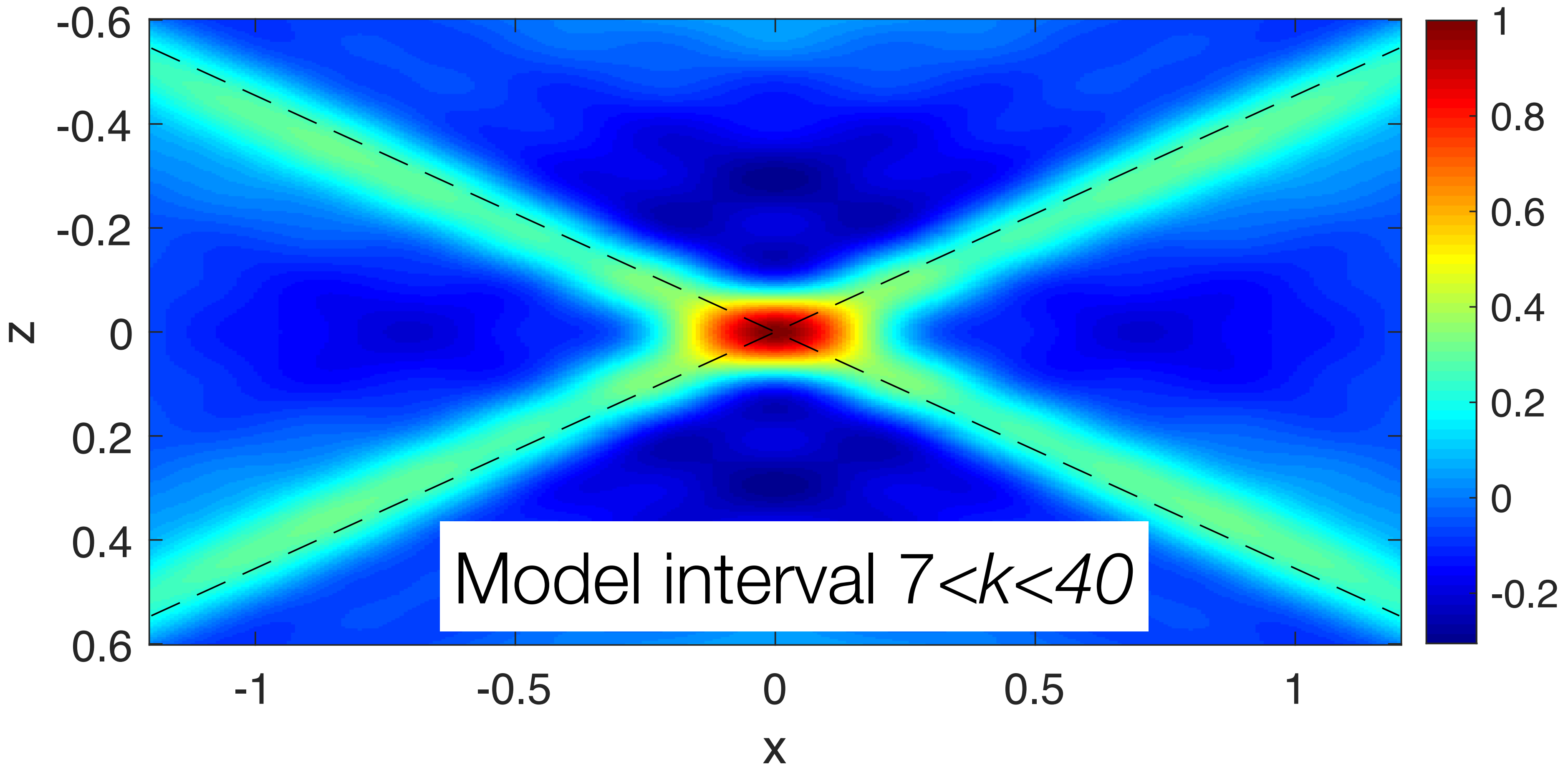
Experiment





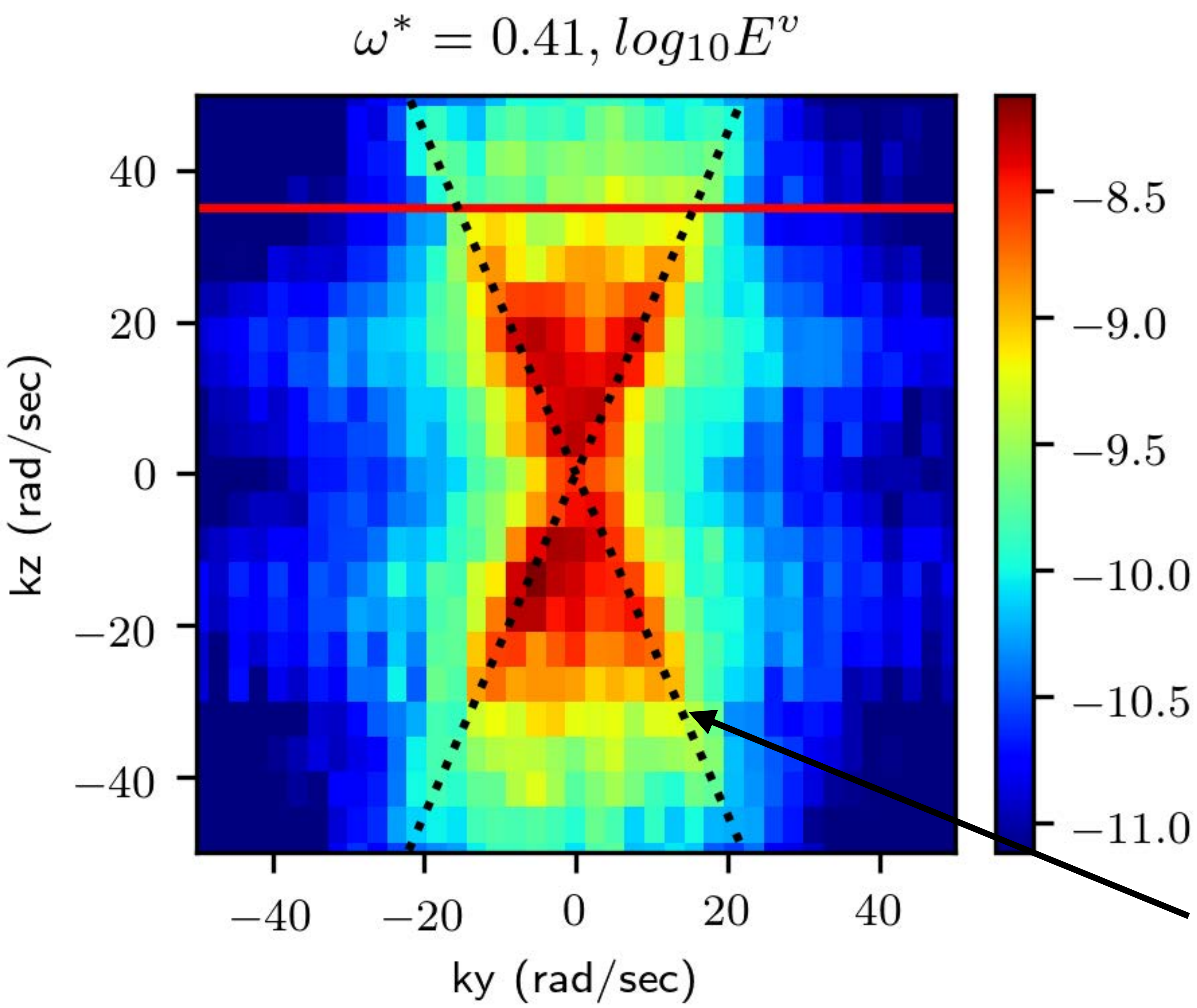
Model:
axisymmetric
superposition
of linear waves
with continuous spectrum





see Savaro et al. PRFluids 2020

looking for the dispersion relation…

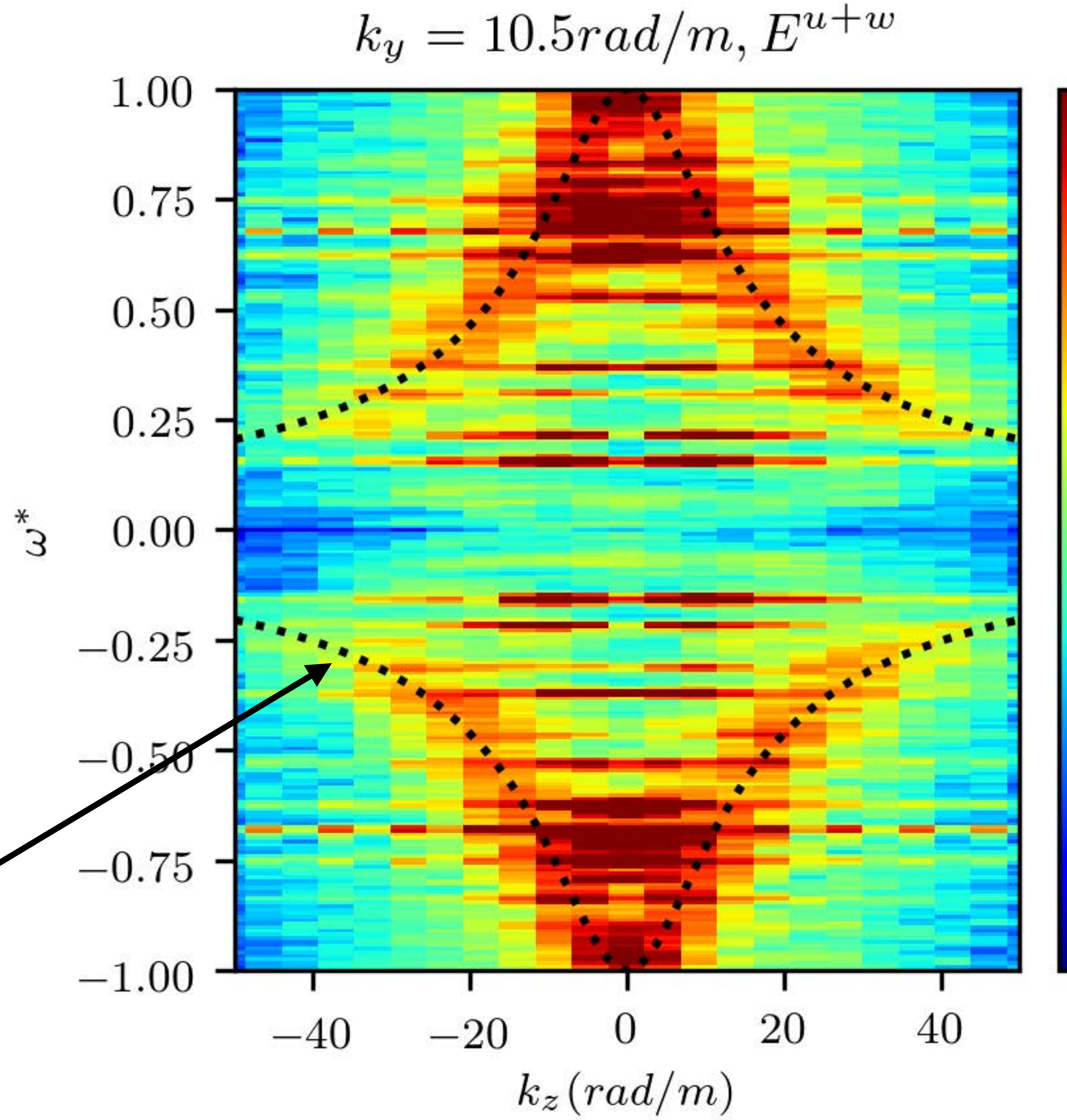


$C(y, z, \omega)$

$E(k_y, k_z, \omega)$

linear wave dispersion relation

$\omega = N \sin \theta$



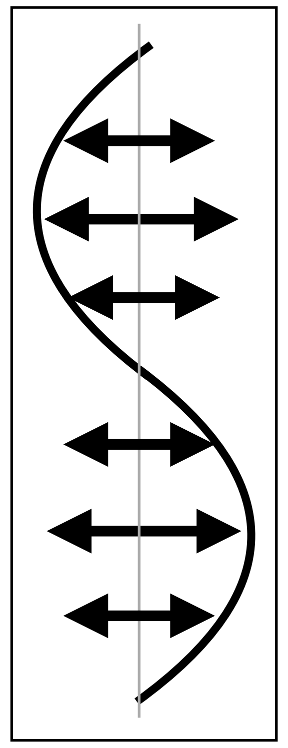
Savaro et al. PRFluids 2020

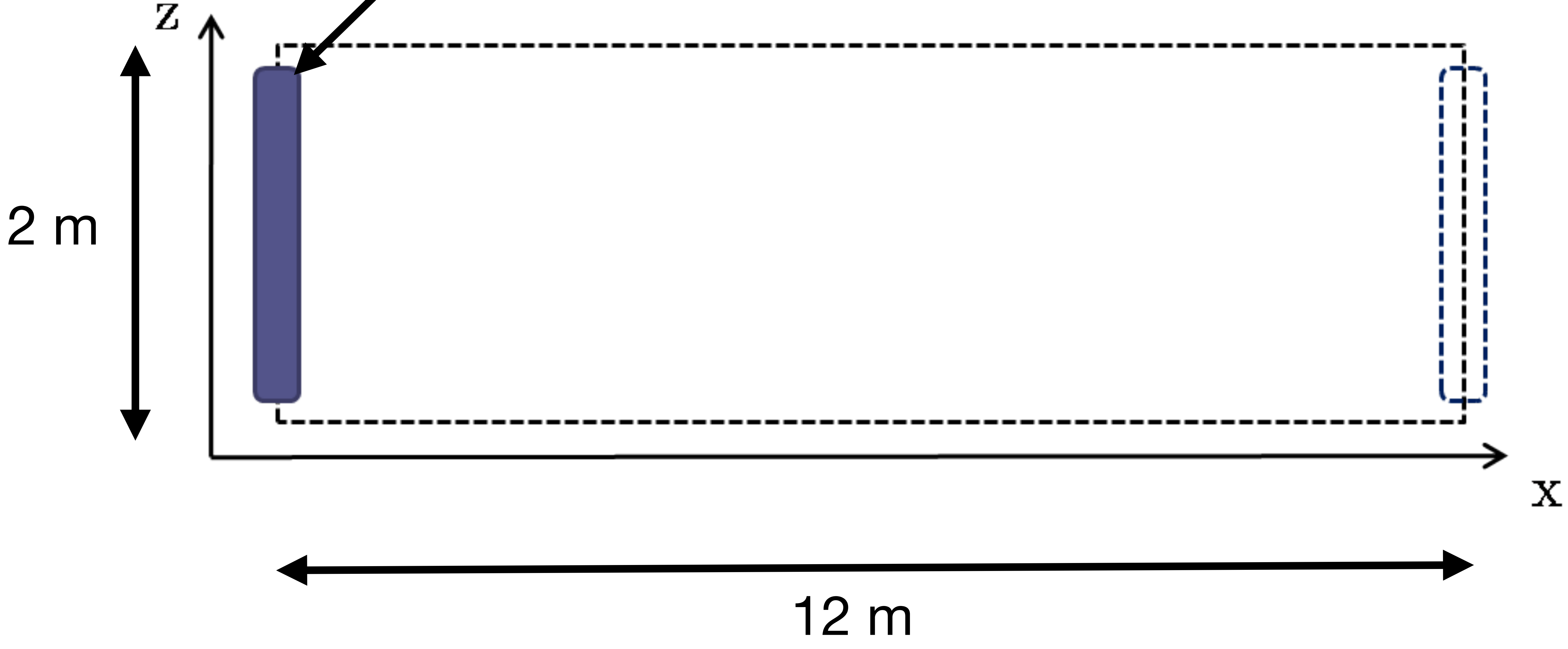
3D numerical simulations

pseudo spectral open source solver FluidSim

https://foss.heptapod.net/fluiddyn/fluidsim

penalization method to mimic the wavemakers
imposing the horizontal velocity

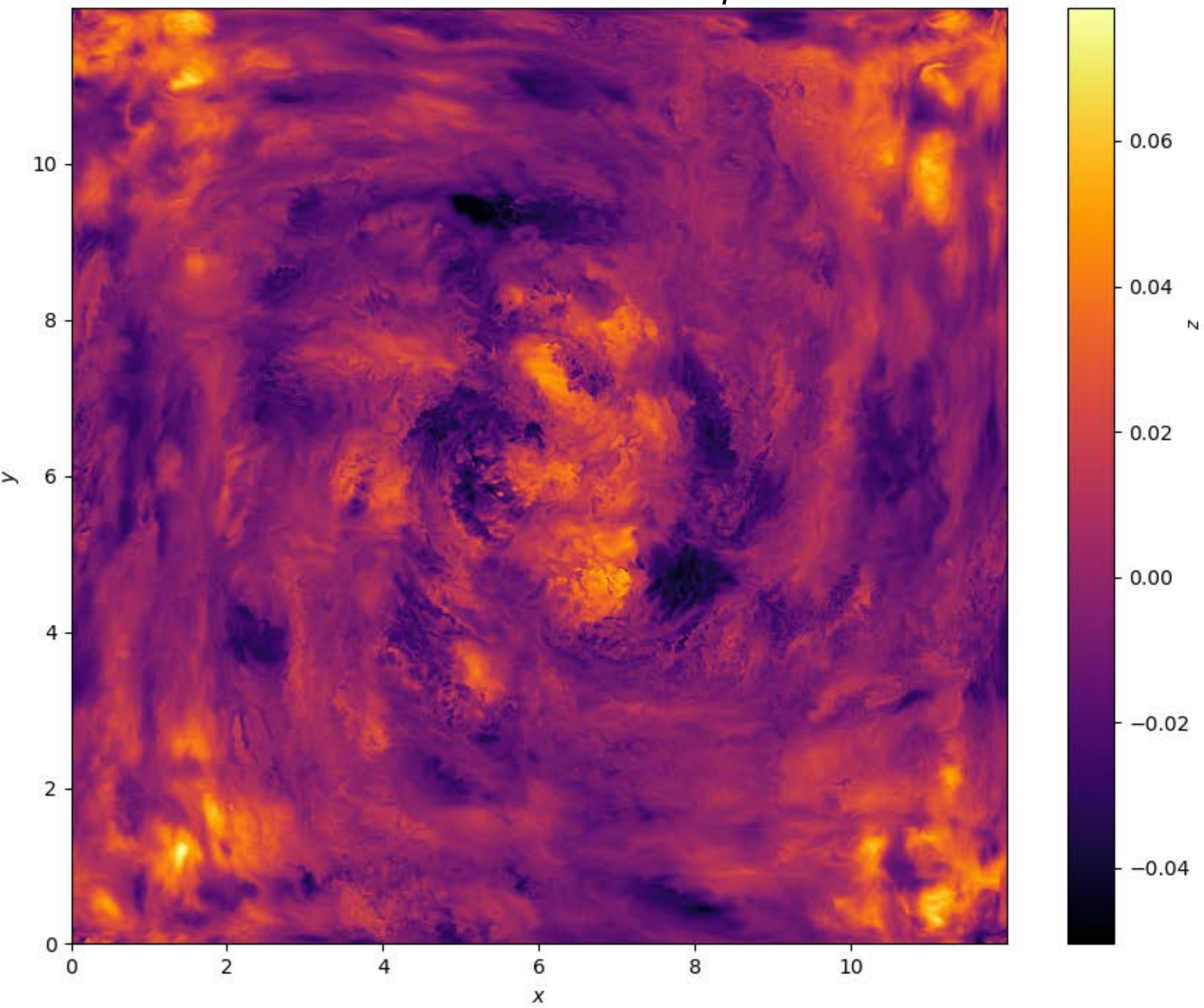


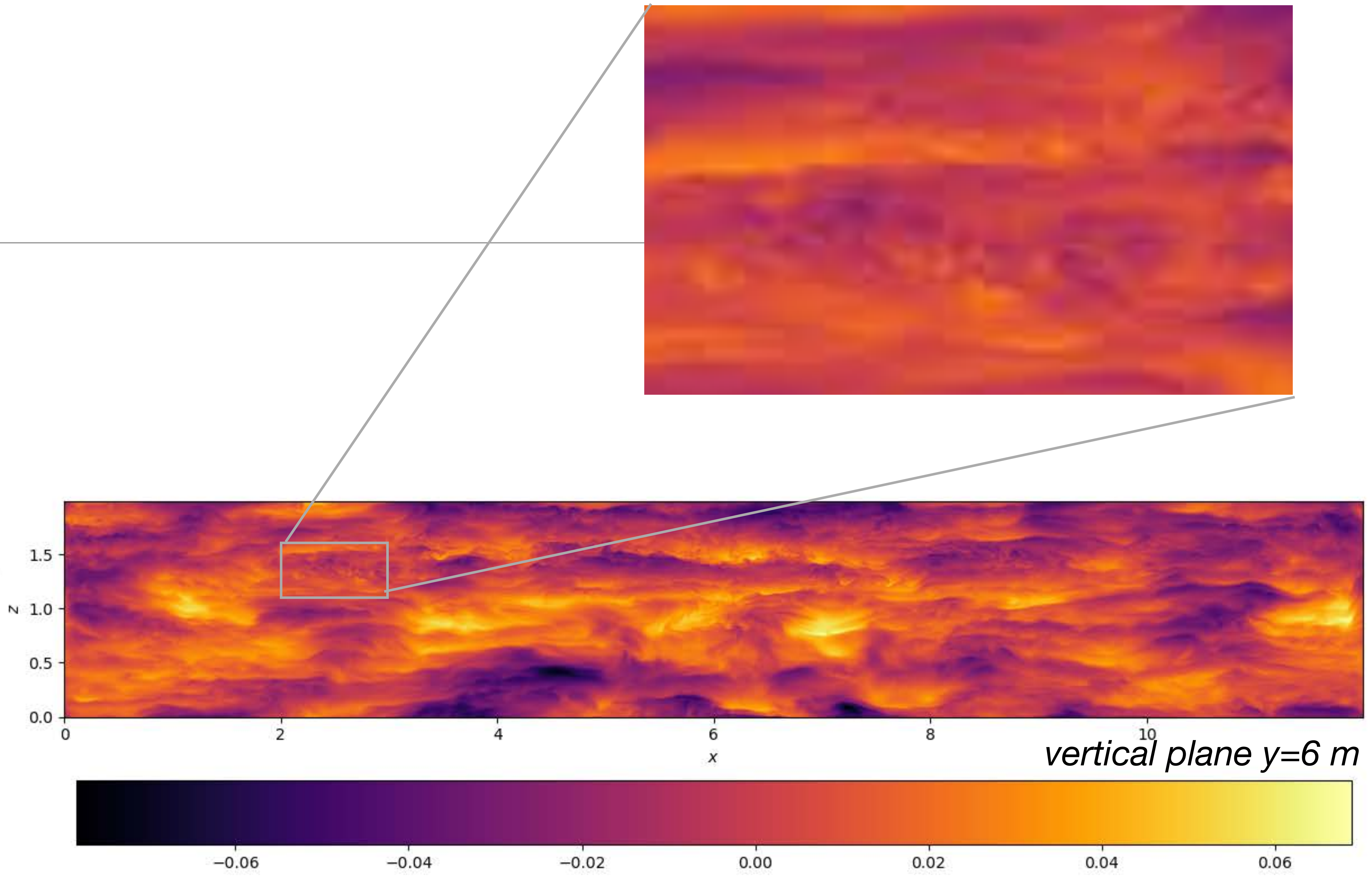


periodic boundary conditions

numerical simulations

horizontal plane z=1.5 m





buoyancy field

resolution: 2304x2304x384 (Occigen)

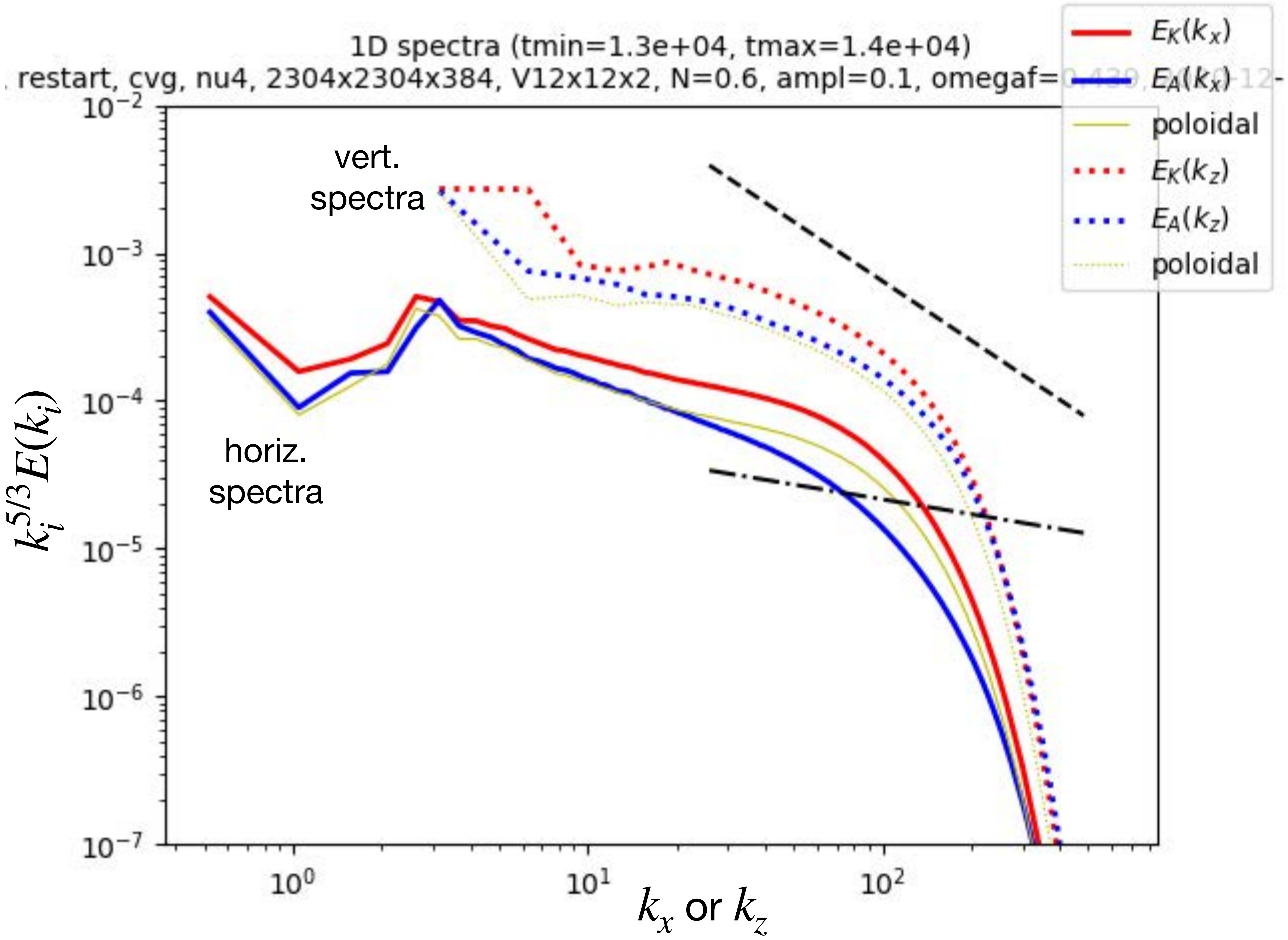
$N = 0.6$ rad/s

$\omega_f = 0.73N$

$A = 10$ cm

parameters similar to experiment (except viscosity…)

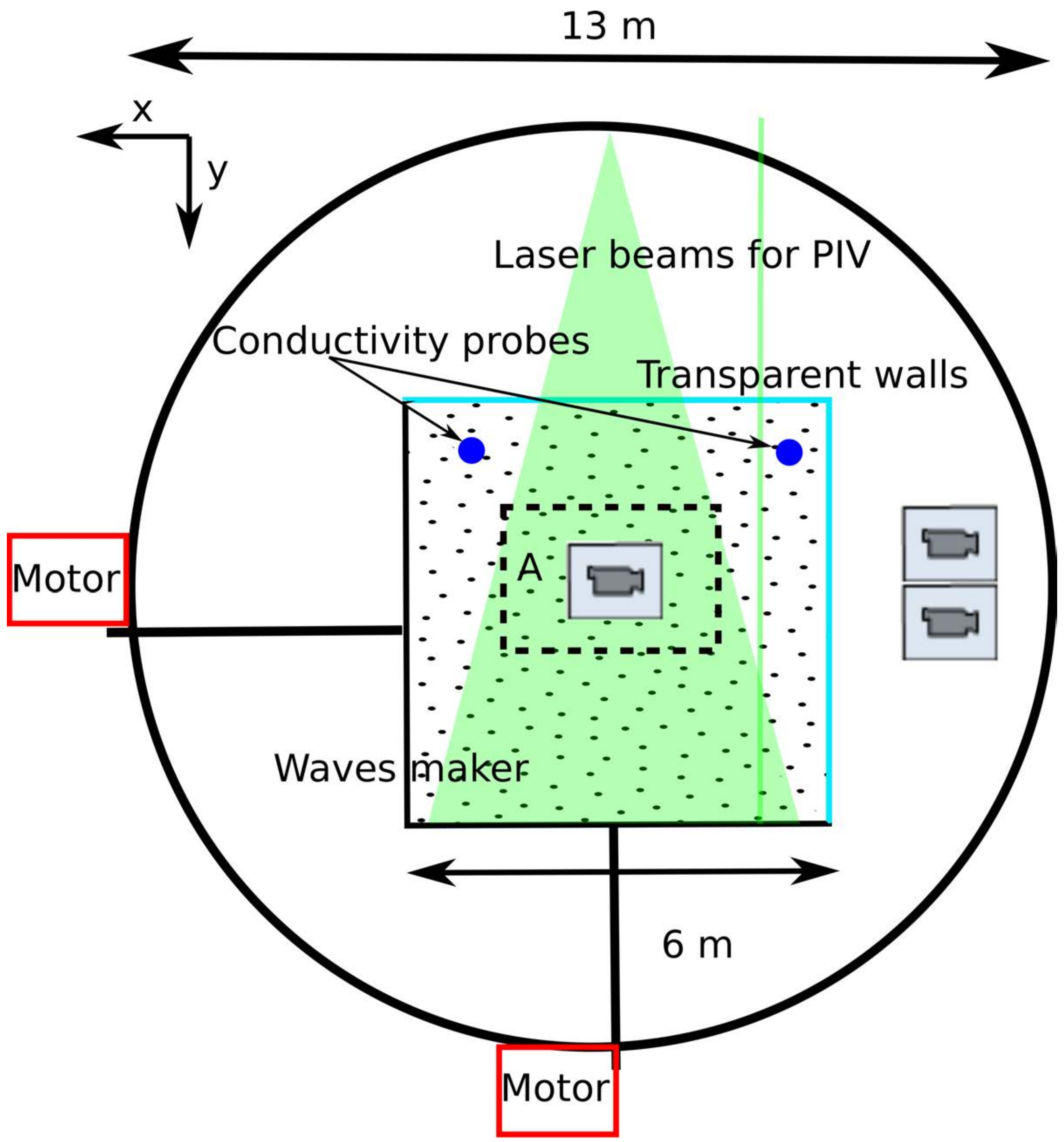
numerical simulations



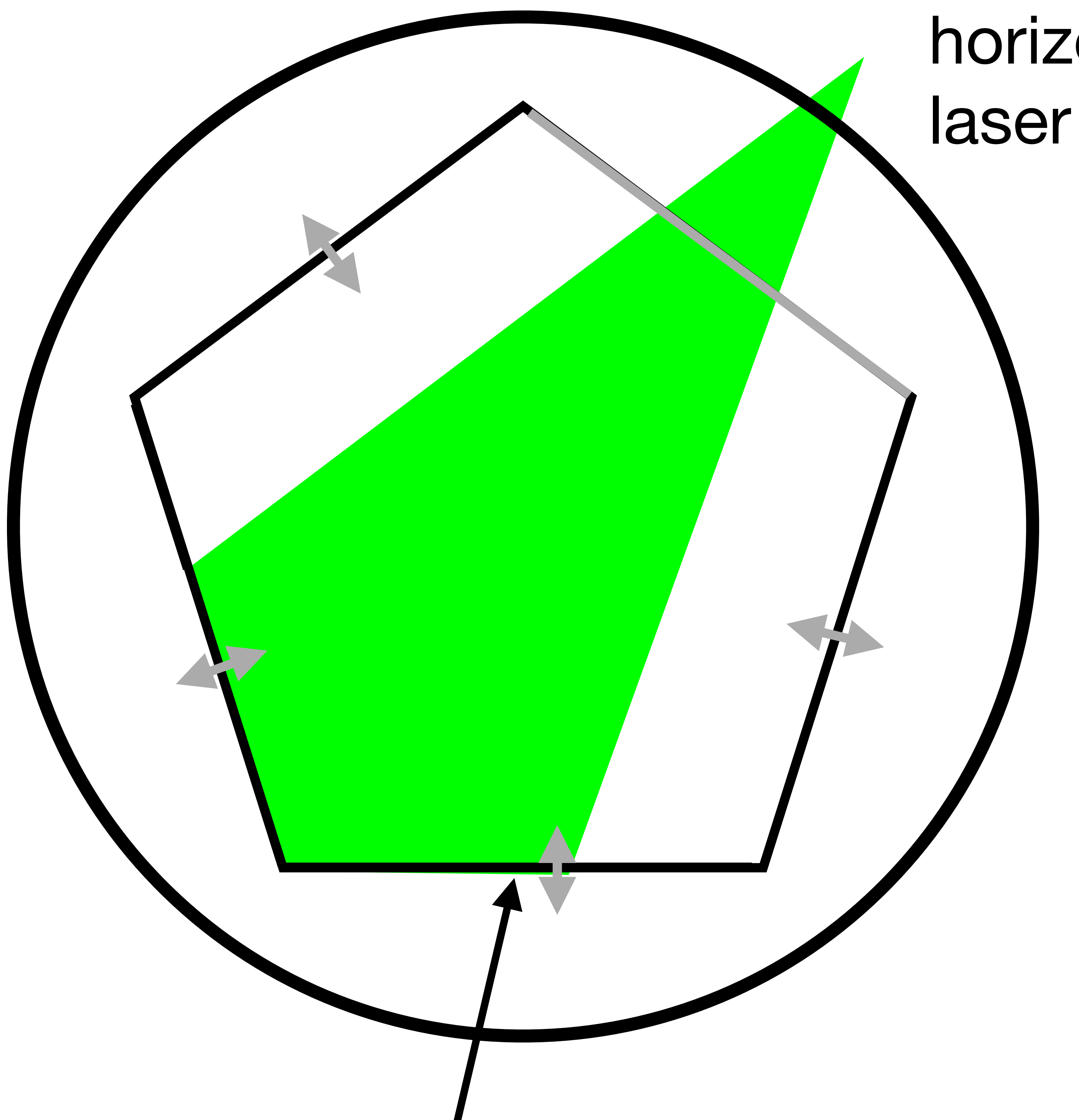
also 2D DNS for higher resolution

less discrete modes ?

square

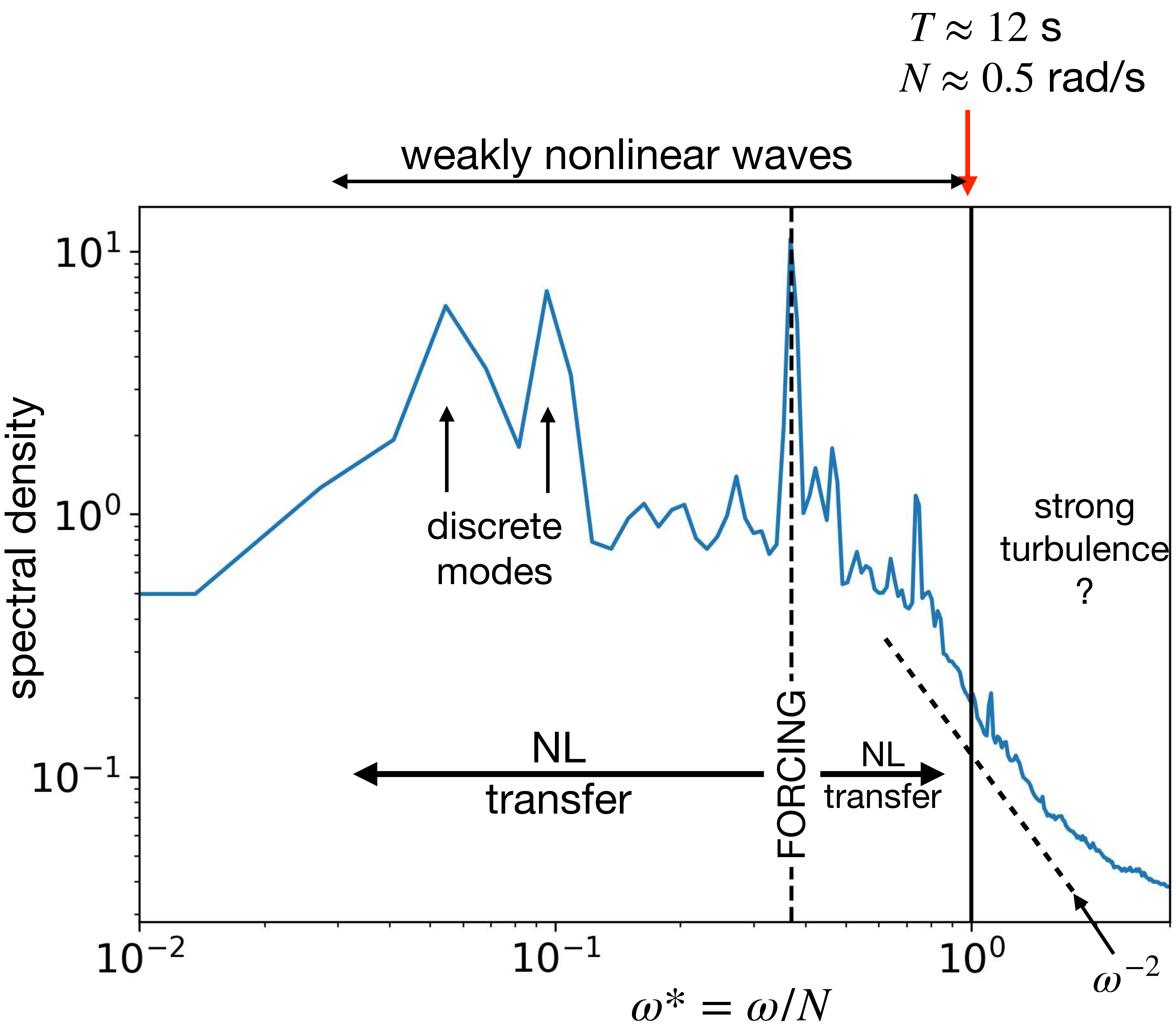


pentagon

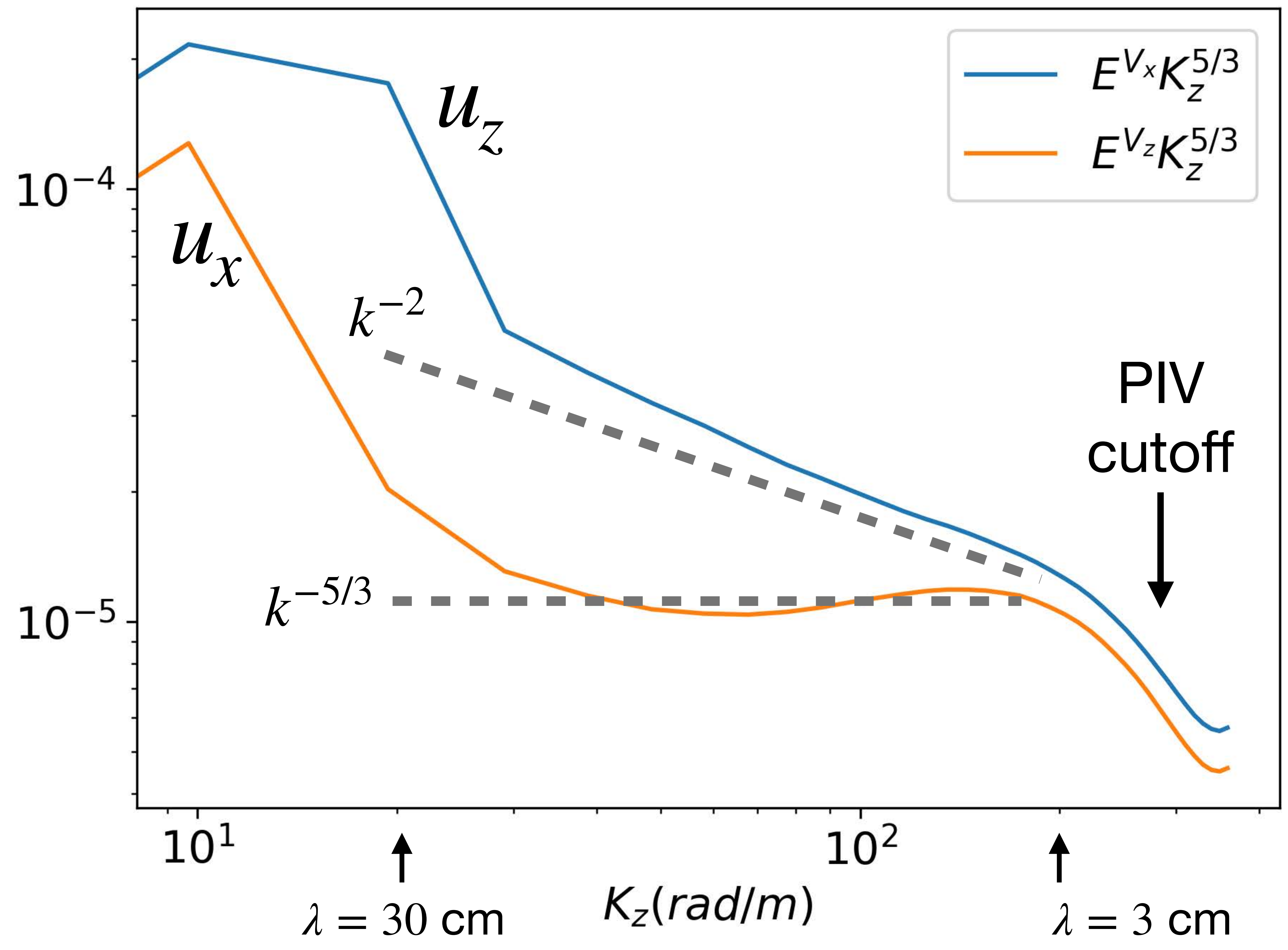


mix water/salt/alcool for optical index matching

Stratified turbulence



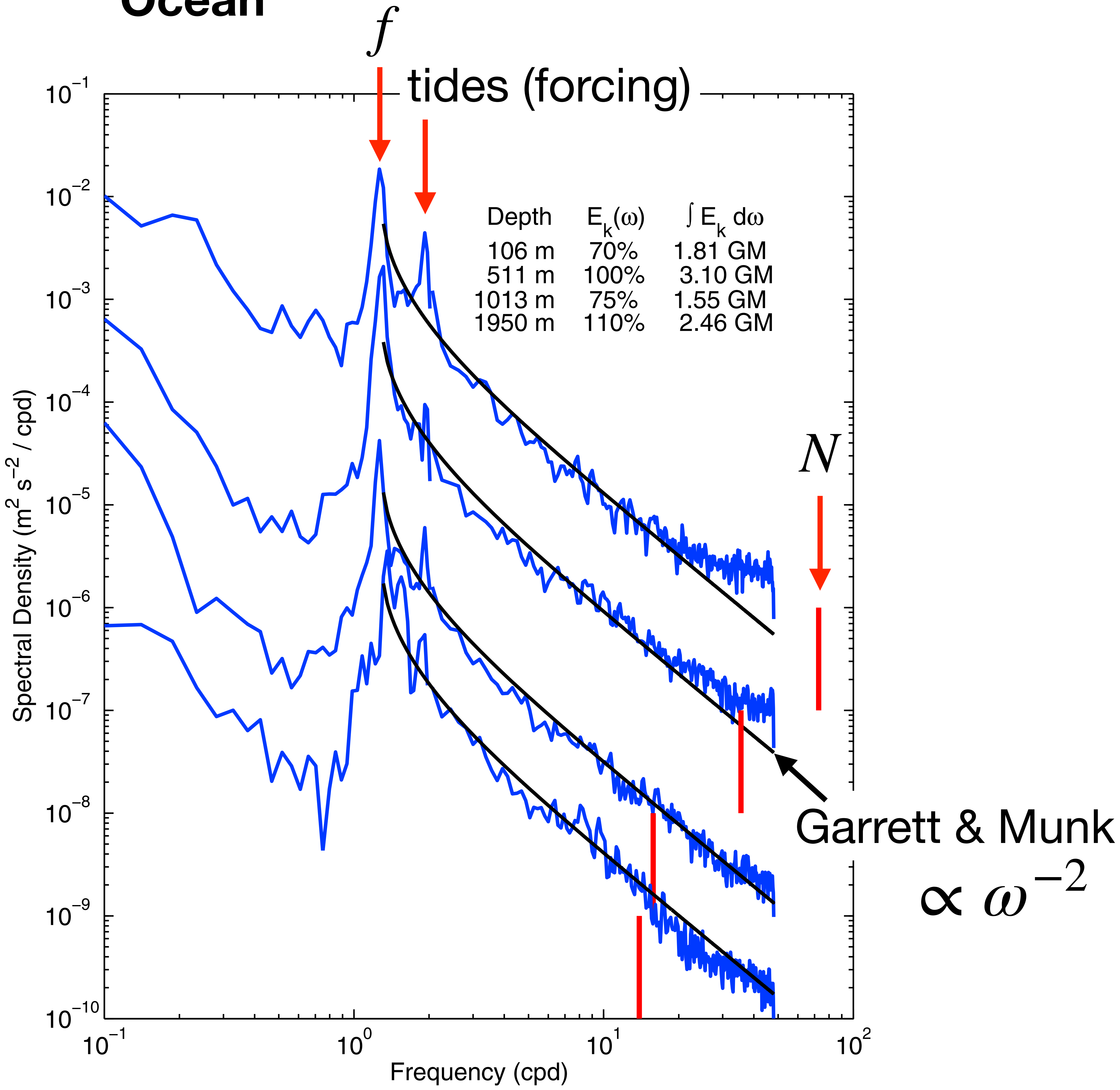
vertical wavenumber spectrum compensated by $k_z^{-5/3}$



anisotropy at intermediate scales
isotropy at small scales (?)

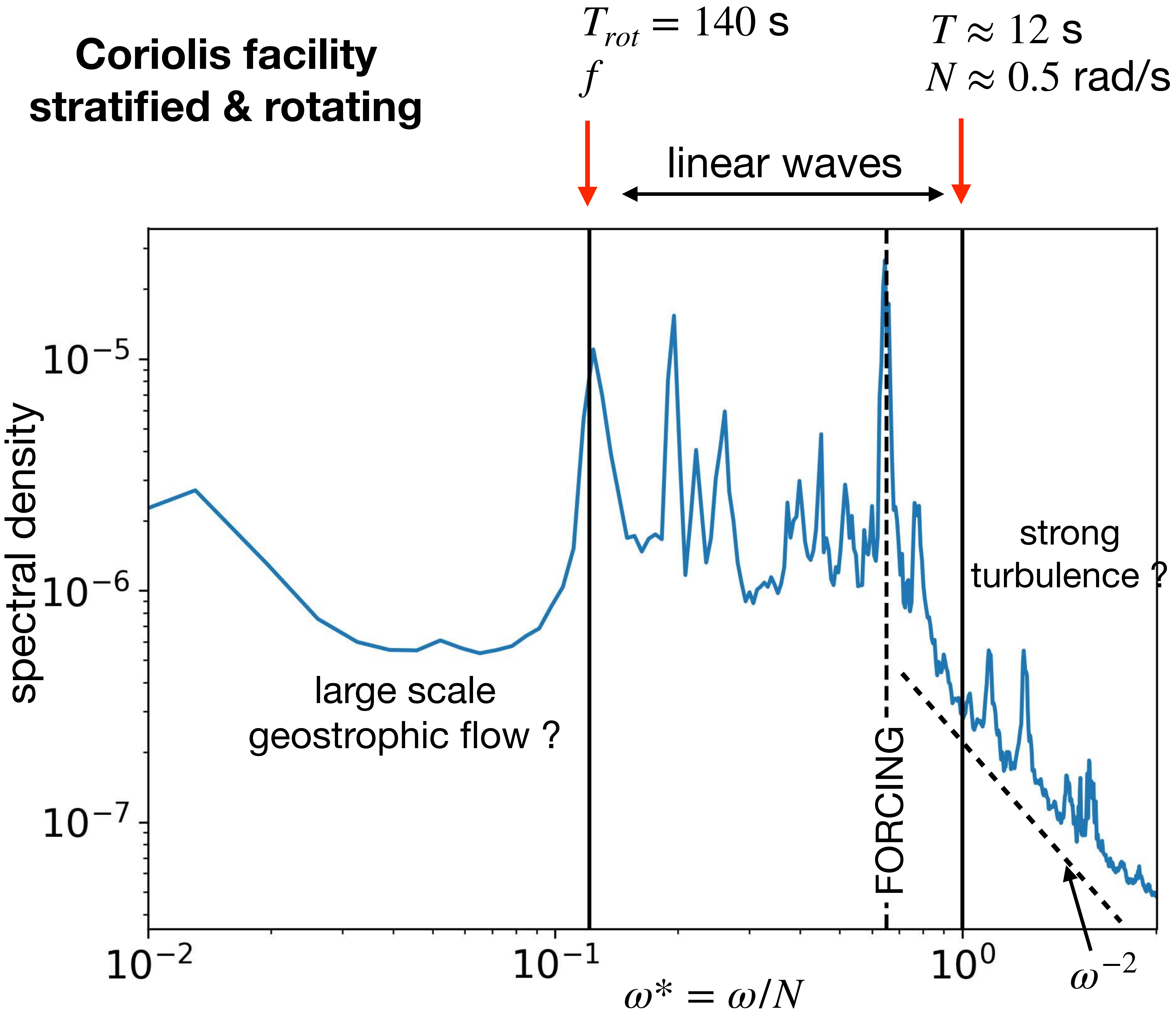
with rotation…

Ocean



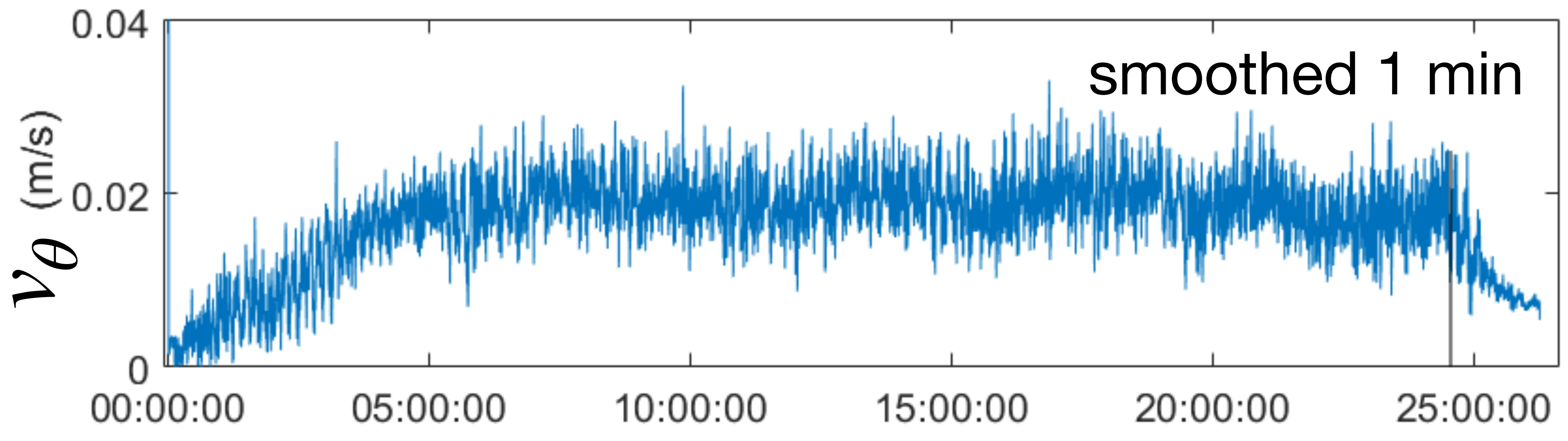
Polzin & Lvov, *Rev. Geophys.* 2011

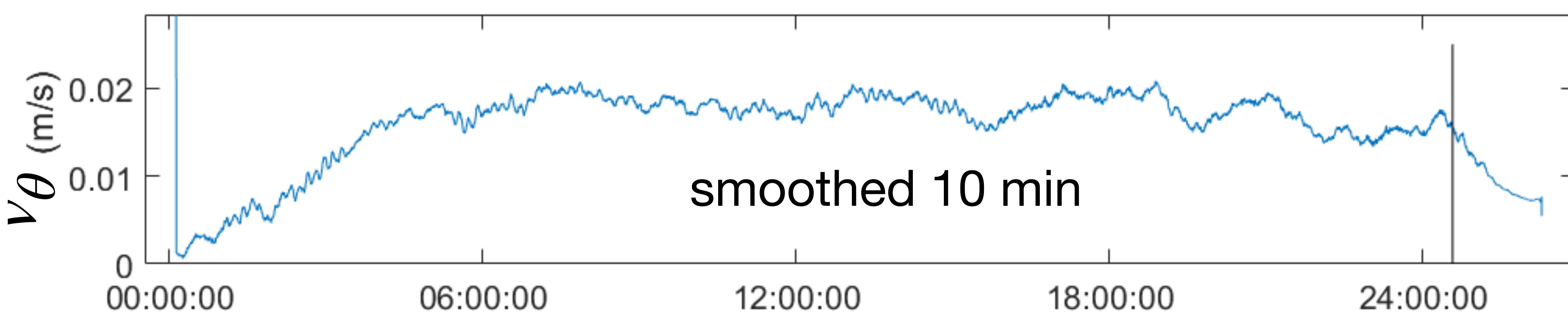
Coriolis facility stratified & rotating



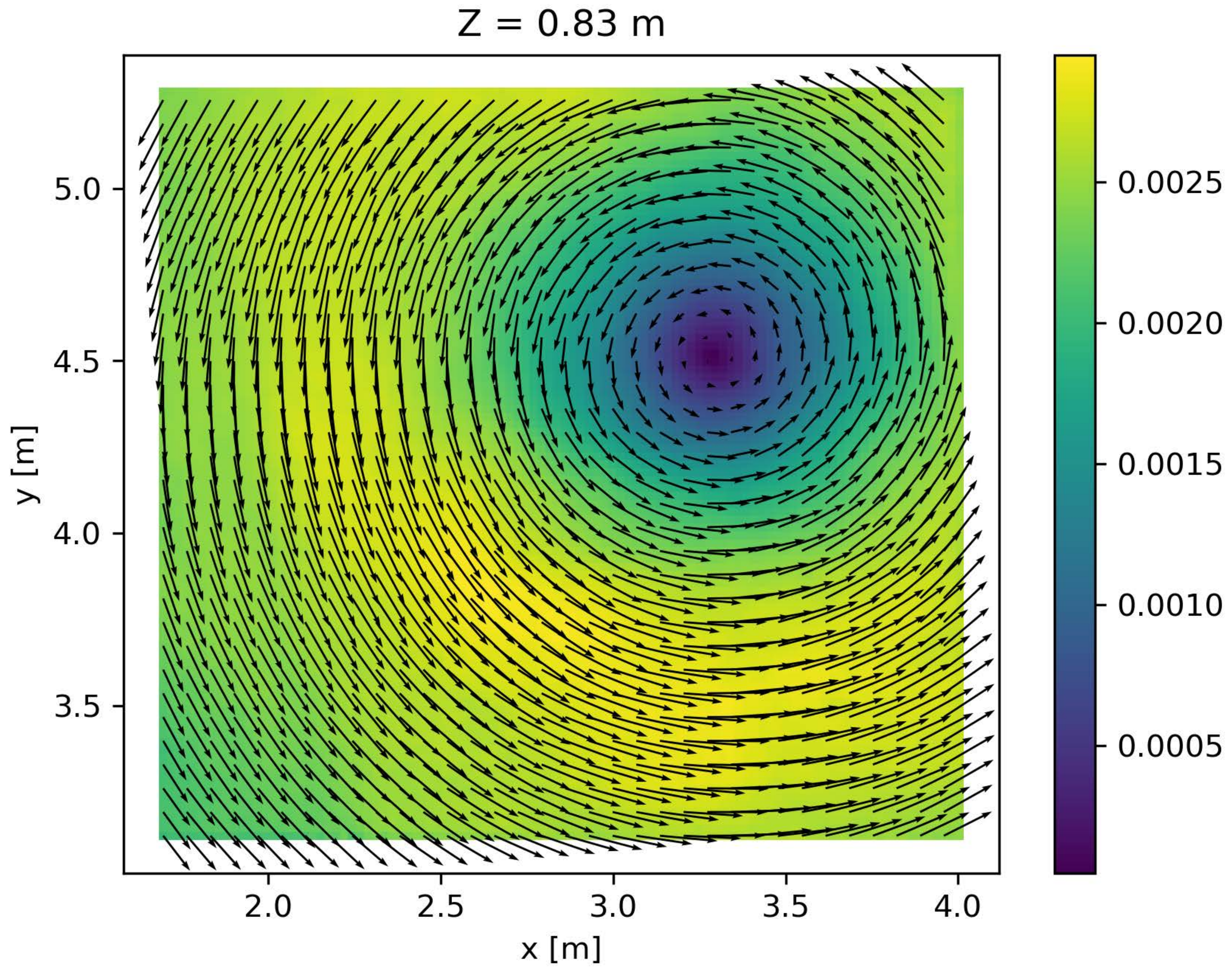
very large scales: vortex (no rotation)

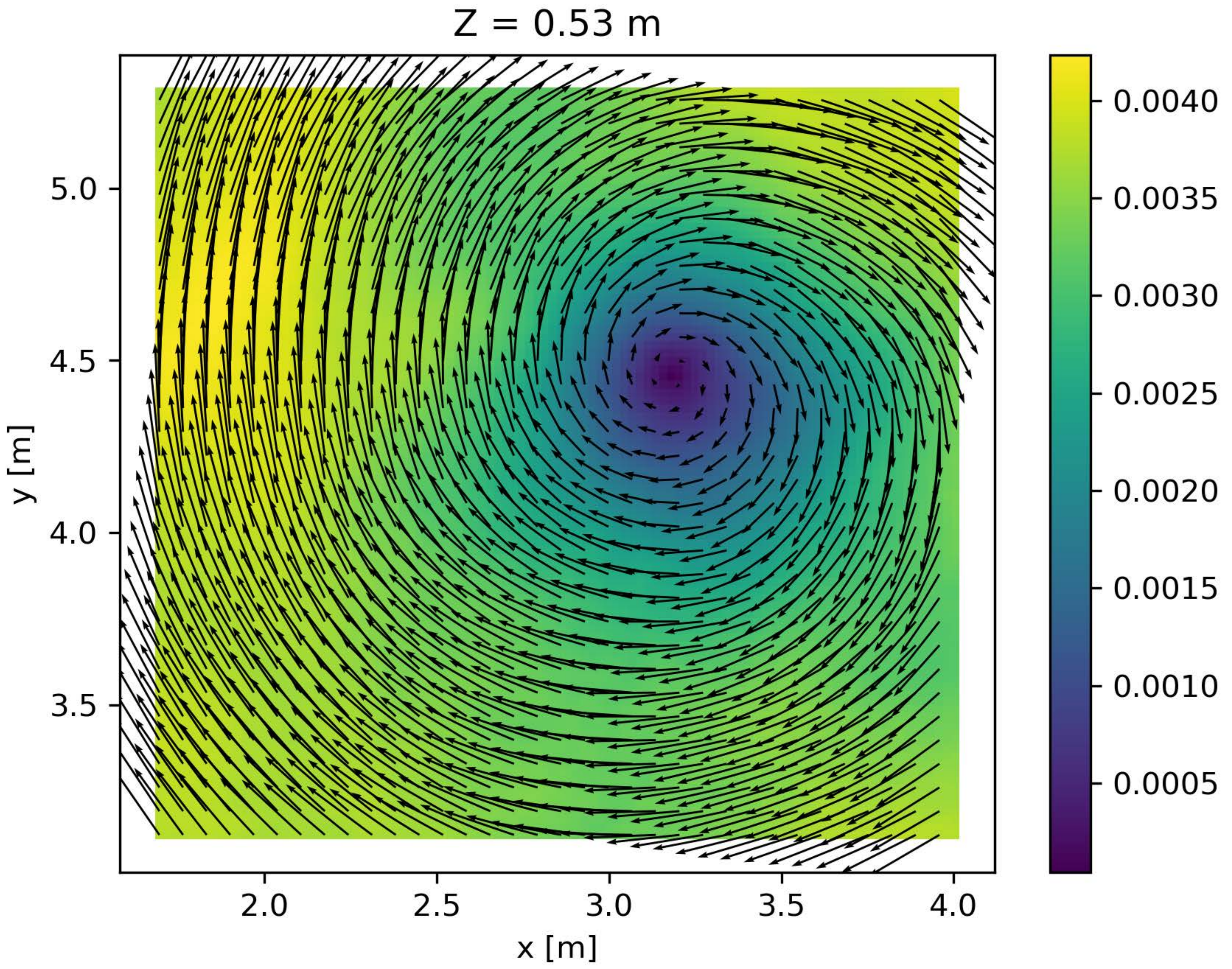
single point velocity ~3m from center





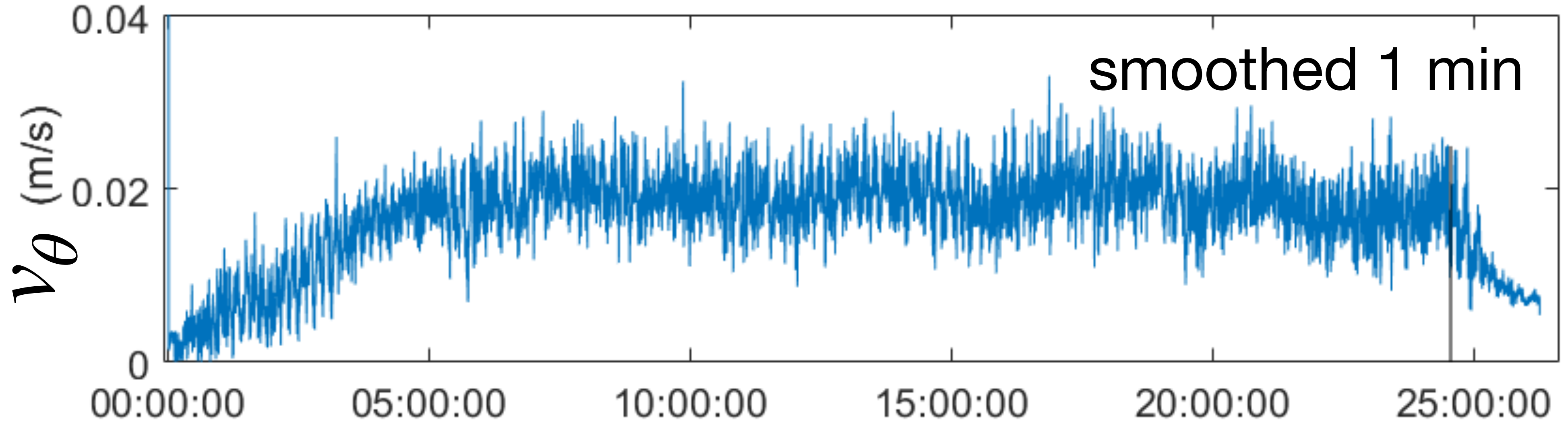
large scale vortex rotating in different direction with z

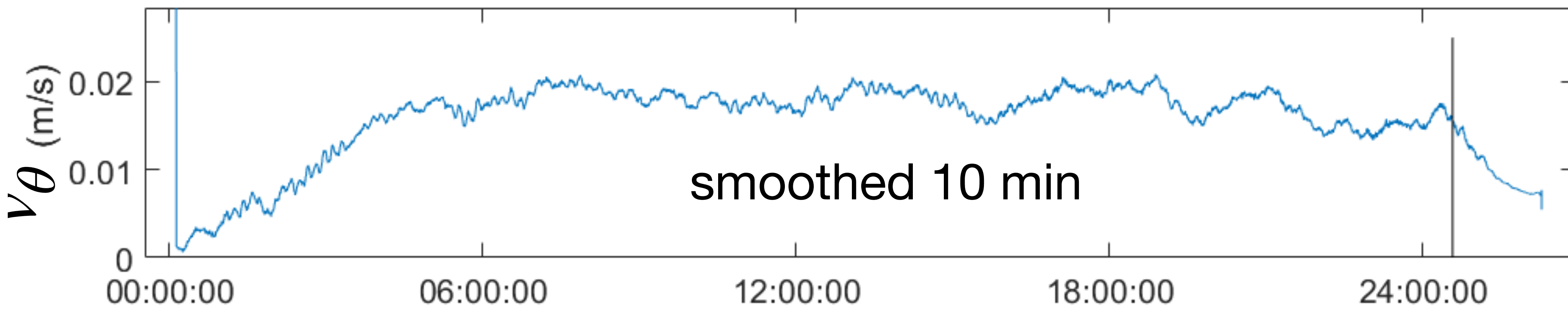




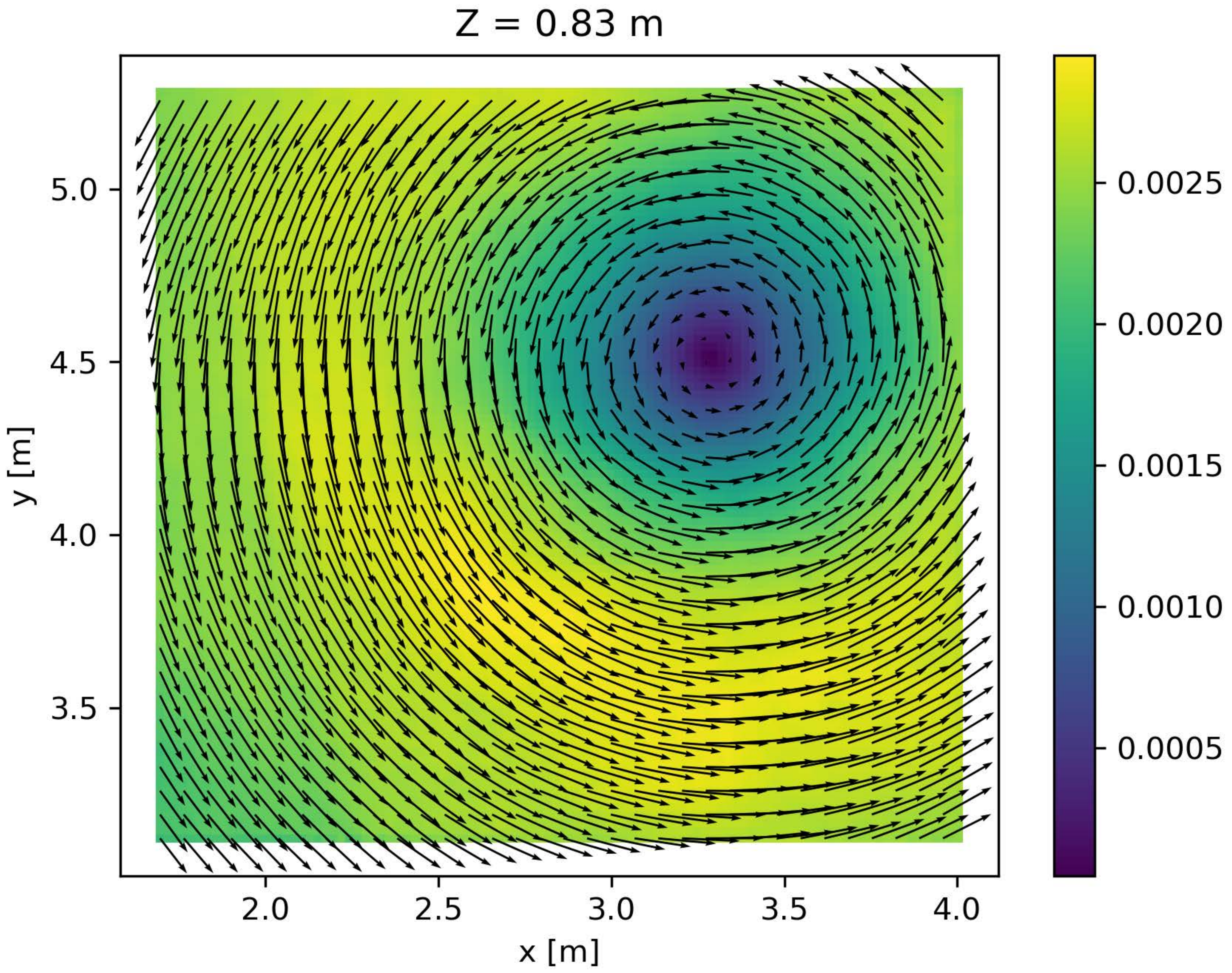
very large scales: vortex

single point velocity ~3m from center

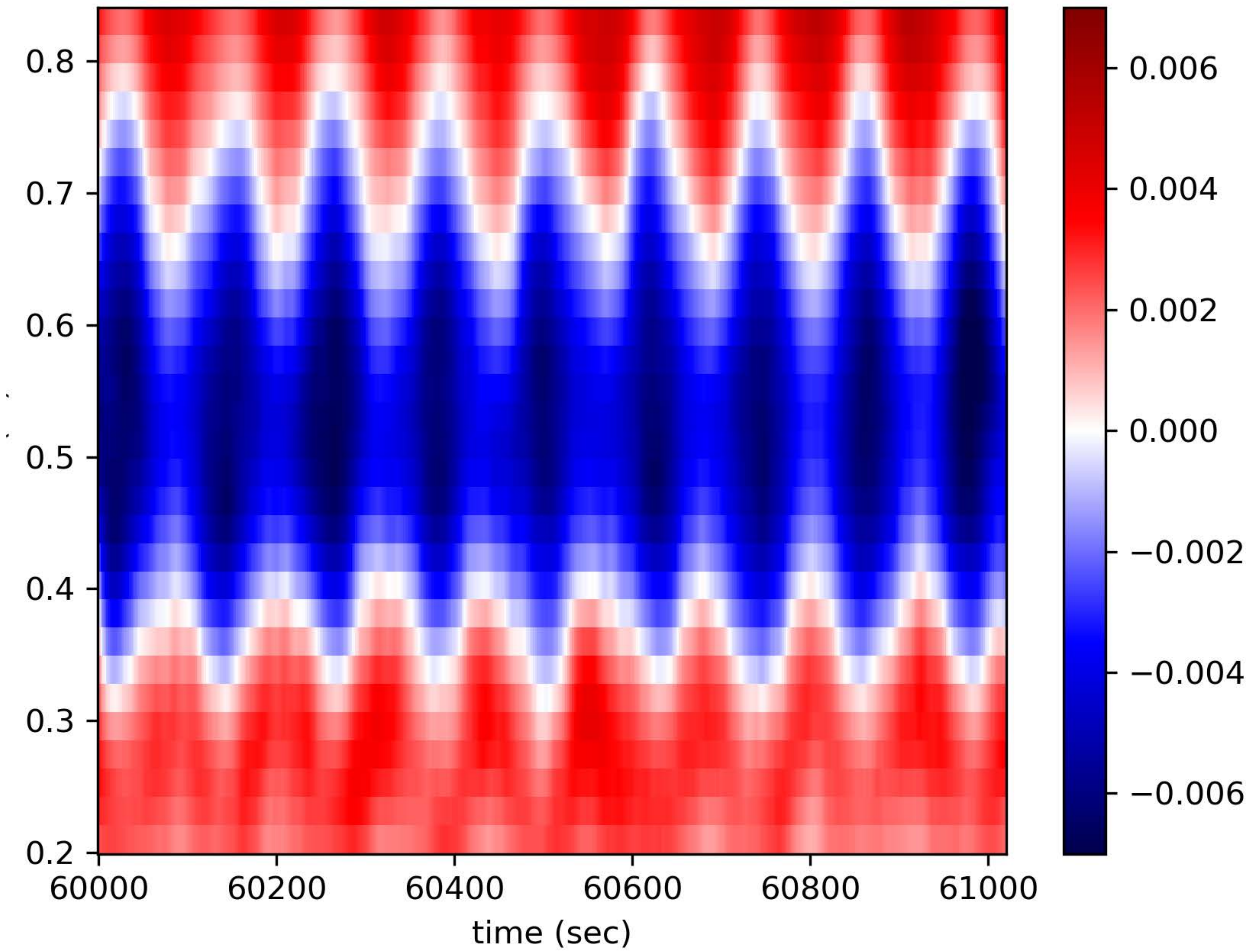




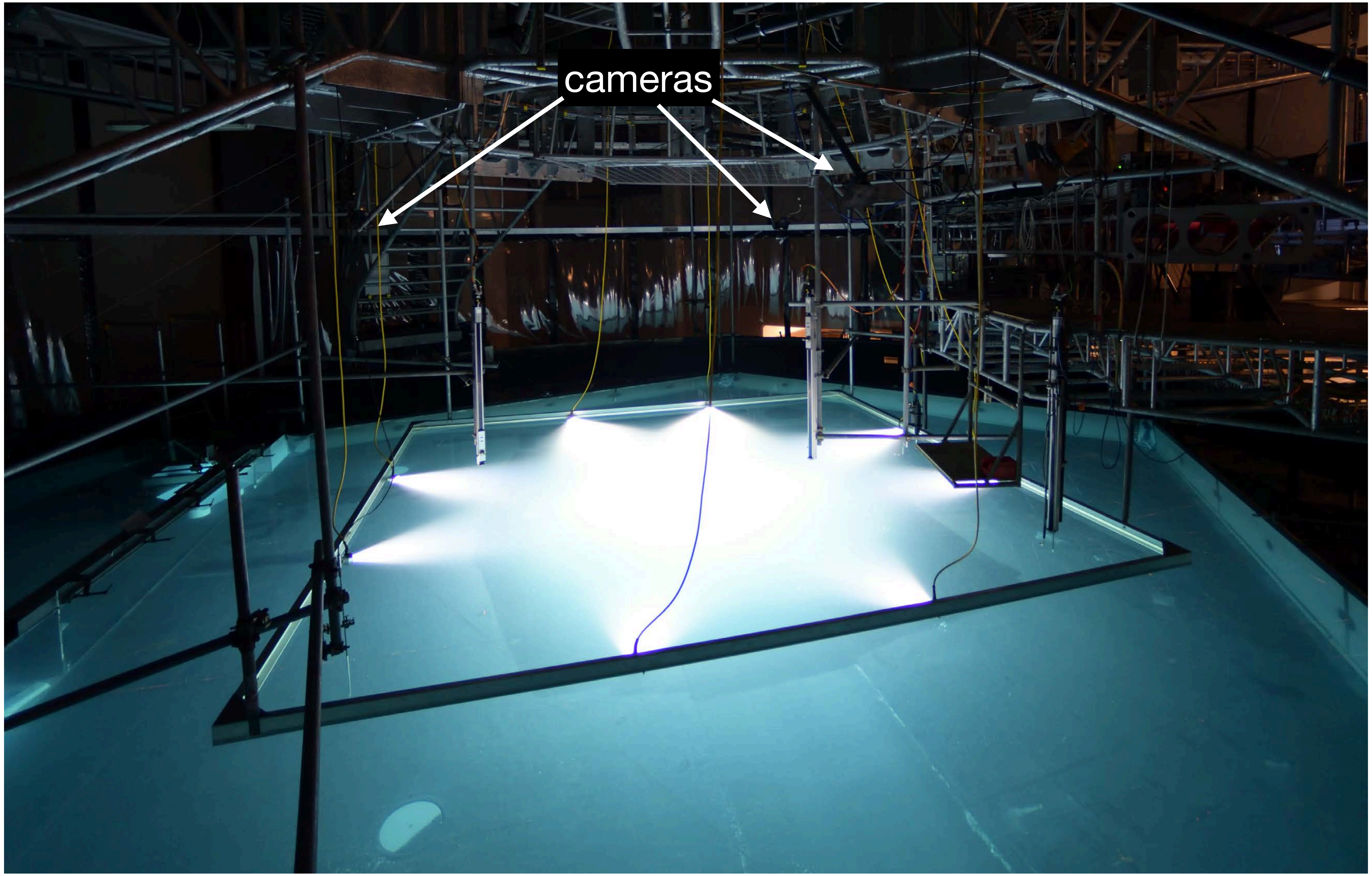
large scale vortex rotating in different direction with z



average vertical vorticity Ω_z



Lagrangian study ?

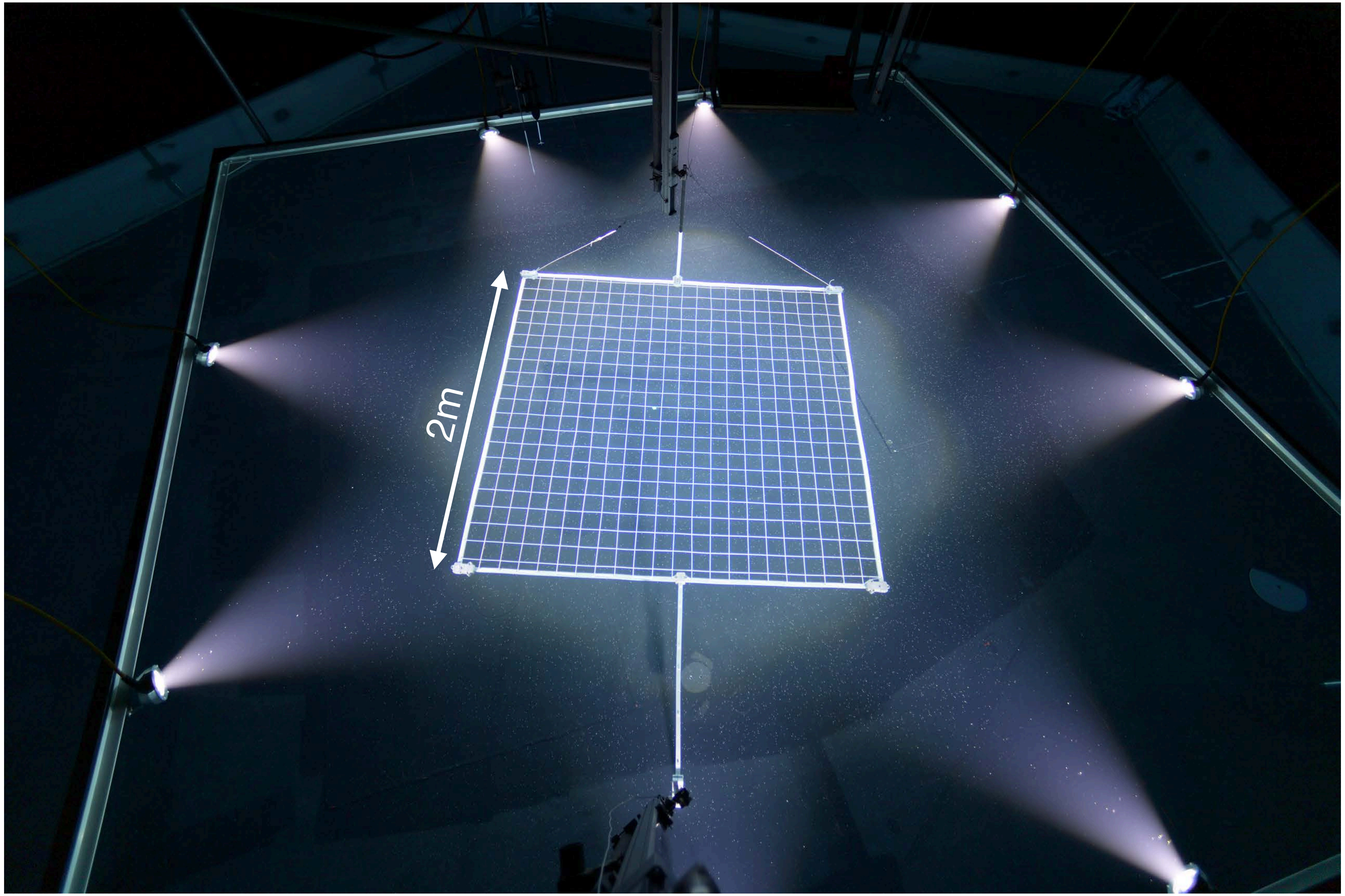


numerical simulations:
Buaria et al. *PRFluids* 2020

tracking individual particles:

- 8x100W LED projector
- 3 cameras (5MPixels)

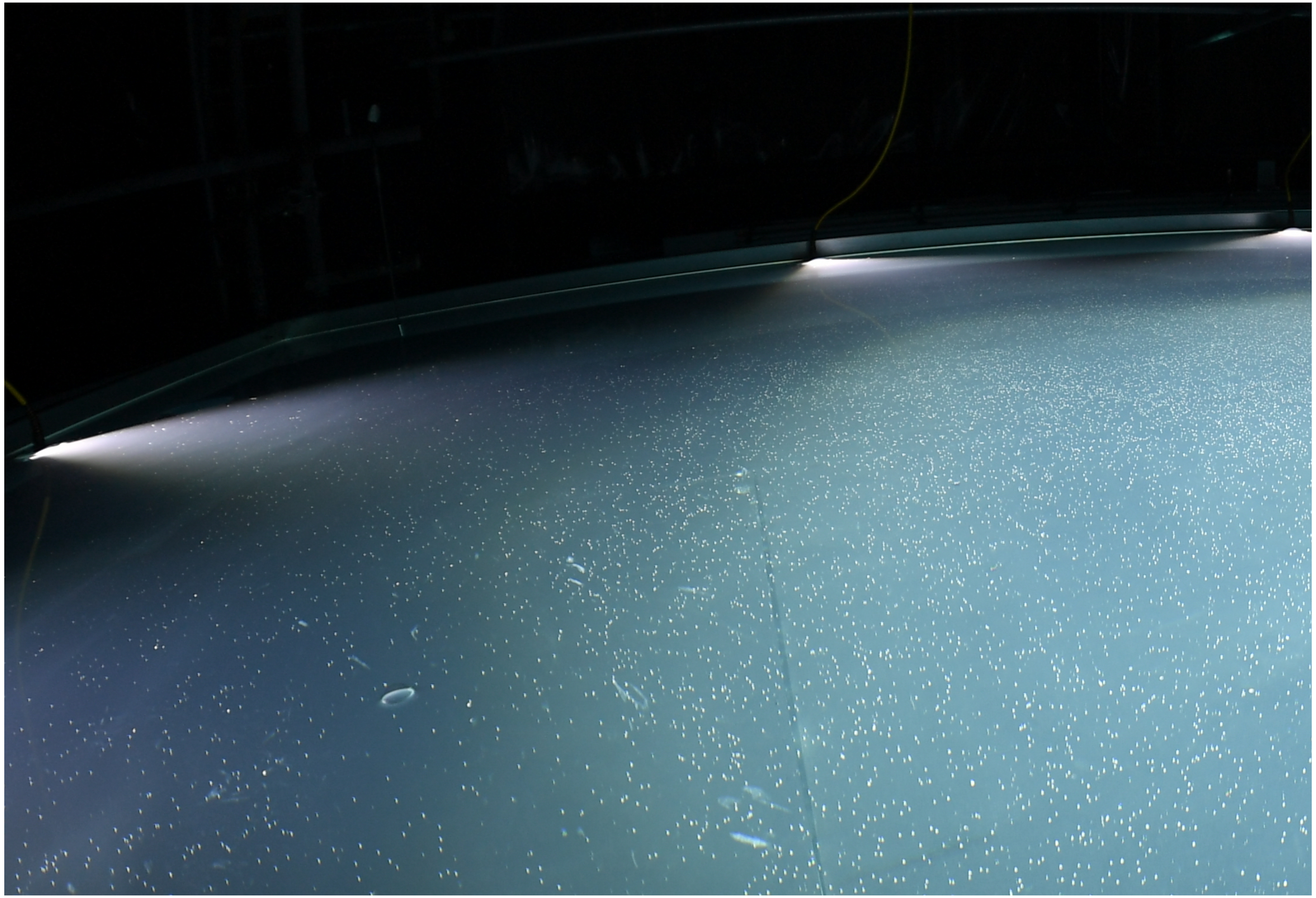
Lagrangian study ?



tracking individual particles:

- 8x100W LED projector
- 3 cameras (5MPixels)
- calibration
  Machicoane et al. *RSI* 2019

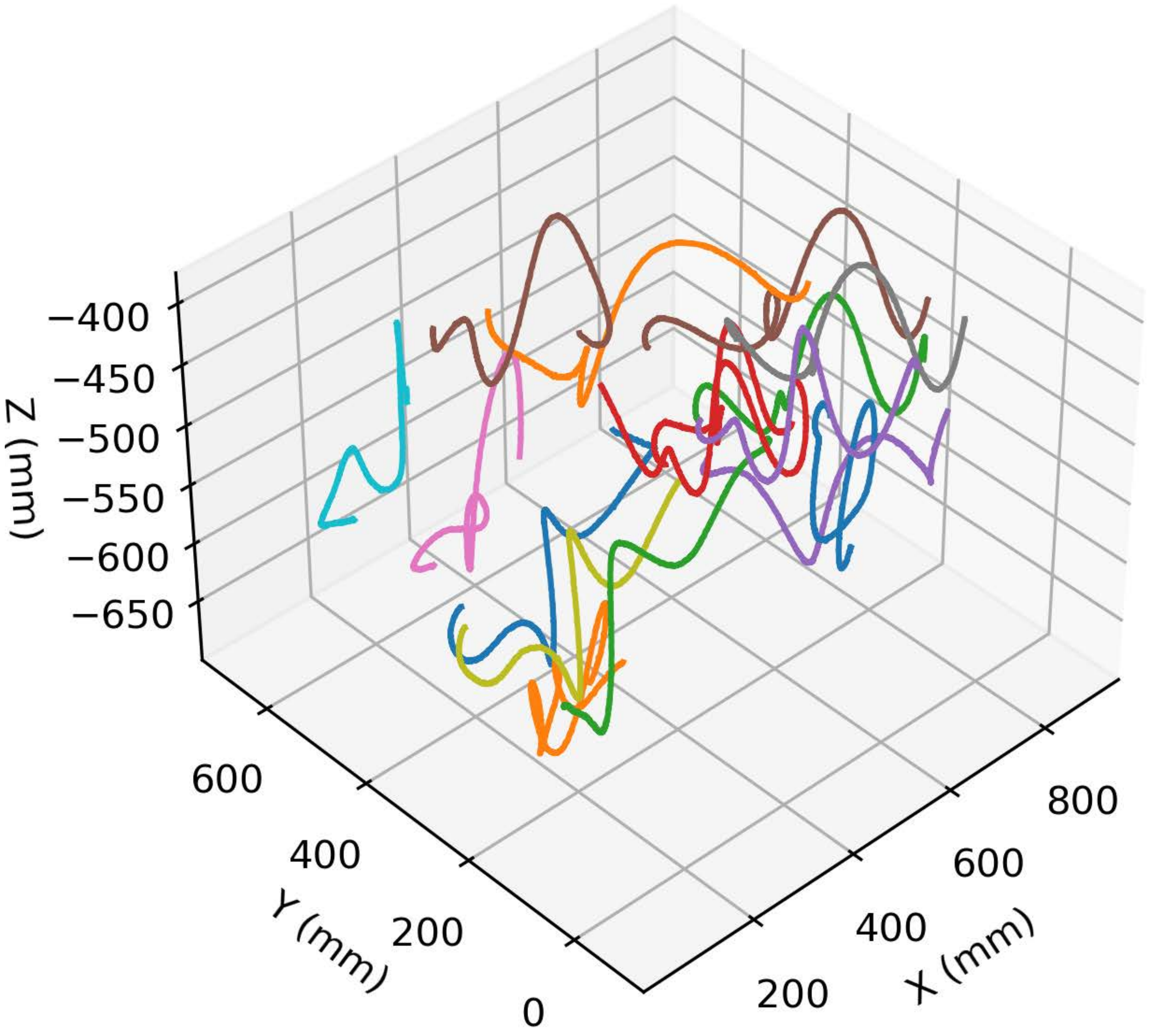
Lagrangian study ?



tracking individual particles:

- 8x100W LED projector
- 3 cameras (5MPixels)
- calibration
  Machicoane et al. *RSI* 2019
- low particle density
  (~1000 particle/image)

Lagrangian study ?

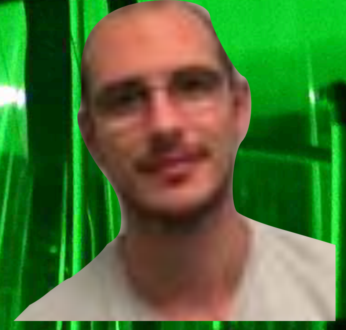


tracking individual particles:

- 8x100W LED projector
- 3 cameras (5MPixels)
- calibration
  Machicoane et al. *RSI* 2019
- low particle density
  (~1000 particle/image)
- PTV software ENS Lyon

work in progress…


M. Calpe-Linares


A. Campagne


G. Davis


J. Reneuve


P. Augier


J. Sommeria

Conclusions

- **observation weak turbulence of internal waves**
  - discrete modes + axisymmetric continuous spectrum
  - impact of domain shape (square & pentagon)
- **strongly stratified turbulence ($N \sim 0.01$, $Re_b \sim 25$) to be confirmed…**

- **in progress:**
  - 2D (vertical plane) & 3D DNS (square): space & time analysis
  - ~130 TB of PIV & PTV images to process…
  - finite size effects in vibrating plates

SIMONS FOUNDATION
European Research Council