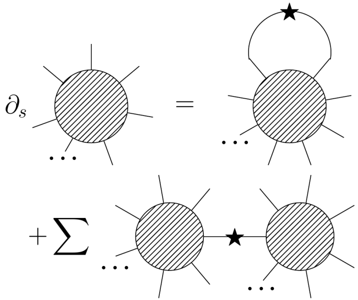
Space-Time Dependence of Correlation Functions in Turbulence



Léonie Canet

NCTR 7, Les Houches

10/02/2025

Presentation outline

1. Why Renormalisation Group for turbulence ?
2. Turbulence as a RG fixed point
3. Time dependence of correlation functions
4. Comparison with direct numerical simulations
5. Closure from symmetries

Acknowledgments

Collaborators

B.Delamotte	N.Wschebor	G.Balarac	V.Rossetto	M.Brachet
Paris	Montevideo	Grenoble	Grenoble	Paris

PhD students and post-docs

C.Pagani M.Tarpin A.Gorbunova C.Fontaine F.Vercesi L.Gosteva

References

M. Tarpin, **LC**, N. Wschebor, Phys. Fluids **30** (2018)

A. Gorbunova, G. Balarac, LC, G. Eyink, V. Rossetto, Phys. Fluids **33** (2021)

C. Pagani, LC, Phys. Fluids **33** (2021)

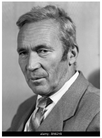
A. Gorbunova, C. Pagani, G. Balarac, LC, V. Rossetto, Phys. Rev. F **6** (2021)

LC, J. Fluid Mech. *Perspectives* **950** (2022)

C. Fontaine, F. Vercesi, M. Brachet, **LC**, PRL **131** (2023)

L. Gosteva, M. Tarpin, N. Wschebor, **LC**, PRE **110** (2024)

Phenomenology of 3D turbulence
Kolmogorov statistical theory ...



Kolmogorov theory 1941

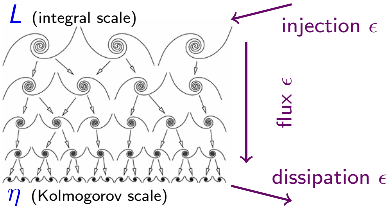
A.N. Kolmogorov

Dokl.Akad.Nauk.SSSR **30**, **31**, **32** (1941)

dimensional analysis:

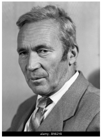
- eddy size $\ell \sim k^{-1}$
- energy flux ϵ
  - ▶ velocity $v \sim (\epsilon \ell)^{1/3}$

Richardson energy cascade



Nazarenko, Lecture Notes in Physics **825** (2011)

Phenomenology of 3D turbulence
Kolmogorov statistical theory . . .



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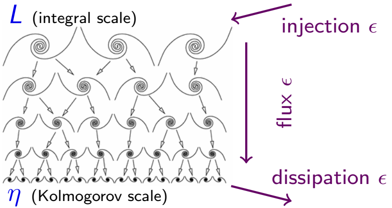


Nazarenko, Lecture Notes in Physics **825** (2011)

- ▶ structure functions

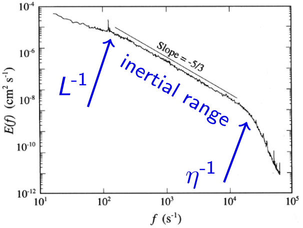
$$S_p(r) \equiv \left\langle \left((\mathbf{v}(\mathbf{r}_0 + \mathbf{r}) - \mathbf{v}(\mathbf{r}_0)) \cdot \hat{\mathbf{r}} \right)^p \right\rangle$$

$$S_p(\ell) = C_p\, \epsilon^{p/3}\, \ell^{p/3} \qquad \text{(K41)}$$

$$S_3(\ell) = -\frac{4}{5}\,\epsilon\,\ell \qquad \text{(exact)}$$

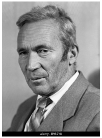
- ▶ kinetic energy spectrum

$$E(k) = C_K \epsilon^{2/3} k^{-5/3}$$



Maurer, Tabeling, Zocchi EPL **26** (1994)

Phenomenology of 3D turbulence
Kolmogorov statistical theory . . . and multi-fractality



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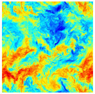

numerical simulations and experiments

$$S_p(\ell) \sim \ell^{\zeta_p} \text{ but } \zeta_p \neq p/3$$

▶ multi-fractality, intermittency

U. Frisch, *Turbulence*, Cambridge University Press

▶ structure functions

$$S_p(r) \equiv \left\langle \left((\mathbf{v}(\mathbf{r}_0 + \mathbf{r}) - \mathbf{v}(\mathbf{r}_0)) \cdot \hat{\mathbf{r}} \right)^p \right\rangle$$

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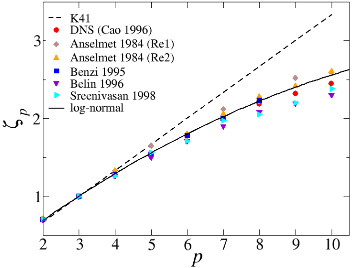
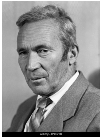


Phenomenology of 3D turbulence
Kolmogorov statistical theory ...



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dimensional analysis:

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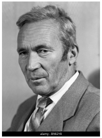
- energy flux ϵ

▶ decorrelation time scale in Eulerian framework

$$\tau_D \sim \epsilon^{-1/3} k^{-2/3} \quad \text{(K41)}$$

Phenomenology of 3D turbulence
Kolmogorov statistical theory ...and random sweeping



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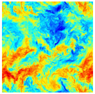

numerical simulations and experiments

$$\tau_D \sim k^{-1}$$

▶ random sweeping effect

Onsager, Z. Physik (1948), Tennekes, JFM (1975)

▶ decorrelation time scale in Eulerian framework

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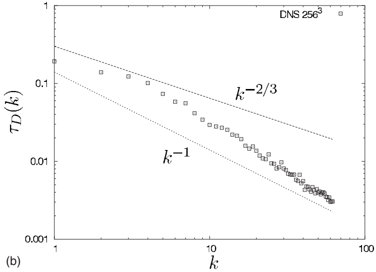

Favier, Godeferd, Cambon, Phys. Fluids **22** (2010)

Random sweeping and space time correlations

- simplified model of advection Kraichnan, Phys. Fluids (1964)
  - velocity decomposition : $\mathbf{v}(t,\mathbf{x}) = \mathbf{U} + \mathbf{u}(t,\mathbf{x})$

    $\mathbf{U}$ large scale, $\sim$ uniform and constant, Gaussian field

    $\mathbf{u}$ small scale, fluctuating field with $|\mathbf{u}| \ll |\mathbf{U}|$

    $\mathbf{U}$ and $\mathbf{u}$ statistically independent

  - advection equation : $\partial_t \mathbf{u} = -(\mathbf{U}\cdot\nabla)\mathbf{u}$

    solution in Fourier space: $\hat{\mathbf{u}}(t,\mathbf{k}) = \hat{\mathbf{u}}(0,\mathbf{k})e^{-i\mathbf{U}\cdot\mathbf{k}t}$

    two-point correlation function:

$$C(t,\mathbf{k}) = \left\langle \hat{\mathbf{u}}(t,\mathbf{k})\cdot\hat{\mathbf{u}}(0,-\mathbf{k})\right\rangle = C(0,\mathbf{k})\exp\left(-\frac{1}{2}U_{\mathrm{rms}}k^2t^2\right)$$

$\longrightarrow$ Gaussian in tk variable

$\longrightarrow$ decorrelation time $\tau_d \sim U_{\mathrm{rms}}^{-1}k^{-1} \neq k^{-2/3}$ (K41)

Random sweeping and space time correlations

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  - velocity decomposition : $\mathbf{v}(t,\mathbf{x}) = \mathbf{U} + \mathbf{u}(t,\mathbf{x})$?

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$\Longrightarrow$ phenomenological model, what about Navier-Stokes equation ?

Challenge: statistical theory of turbulence from “first principles”

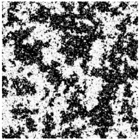
Statistical theory of turbulence
Why the Renormalisation Group ?

- many similarities between critical phenomena
  and homogeneous isotropic turbulence

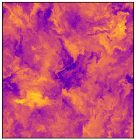
Nelkin, Phys. Rev. A **9** (1974), Rose, Sulem, J. de Phys. **39** (1978)

Eyink, Goldenfeld, Phys. Rev. E **50** (1994), Gawędzki, Nucl. Phys. B **58** (1997)

- scale invariance, self-similarity
- universality
- anomalous critical exponents



2D Ising magnet

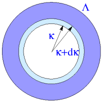


turbulence

Renormalisation Group invented for critical phenomena

$\Longrightarrow$ Wilson's RG

Wilson, Kogut, Phys. Rep. C **12** (1974)



- progressive averaging of fluctuation modes
- build effective theory at scale κ

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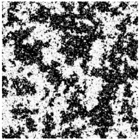

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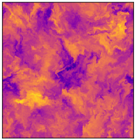


turbulence

Renormalisation Group invented for critical phenomena

scale invariance $\Longleftrightarrow$ fixed point of the RG

How to achieve progressive averaging
of fluctuations modes ?
(in a smooth and rigorous way)

Path integral of stochastic equation
Martin-Siggia-Rose-Janssen-de Dominicis formalism

- generic Langevin equation

$$\partial_t \phi(t,\mathbf{x}) + \mathcal{F}[\phi(t,\mathbf{x})] = \eta(t,\mathbf{x})\,,$$
$$\langle \eta(t,\mathbf{x})\eta(t',\mathbf{x}')\rangle = 2\delta(t-t')D(|\mathbf{x}-\mathbf{x}'|)\,.$$

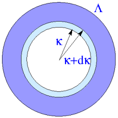
- path integral formulation

Martin, Siggia, Rose, PRA **8** (1973), Janssen, Z. Phys. B **23** (1976), de Dominicis, J. Phys. Paris **37** (1976)

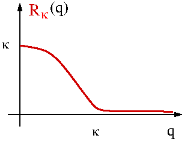
$$\mathcal{P}[\phi] = \int \mathcal{D}\eta \mathcal{P}[\eta]\delta(\phi - \phi_\eta) = \int \mathcal{D}\bar{\phi}\; e^{-\mathcal{S}[\phi,\bar{\phi}]}$$
$$\mathcal{S}[\phi,\bar{\phi}] = \int_{t,\mathbf{x}} \Big\{ \bar{\phi}\, \underbrace{(\partial_t \phi + \mathcal{F}[\phi])}_{\text{deterministic}} \Big\} + \int_{t,\mathbf{x},\mathbf{x}'} \bar{\phi}(t,\mathbf{x})\, \underbrace{D(|\mathbf{x}-\mathbf{x}'|)}_{\text{noise}}\, \bar{\phi}(t,\mathbf{x}')$$
$$\mathcal{Z}[J,\bar{J}] = \int \mathcal{D}\phi \mathcal{P}[\phi] = \int \mathcal{D}\phi\mathcal{D}\bar{\phi}\; e^{-\mathcal{S}[\phi,\bar{\phi}] + \int_{t,\mathbf{x}} \left(J\phi + \bar{J}\bar{\phi}\right)}$$

Functional Renormalisation Group

- based on Wilson's RG ideas



  - progressive integration of fluctuation modes
  - Effective average action Γ_κ instead of effective action $\mathcal{S}_\kappa$

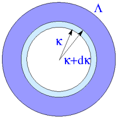


- separation of fluctuation modes

$$\mathcal{Z}_\kappa = \int \mathcal{D}\varphi \, e^{-\mathcal{S}[\varphi] - \Delta\mathcal{S}_\kappa[\varphi] + \int_{t,\mathbf{x}} J\varphi} \quad \text{with } \Delta\mathcal{S}_\kappa = \frac{1}{2}\int_{\mathbf{q}} \varphi R_\kappa(\mathbf{q})\varphi$$

Functional Renormalisation Group

- based on Wilson's RG ideas



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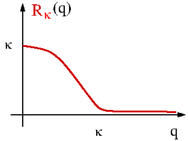

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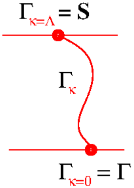
- effective average action: Legendre transform of $\mathcal{W}_\kappa = \ln \mathcal{Z}_\kappa$

$$\Gamma_\kappa[\psi] + \Delta\mathcal{S}_\kappa[\psi] = -\mathcal{W}_\kappa[J] + \int_{t,\mathbf{x}} J\psi \qquad \text{with } \psi = \langle\varphi\rangle = \frac{\partial \mathcal{W}_\kappa}{\partial J}$$

- exact RG equation for Γ_κ

Wetterich, Phys. Lett. B **301** (1993), Dupuis, *et al*, Phys. Rep. 910 (2021)

$$\partial_\kappa \Gamma_\kappa = \frac{1}{2}\mathrm{Tr}\int_{\mathbf{q}} \partial_\kappa R_\kappa(\mathbf{q}) \left[\Gamma_\kappa^{(2)} + R_\kappa\right]^{-1}(-\mathbf{q})$$



Functional Renormalisation Group

- exact RG equation for effective average action   Wetterich, Phys. Lett. B **301** (1993)

$$\partial_\kappa \Gamma_\kappa = \frac{1}{2}\mathrm{Tr}\int_{\mathbf{q}} \partial_\kappa R_\kappa(\mathbf{q})\left[\Gamma_\kappa^{(2)} + R_\kappa\right]^{-1}(-\mathbf{q})$$

- complementary non-pertubative and accurate approximation schemes
  - derivative expansion
  - vertex expansion

Dupuis, *et al*, Phys. Rep. 910 (2021)

3D Ising

	ν	η	ω
conformal bootstrap	0.629971(4)	0.0362978(20)	0.82958(23)
FRG $\mathcal{O}(\partial^6)$	0.63007(10)	0.03648(18)	0.832(14)*
Monte Carlo	0.63002(10)	0.03627(10)	0.832(6)
RG 6-loop	0.6304(13)	0.0335(25)	0.799(11)

$* : \mathcal{O}(\partial^4)$

Balog, Chaté, Delamotte, Wschebor, PRL **103** (2019)

3D $\mathcal{O}(4)$

	ν	η	ω
conformal bootstrap	0.7472(87)	0.0378(32)	0.817(30)
FRG $\mathcal{O}(\partial^4)$	0.7478(9)	0.0360(12)	0.761(12)
Monte Carlo	0.7477(8)	0.0360(4)	0.765
RG 6-loop	0.741(6)	0.0350(45)	0.774(20)

de Polsi, Balog, Tissier, Wschebor, PRE **101** (2020)

Path integral representation of stochastic Navier-Stokes equation

- stochastic Navier-Stokes equation

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha \pi - \nu \nabla^2 v_\alpha = f_\alpha \qquad \text{with} \qquad \partial_\alpha v_\alpha = 0$$

$\mathbf{f}$: Gaussian random forcing of zero mean and covariance

$$\left\langle f_\alpha(t,\mathbf{x}) f_\beta(t',\mathbf{x}')\right\rangle = 2\delta_{\alpha\beta}\delta(t-t') N_L(|\mathbf{x}-\mathbf{x}'|)$$

- Path integral for stochastic Navier-Stokes equation

LC, J. Fluid Mech. **950** (2022)

$$\mathcal{Z} = \int \mathcal{D}\mathbf{v}\,\mathcal{D}\bar{\mathbf{v}}\,\mathcal{D}\pi\,\mathcal{D}\bar{\pi}\; e^{-\mathcal{S}_{\rm NS} + \text{ source terms}}$$

$$\mathcal{S}_{\rm NS} = \int_{t,\mathbf{x}} \bar{v}_\alpha \underbrace{\left[\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha \pi - \nu \nabla^2 v_\alpha\right]}_{\text{deterministic}} + \bar{\pi} \underbrace{\left[\partial_\alpha v_\alpha\right]}_{\text{constraint}} - \int_{t,\mathbf{x},\mathbf{x}'} \bar{v}_\alpha \underbrace{N_L(|\mathbf{x}-\mathbf{x}'|)}_{\text{noise}} \bar{v}_\alpha$$

Renormalisation Group
for stochastic Navier-Stokes equation

▶ Original Wilsonian RG intimately linked with the "ε-expansion"

Wilson, Kogut, *The RG and the ε-expansion* , Phys. Rep. C **12** (1974)

$\cdots$ but for stochastic Navier-Stokes equation

$$\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha + \frac{1}{\rho}\partial_\alpha \pi - \nu \nabla^2 v_\alpha = f_\alpha \qquad \text{with} \qquad \partial_\alpha v_\alpha = 0$$

absence of a small expansion parameter ε!

▷ introduction of an ε via forcing covariance $N_L(\mathbf{k}) \propto k^{d-\varepsilon}$

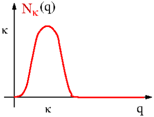
de Dominicis, Martin, PRA **19** (1979), Fournier, Frisch, PRA **28** (1983), Yakhot, Orszag, PRL **57** (1986)

- 3D kinetic energy spectrum $E(k) \propto k^{1-2\varepsilon/3} \Longrightarrow$ K41 scaling for $\varepsilon \to 4$!
- "freezing" mechanism should occur for $\varepsilon > 4$ for universality

use alternative approximation scheme
within the FRG formalism

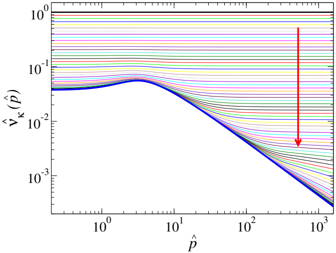
FRG fixed point for large-scale forcing

- RG fixed point for 3D homogeneous isotropic stationary turbulence
  - for a large-scale forcing
  - simple approximation

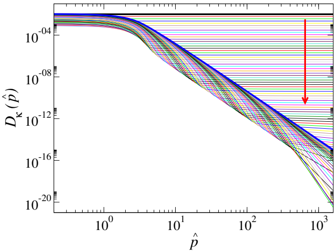


Tomassini, Phys. Lett. B **411** (1997), Mejía-Monasterio, Muratore-Ginnaneschi, PRE **86** (2012)
LC, Delamotte, Wschebor, PRE **93** (2016), LC, J. Fluid Mech. *Perspectives* **950** (2022)

renormalised viscosity $\nu \to \nu_{\rm eff}(p)$



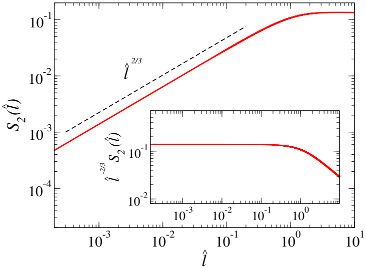
renormalised forcing $D \to D_{\rm eff}(p)$



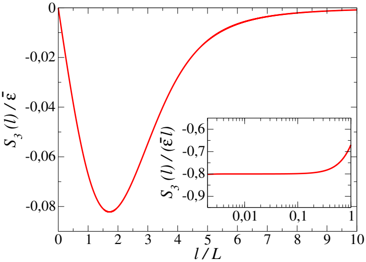
FRG fixed point for large-scale forcing

statistical properties: universal and with K41 scaling

second-order structure function



third-order structure function



$$S_2(\ell) \sim C_2(\epsilon\ell)^{2/3}$$

FRG: $C_2 \simeq 2.06$

experiments: $C_2 \simeq 2.0 \pm 0.4$

K.R. Sreenivasan, Phys. Fluids **7** (1995)

$$S_3(\ell) \sim C_3(\epsilon\ell)$$

FRG: $C_3 = -0.80$

exact: $C_3 = -4/5$

LC, J. Fluid Mech. *Perspectives* **950** (2022)

Time dependence of generic n-point correlation functions

- space-time n-point connected correlation functions

$$C^{(n)}_{\alpha_1 \ldots \alpha_n}(\{t_i, \mathbf{x}_i\}) \equiv \left\langle v_{\alpha_1}(t_1, \mathbf{x}_1) \cdots v_{\alpha_n}(t_n, \mathbf{x}_n) \right\rangle_c$$

Time dependence of generic n-point correlation functions

- space-time n-point connected correlation functions

$$C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$$

- exact asymptotic behaviour in the limit of all $|\mathbf{k}_i|$ large:

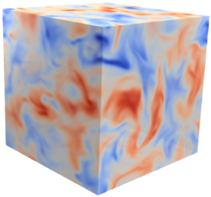
$$C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\mathbf{k}_i\}) \propto \begin{cases} \exp\left(-\alpha_0 \frac{L^2}{\tau^2}\left|\sum_\ell \mathbf{k}_\ell t_\ell\right|^2 + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t_i \ll \tau \\ \exp\left(-\alpha_\infty \frac{L^2}{\tau}\,|t|\,\sum_{k\ell} \mathbf{k}_k\cdot\mathbf{k}_\ell + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t_i \gg \tau \end{cases}$$

  - small times regime $\Longrightarrow$ random sweeping
  - rigorous and generalised for any n-point correlations
  - prediction of a new regime at large time

Presentation outline

Comparison with Direct Numerical Simulations

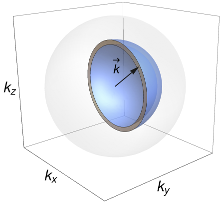
- 3D homogeneous isotropic incompressible flow
- large-scale random forcing
- all scales resolved down to $k_{\max}\eta \simeq 1.5$



► computation of space-time correlations

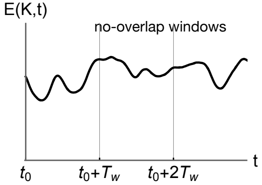
▷ spatial average

over spherical shells (isotropy)



▷ temporal average

over separate time windows in the stationary state



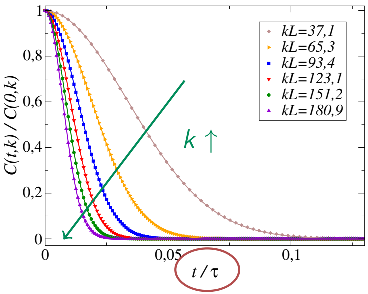
Gorbunova, Balarac, LC, Eyink, Rossetto, Phys. Fluids **33** (2021)

Two-point correlation function at large wave numbers
Small time delays: random sweeping effect

- result from functional renormalisation group (FRG):

$$C(t, \mathbf{k}) = C(0, \mathbf{k}) \underbrace{\exp\left(-\alpha_0 \, (L/\tau)^2 \, (kt)^2 \right)}_{\text{Gaussian in } kt}$$

- results from direct numerical simulations (DNS):



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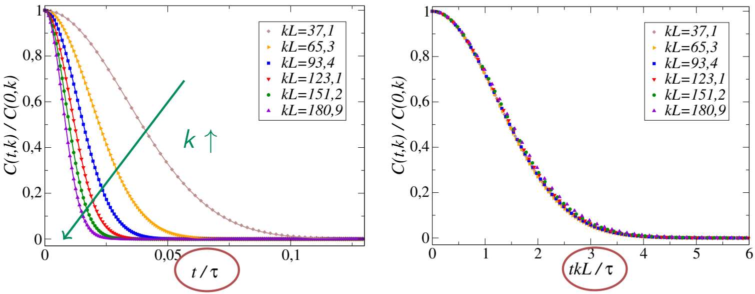
$$C(t,\mathbf{k}) = C(0,\mathbf{k}) \underbrace{\exp\left(-\alpha_0 (L/\tau)^2 (kt)^2\right)}_{\text{Gaussian in } kt}$$

$\Longrightarrow$ $C(t,\mathbf{k})/C(0,\mathbf{k})$ should collapse onto a single Gaussian against kt

- results from direct numerical simulations (DNS):



Two-point correlation function at large wave numbers
Small time delays: random sweeping effect

- result from functional renormalisation group (FRG):

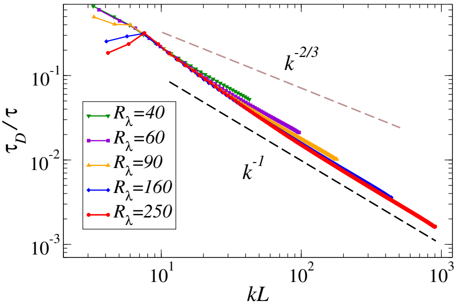
$$C(t,\mathbf{k}) = C(0,\mathbf{k})\ \underbrace{\exp\left(-\alpha_0\,(L/\tau)^2\,(kt)^2\right)}_{\text{Gaussian in } kt}$$

- decorrelation time τ_D
  - from Gaussian fit $\exp(-(t/\tau_D)^2)$
  - from FRG:

    $$\tau_D = (\sqrt{\alpha_0}(L/\tau)k)^{-1}$$

  - similar to

    Favier, Cambon *et al*, Phys. Fluids **22** (2010)



  - implies frequency spectrum $E(\omega) \sim \omega^{-5/3}$

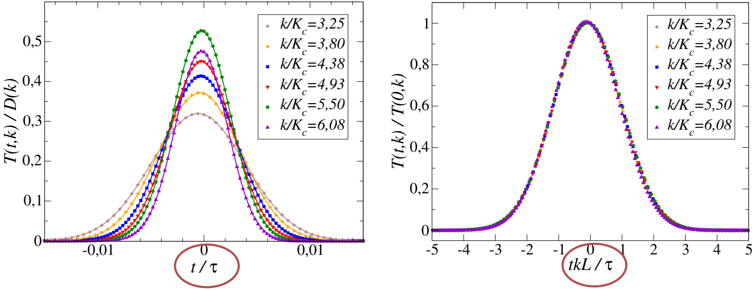
    Chevillard, Roux, Leveque, Mordant, Pinton, Arneodo, PRL **95** (2005).

Three-point correlation function at large wave numbers
Small time delays

- advection-velocity correlation function from FRG

$$T(t,\mathbf{k}) = -ik_n P^{\perp}_{\ell m} \sum_{\mathbf{k}'} C^{(3)}_{mn\ell}(t,\mathbf{k}',t,\mathbf{k}-\mathbf{k}')\Big|_{\text{large }\mathbf{k}'} \propto \exp\left(-\alpha_0 (L/\tau)^2 |\mathbf{k}|^2 t^2\right)$$

- results from direct numerical simulations



Gorbunova, Balarac, LC, Eyink, Rossetto, Phys. Fluids **33** (2021)

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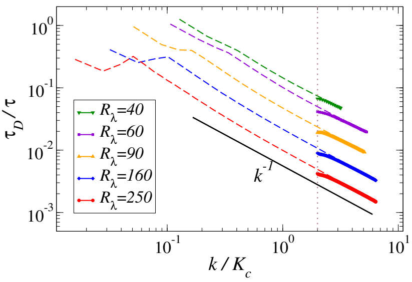


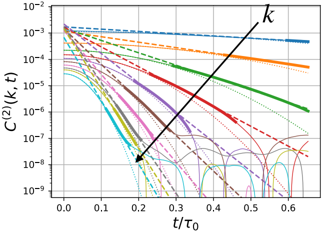
$\Longrightarrow$ same coefficient α_0 as for two-point function

Gorbunova, Balarac, LC, Eyink, Rossetto, Phys. Fluids **33** (2021)

What about large time delays ?

$$C(t,\mathbf{k}) = C(0,\mathbf{k}) \underbrace{\exp\left(-\alpha_\infty\,(L^2/\tau)\,k^2|t|\right)}_{\text{exponential in } k^2 t}$$

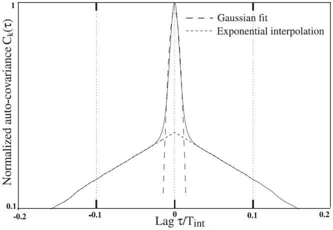
- hard to observe in DNS of Navier-Stokes flow



Gorbunova, Balarac, LC, Eyink, Rossetto,

Phys. Fluids **33** (2021)

- experiment of turbulence air jet



but correlations of modulus

Poulain, Mazellier, Chevillard, Gagne, Baudet,

Eur. Phys. J. B **53** (2006)

Passively advected scalars in turbulent flows

- diffusion-advection equation

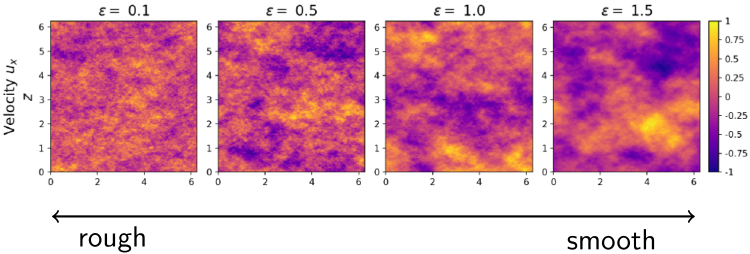
$$\partial_t \theta + \mathbf{v} \cdot \nabla \theta - \kappa \nabla^2 \theta = f$$

$\mathbf{v}$ synthetic random field Kraichnan, Phys. Fluids **11** (1968)

$$\left\langle \hat{v}_i(t, \mathbf{k}) \hat{v}_j(t', -\mathbf{k}) \right\rangle = P_{ij}^{\perp}(\mathbf{k}) \frac{D_0}{k^{d+\varepsilon}} T_\tau(t - t')$$

©Walter Baxter

τ: finite correlation time, Kraichnan limit $T_\tau(t - t') \xrightarrow{\tau \to 0} \delta(t - t')$



Correlations in time-correlated Kraichnan model

- result from functional renormalisation group

$$C_\theta(t,\mathbf{k}) \propto \begin{cases} \exp\Big(-\gamma_0 (L/\tau)^2\,(tk)^2\Big) & |t| \ll \tau \\ \exp\Big(-\gamma_\infty (L^2/\tau)\,|t|\,k^2\Big) & |t| \gg \tau \end{cases}$$

Pagani and LC

Phys. Fluids **33** (2021)

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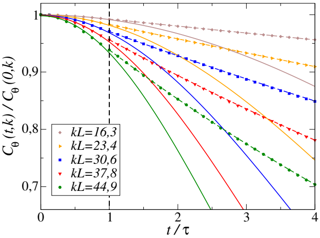
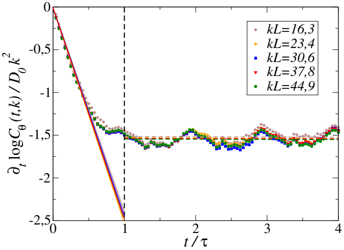
Pagani and LC

Phys. Fluids **33** (2021)

- results from direct numerical simulations





- Gaussian at $|t| \leq \tau$
- exponential at $|t| \geq \tau$

$$\frac{1}{k^2}\frac{d}{dt}\log\left[C_\theta(t,\mathbf{k})\right] \propto \begin{cases} -2\gamma_0 (L/\tau)^2\, t \\ -\gamma_\infty (L^2/\tau) \end{cases}$$

Gorbunova, Pagani, Balarac, LC, Rossetto, Phys. Rev. F **6** (2021)

Correlations in delta-correlated Kraichnan model $\tau \to 0$

- result from functional renormalisation group (FRG)

$$C_\theta(t, \mathbf{k}) = C_\theta(0, \mathbf{k}) \exp\left(-\kappa_{\rm ren} k^2 |t|\right)$$

$$\kappa_{\rm ren} = \kappa + \underbrace{\frac{d-1}{2d} \int_{\mathbf{k}} \frac{D_0}{(k^2 + m^2)^{(d+\varepsilon)/2}} d^d\mathbf{k}}_{\text{determined by velocity only}}$$

- exponential decay for all times
- exact expression for $\kappa_{\rm ren}$

Pagani and LC, Phys. Fluids **33** (2021)

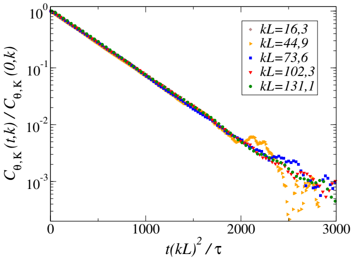
similar to eg Kraichnan, PRL **72** (1994), Mitra, Pandit, PRL **95** (2005)

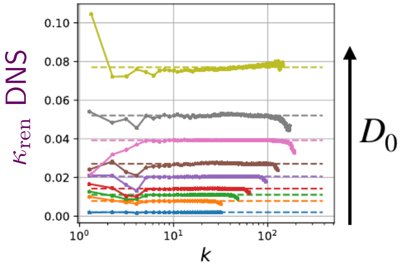
Correlations in delta-correlated Kraichnan model $\tau \to 0$

- result from functional renormalisation group (FRG)

$$C_\theta(t, \mathbf{k}) \propto \exp\left(-\kappa_{\rm ren} k^2 |t|\right) \,, \quad \kappa_{\rm ren} = \kappa + \frac{1}{3} \int_{\mathbf{k}} \frac{D_0}{(k^2 + m^2)^{(3+\varepsilon)/2}} d^3\mathbf{k}$$

- results from direct numerical simulations (DNS)





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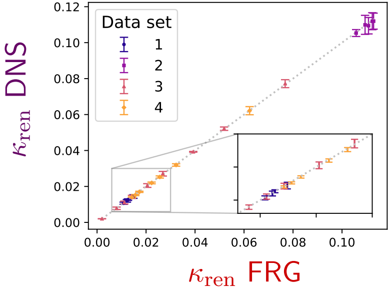


Gorbunova, Pagani, Balarac, LC, Rossetto, Phys. Rev. F **6** (2021)

Presentation outline

Key ingredient for closure: Extended symmetries and Ward identities

- extended symmetry: infinitesimal transformation such that the variation of $\mathcal{S}$ is linear in the fields

$\Longrightarrow$ yields exact functional Ward identities:

$$\delta\Gamma[\psi,\bar\psi] = \delta\mathcal{S}[\varphi,\bar\varphi]\Big|_{\psi=\langle\varphi\rangle,\bar\psi=\langle\bar\varphi\rangle}$$

- infinite set of exact identities for vertices

$$\Gamma^{(m,n)}_{\alpha_1\cdots\alpha_{m+n}}(t_1,\mathbf{x}_1,\cdots,t_{m+n},\mathbf{x}_{m+n}) = \frac{\delta^{m+n}\Gamma}{\underbrace{\delta\psi_{\alpha_1}(t_1,\mathbf{x}_1)\cdots}_{m\ \psi}\underbrace{\delta\bar\psi_{\alpha_{m+1}}(t_{m+1},\mathbf{x}_{m+1})\cdots}_{n\ \bar\psi}}$$

$$\left\{\Gamma^{(m,n)}_{\alpha_1\cdots\alpha_{m+n}}\right\} \Longleftrightarrow \left\{C^{(m,n)}_{\alpha_1\ldots\alpha_{m+n}}\right\}$$

Extended symmetries and Ward identities of the Navier-Stokes action

- time-gauged Galilean invariance: $\mathcal{G} = \begin{cases} \mathbf{x} \to \mathbf{x} + \vec{\epsilon}(t) \\ \mathbf{v} \to \mathbf{v} - \partial_t \vec{\epsilon}(t) \end{cases}$

infinite set of exact Ward identities for all vertices with $\mathbf{q} = 0$ on a $\mathbf{u}$

$$\Gamma^{(\mathbf{m+1},n)}_{\alpha\alpha_1\cdots\alpha_{n+m}}(\omega, \mathbf{q} = 0; \{\nu_i, \mathbf{p}_i\}) = \mathcal{D}_\alpha(\omega)\Gamma^{(\mathbf{m},n)}_{\alpha_1\cdots\alpha_{n+m}}(\{\nu_i, \mathbf{p}_i\})$$

[$\mathcal{D}_\alpha(\omega)$ shift operator]

- time-gauged shift symmetry: $\mathcal{R} = \begin{cases} \delta\bar{v}_\alpha(t,\mathbf{x}) & = \bar{\epsilon}_\alpha(t) \\ \delta\bar{\pi}(t,\mathbf{x}) & = v_\beta(t,\mathbf{x})\bar{\epsilon}_\beta(t) \end{cases}$
  - not identified yet! LC, B. Delamotte, N. Wschebor, Phys. Rev. E **91** (2015)

infinite set of exact Ward identities for all vertices with $\mathbf{q} = 0$ on a $\bar{\mathbf{u}}$

$$\Gamma^{(m,n)}_{\alpha_1\cdots\alpha_{m+n}}(\nu_1, \mathbf{p}_1, \cdots, \nu_{m+1}, \mathbf{q} = 0, \cdots) = 0$$

Space-time correlations from Functional Renormalisation Group

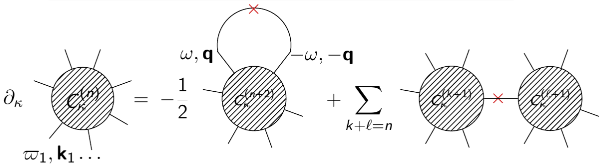
- space-time *n*-point connected correlation functions

$$C^{(n)}_{\alpha_1\ldots\alpha_n}(\{t_i,\mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$$

- exact (but infinite hierarchy of) FRG flow equations for $C^{(n)}$
  - derived from flow equation for generating functional $\mathcal{W}_\kappa = \ln \mathcal{Z}_\kappa$

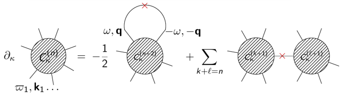
$$\partial_\kappa \mathcal{W}_\kappa = -\frac{1}{2}\mathrm{Tr}\int_{t_x,t_y,\mathbf{x},\mathbf{y}} \partial_\kappa[R_\kappa]_{\alpha\beta}(\mathbf{x}-\mathbf{y})\left\{\frac{\delta^2\mathcal{W}_\kappa}{\delta j_\alpha(t_x,\mathbf{x})\delta j_\beta(t_y,\mathbf{y})} + \frac{\delta\mathcal{W}_\kappa}{\delta j_\alpha(t_x,\mathbf{x})}\frac{\delta\mathcal{W}_\kappa}{\delta j_\beta(t_y,\mathbf{y})}\right\}$$

Polchinski, Nucl. Phys. B **231** (1984), Wetterich, Phys. Lett. B **301** (1993)



Exact closure in the large wave-number limit

- flow for $C^{(n)}_{\alpha_1\dots\alpha_n}(\{t_i,\mathbf{x}_i\}) \equiv \Big\langle v_{\alpha_1}(t_1,\mathbf{x}_1)\cdots v_{\alpha_n}(t_n,\mathbf{x}_n)\Big\rangle_c$



exact (but infinite hierarchy of) flow

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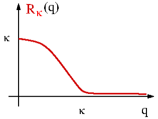
exact (but infinite hierarchy of) flow

asymptotic flow at large wavenumber

(1) large wave-number expansion: all $|\mathbf{k}_i|$ and $\left|\sum_i \mathbf{k}_i\right| \gg \kappa$



▷ $\partial_\kappa R_\kappa(\mathbf{q})$: $|\mathbf{q}| \lesssim \kappa \Longrightarrow |\vec{q}| \ll |\vec{k}_i|$

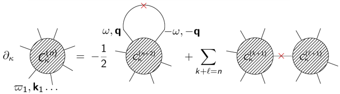
$\Longrightarrow$ set $\vec{q} = 0$ in all vertices

asymptotically exact for $|\mathbf{k}_i| \gg \kappa \sim L^{-1}$ and in a scaling regime

Blaizot, Wschebor, Mendez-Galain, Phys. Lett B **832** (2006), Tarpin, LC, Wschebor, Phys. Fluids **30** (2018)

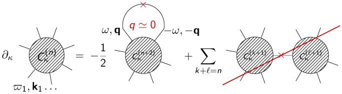
Exact closure in the large wave-number limit

- flow for $C^{(n)}_{\alpha_1\ldots\alpha_n}(\{t_i, \mathbf{x}_i\}) \equiv \left\langle v_{\alpha_1}(t_1, \mathbf{x}_1) \cdots v_{\alpha_n}(t_n, \mathbf{x}_n) \right\rangle_c$



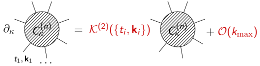
exact (but infinite hierarchy of) flow

large k_i



asymptotic flow at large wavenumber

extended symmetries



closed flow at large wavenumber

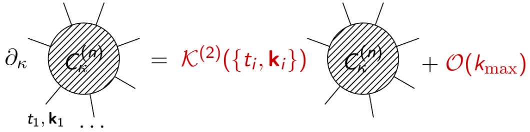
with $\mathcal{K}^{(2)}(\{\omega_i, \mathbf{k}_i\}) = \int_\omega J(\omega)\mathcal{D}_\alpha \mathcal{D}_\alpha$ and $J(\omega) = \int_{\mathbf{q}} \kappa \tilde{\partial}_\kappa C^{(2)}_\kappa(\omega, \mathbf{q})$

(2) Ward identities related to extended symmetries

- time-gauged Galilee
- time-gauged response shift

infinite set of exact Ward identities for all vertices with a $\mathbf{q} = 0$

Exact closure in the large wave-number limit



- kernel: $\mathcal{K}^{(2)}(\{\omega_i, \mathbf{k}_i\}) = \int_\omega J(\omega) \mathcal{D}_\mu \mathcal{D}_\mu$ with $J(\omega) = \int_{\mathbf{q}} \kappa \tilde{\partial}_\kappa C_\kappa^{(2)}(\omega, \mathbf{q})$
- Fourier inverse in real time:

$$\mathcal{K}^{(2)}(\{t_i, \mathbf{k}_i\}) = \int_\omega J(\omega) \sum_{k,\ell} \frac{\vec{k}_k \cdot \vec{k}_\ell}{\omega^2} \left(e^{i\omega(t_k - t_\ell)} - e^{i\omega t_k} - e^{-i\omega t_\ell} + 1 \right)$$

- at a fixed point, in the small and large time limits:

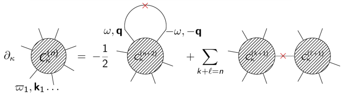
$$\mathcal{K}^{(2)}(\{\hat{t}_i, \hat{\mathbf{k}}_i\}) \xrightarrow{\hat{t}_i \ll 1} I_0^* \left| \sum_\ell \hat{\mathbf{k}}_\ell \hat{t}_\ell \right|^2$$

$$\mathcal{K}^{(2)}(\{t_i, \mathbf{k}_i\}) \xrightarrow{\hat{t}_i \gg 1} I_\infty^* \sum_{k,\ell} \vec{k}_k \cdot \vec{k}_\ell \left(|t_k| + |t_\ell| - |t_\ell - t_k| \right)$$

Tarpin, LC, Wschebor, Phys. Fluids **30**, 055102 (2018)

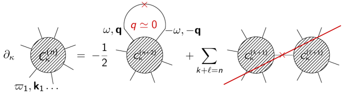
Exact closure in the large wave-number limit

- flow for $C^{(n)}_{\alpha_1\ldots\alpha_n}(\{t_i, \mathbf{x}_i\}) \equiv \left\langle v_{\alpha_1}(t_1, \mathbf{x}_1)\cdots v_{\alpha_n}(t_n, \mathbf{x}_n)\right\rangle_c$



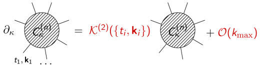
exact (but infinite hierarchy of) flow

large k_i



asymptotic flow at large wavenumber

extended symmetries



closed flow at large wavenumber

fixed point

analytical solution

$$C^{(n)}_{\alpha_1\ldots\alpha_n}(\{t_i, \mathbf{x}_i\}) = C^{(n)}_{\alpha_1\ldots\alpha_n}(\{0, \mathbf{x}_i\}) \times \textbf{dominant term}$$

(3) solution at the fixed point

$$C^{(n)}_{\alpha_1\ldots\alpha_n}(\{t_i, \mathbf{k}_i\}) \propto \begin{cases} \exp\left(-\alpha_0 \frac{L^2}{\tau^2}\left|\sum_\ell \mathbf{k}_\ell t_\ell\right|^2 + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t_i \ll \tau \\ \exp\left(-\alpha_\infty \frac{L^2}{\tau}\,|t|\,\sum_{k\ell} \mathbf{k}_k\cdot\mathbf{k}_\ell + \mathcal{O}(|\mathbf{k}_{\max}|L)\right) & t_i \gg \tau \end{cases}$$

M. Tarpin, LC, N. Wschebor, Phys. Fluids **30** (2018), LC, J. Fluid Mech. *Perspectives* **950** (2022)

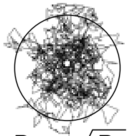
Interpretation of the two regimes of decorrelation

single-particle dispersion

- Lagrangian mean-square displacement

$$\langle |\mathbf{r}(t)|^2 \rangle \sim \begin{cases} U_{\rm rms}^2 t^2 & |t| \ll \tau_0 \quad \text{ballistic transport} \\ 2D|t| & |t| \gg \tau_0 \quad \text{diffusive transport} \end{cases}$$

Taylor, Proc. Lond. Math. Soc. **2** (1922)



$R \propto \sqrt{Dt}$

D: eddy diffusivity

- Eulerian correlation function of scalars

$$C(t,\mathbf{k}) \sim \exp\left(-\frac{1}{2}k^2\langle|\mathbf{r}(t)|^2\rangle\right) \sim \begin{cases} \exp\left(-\frac{1}{2}U_{\rm rms}^2 k^2 t^2\right) & |t| \ll \tau_0 \\ \exp\left(-Dk^2|t|\right) & |t| \gg \tau_0 \end{cases}$$

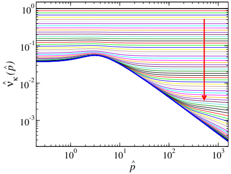
Kraichnan, Phys. Fluids **7** (1964)

$\Longrightarrow$ similar to FRG results !

$$C(t,\mathbf{k}) \sim \begin{cases} \text{Gaussian in } kt & |t| \ll \tau_0 \\ \text{exponential in } k^2 t & |t| \gg \tau_0 \end{cases}$$

Summary

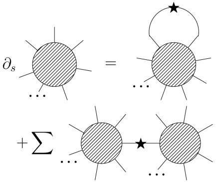
FRG fixed point for turbulence



- fixed point for large-scale forcing
- effective viscosity and forcing amplitude

  $\Longrightarrow$ K41 scaling for simple approximation

time dependence of *n*-point correlations at large *p*



- exact closure of FRG equations based on extended symmetries

  $\Longrightarrow$ small times: $\exp(-\alpha_0(kt)^2)$

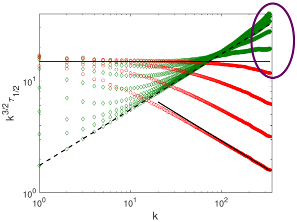
  $\Longrightarrow$ large times: $\exp(-\alpha_\infty k^2|t|)$

New fixed point for Burgers-KPZ equation

- one-dimensional Burgers equation with stochastic force

$$\partial_t v + \lambda v \partial_x v = \nu \partial_x^2 v + \sqrt{D}\partial_x f$$

- decorrelation time from the two-point function $C(t,k)$: $\tau_{1/2} \sim k^{-z}$



Inviscid Burgers: $z = 1$
($\nu = 0$)

Kardar-Parisi-Zhang: $z = 3/2$

Edwards-Wilkinson: $z = 2$
($\lambda = 0$)

unexplained scaling regime!

Cartes, T., Pandit, Brachet
Phil. Trans. A **380** (2022)

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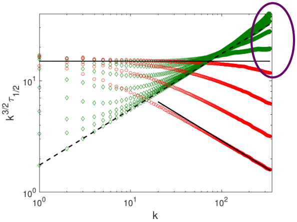

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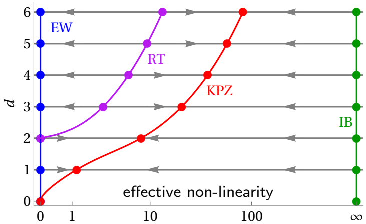
new fixed point of the Burger-KPZ equation

Fontaine, Vercesi, Brachet, LC, PRL **131** (2023)

Gosteva, Tarpin, Wschebor, LC, PRE **110** (2024)



Intermittency corrections in shell models

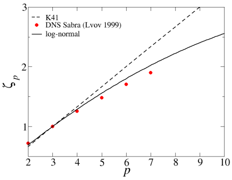
► toy model for turbulence: Sabra shell model

▷ scalar velocity modes $v_n(t) \in \mathbb{C}$ on discrete shells $k_n = k_0 \lambda^n$

$$\begin{cases} \dfrac{dv_n}{dt} = B_n[v, v^*] - \nu k_n^2 v_n + f_n \\ B_n[v, v^*] = i\left[a k_{n+1} v_{n+2} v_{n+1}^* + b k_n v_{n+1} v_{n-1}^* - c k_{n-1} v_{n-1} v_{n-2}\right] \end{cases}$$

▷ features intermittency similar to NS turbulence

L'vov, Podivilov, Pomyalov, Procaccia, Vandembroucq, PRE **58** (1998)



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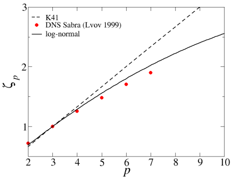

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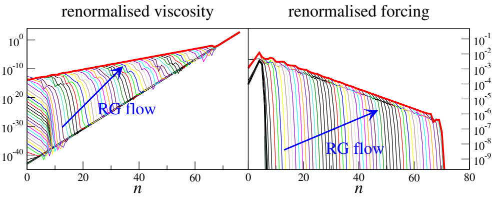
▶ FRG: fixed point in inverse RG flow with anomalous exponents

- RG flow from large to small scales



- exponent for S_2

$$\zeta_2^{\mathrm{K41}} = 2/3$$
$$\zeta_2^{\mathrm{FRG}} \simeq 0.74 \pm 0.03$$
$$\zeta_2^{\mathrm{DNS}} \simeq 0.720 \pm 0.008$$

Fontaine, Tarpin, Bouchet, LC, SciPost Phys. **15** (2023)


Thank you for your attention !
LPMMC