# Wave Turbulence of Gravity-capillary surface waves.

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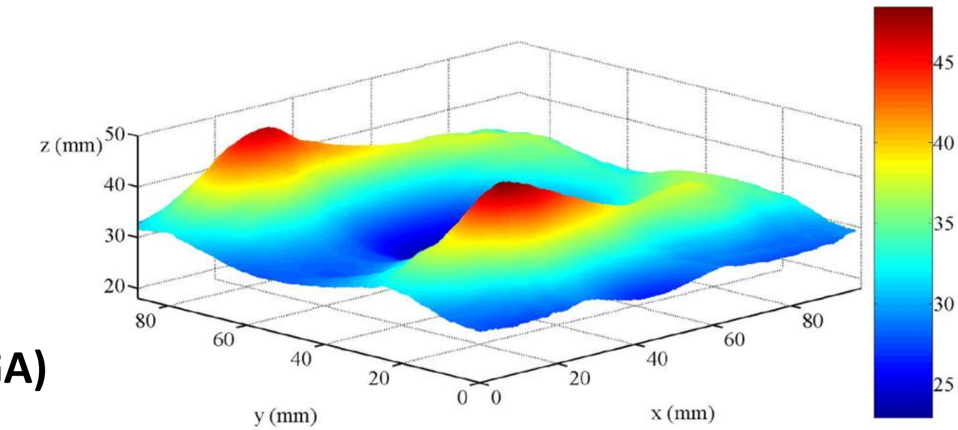


**Acknowledgments:** **Nicolas Mordant (LEGI /UJF Grenoble)**
**Claudio Falcon (DFI/Universidad del Chili)**

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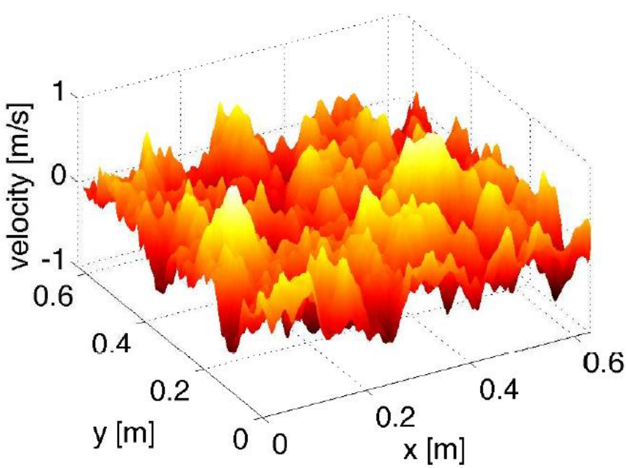
# Wave turbulence

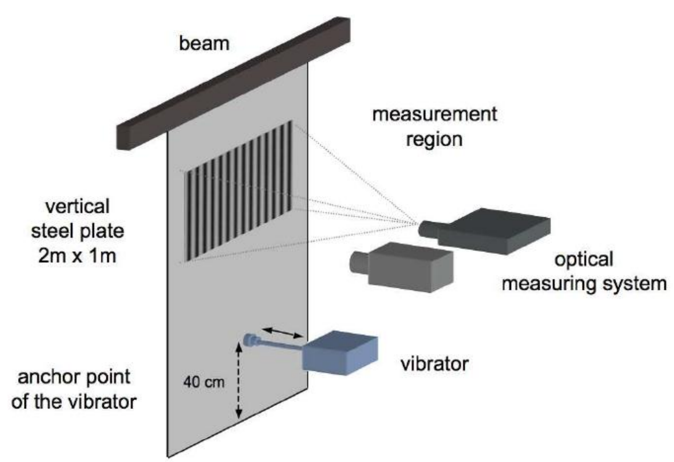
- Concept of turbulence is in fact more general for out equilibrium systems:
  Scale invariant and non linear process which transfers energy from a forcing scale to a dissipation scale.

- In particular : **Wave turbulence**
  Field of waves maintained by a stationary forcing and interacting due to non-linear effects.

- Numerous examples : **Alfven waves** in plasma, **Internal waves** in the ocean, **Spin waves** in solid state physics,
  **Elastic waves**
  in a thin metal plate
  ( bending modes) :

*Cobelli, Petitjeans, Maurel, Pagneux & Mordant*
***PRL** (103), 2009*



# Weak turbulence theory

- Wave turbulence "simpler" than hydrodynamic turbulence.
  For **weak non-linearity** and random waves ( isotropy, uniform distribution of phases and Gaussian distribution of amplitudes ) analytical treatment becomes possible.

- Decomposition of the wave field in Fourier space as a function of wave number **k**.

- Hamiltonian formalism: canonical amplitude $a_{\mathbf{k}}(t)$
  Used to define occupation number like a number of quasi-particles. $n_{\mathbf{k}} = \langle a_{\mathbf{k}}(t) a^*_{\mathbf{k}}(t) \rangle$

- Kinetic equation involving a forcing term, a dissipation term and an interaction term: exchange between the different waves numbers defined as an integral over all the values of **k.**

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = St(n_{\mathbf{k}}) - 2\gamma_{\mathbf{k}} n_{\mathbf{k}} + F_{\mathbf{k}}(t)$$


**Ref**:
- S. Nazarenko **Wave Turbulence** (2011) (Springer-Verlag)
- V.E. Zakharov, V.S. L'vov and G. Falkovich, ***Kolmogorov Spectra of Turbulence :Wave Turbulence***, (1992) (Series in Nonlinear Dynamics, Springer).

# Surface wave turbulence

- Study of a turbulent field of waves on a liquid surface, for a stationary excitation.
- Wave turbulence => when amplitude of forcing is high enough, observation of a **Turbulent cascade.**
  Agreement with Wave Turbulence theory (Weak Turbulence) ?
- Inside inertial range, Weak Turbulence Theory predicts power law behavior of spectra in Fourier space.
  Depending on the expression of interaction parameter.
  * Kind of the repelling force : Gravity at large scale
  Surface tension at small scale

  * Kind of weakly non-linear resonant interactions :
  3 waves (triads)
  4 waves (quartets).

  * Viscous dissipation assumed at a very small scale.

# Surface waves

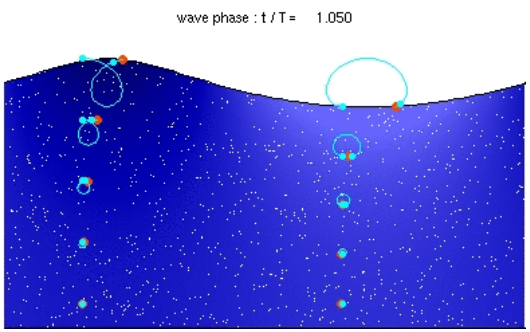
- Waves at a liquid surface : most obvious way to observe Wave Turbulence in laboratory.

- At large scale restoring force is the weight => gravity waves
  For small scales restoring force is surface tension, which acts against an increases of the area of the liquid surface.
  Intermediate scale corresponds to the capillary length $L_c = \sqrt{\frac{\gamma}{\rho g}}$
  Lc ~ 2 mm

- For a potential flow of ideal incompressible deep fluid with a free surface.
  Without vorticity and dissipation, non linearity neglected

  Dispersion relation : $\omega(k) = \sqrt{gk + (\gamma/\rho)k^3}$

  Streamlines : circles




- Applications : oceanography, wave energy, exchanges between the Atmosphere and the Sea ...

# Kolmogorov-Zakharov (K.Z.) spectra

- In stationary regime, at scales where dissipation and forcing are negligible, Weak Turbulence Theory predicts power law spectra for $n_k$ depending on the expression of interaction parameter
  => power law spectra for the water elevation η(x,t).

$$E_k = \int e^{ikz} \langle \eta(\boldsymbol{x}, t) \eta(\boldsymbol{x} + \boldsymbol{w} z, t) \rangle \, dz \qquad E_\omega = \int e^{i\omega t'} \langle \eta(\boldsymbol{x}, t) \eta(\boldsymbol{x}, t + t') \rangle \, dt'$$

**Gravity** waves : $E_k \propto k^{-5/2}$ $E_\omega \propto \omega^{-4}$ 4 waves interactions

**Capillary** waves : $E_k \propto k^{-15/4}$ $E_\omega \propto \omega^{-17/6}$ 3 waves interactions

# Motivations and outlines

- Wave Turbulence Theory depends on strong hypotheses (small non-linearity, infinite systems, initial random conditions, absence of dissipation in the inertial range …).
- Test relevance and applicability of these concepts in controlled laboratory experiments.
  Application : Energy balance, Non-linear dissipation of swell …

Outlines.

I. Laboratory investigation of gravity-capillary wave turbulence.
   Review on Previous results.

II. Gravity wave turbulence in large basins

III. Capillary wave turbulence:
   solving the question on the energy scaling.

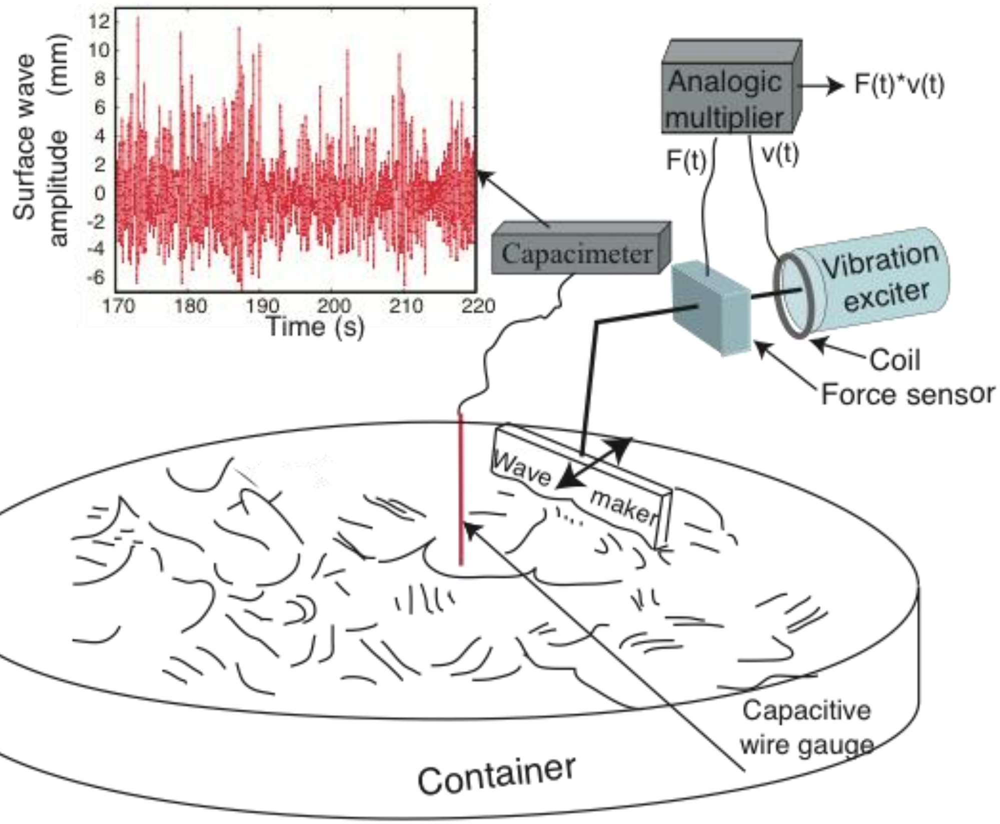
IV. Spatiotemporal measurement of capillary wave turbulence.

# I-Laboratory investigation of gravity-capillary wave turbulence

One point measurement of the wave height η(t)

$$\eta(t) \Rightarrow S_\eta(f) = \frac{1}{T}\left|\int \eta(t)e^{i\omega t}dt\right|^2$$

*Falcon, Laroche, Fauve* ***PRL (98)*** *2007*

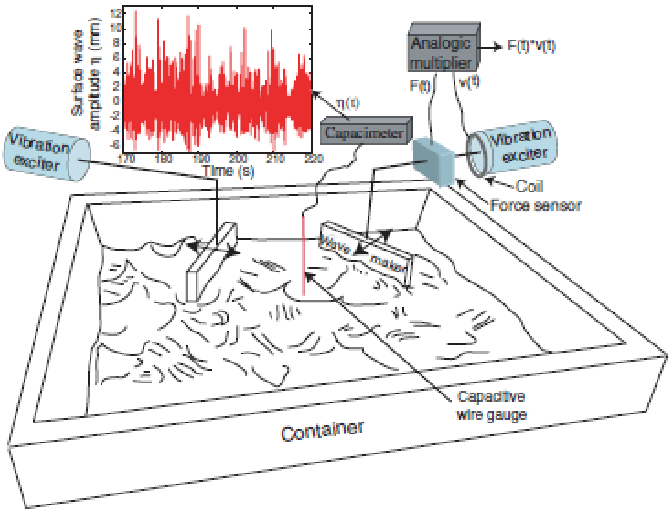



Waves created by paddles moved by electromagnetic shakers.
Forcing: band pass filtered noise
Random in amplitude and frequency
in the gravity waves range (below 10Hz)

Measurement of the injected power

$$\varepsilon_I = \frac{<I(t)>}{\rho S} = \frac{<F(t)v(t)>}{\rho S}$$

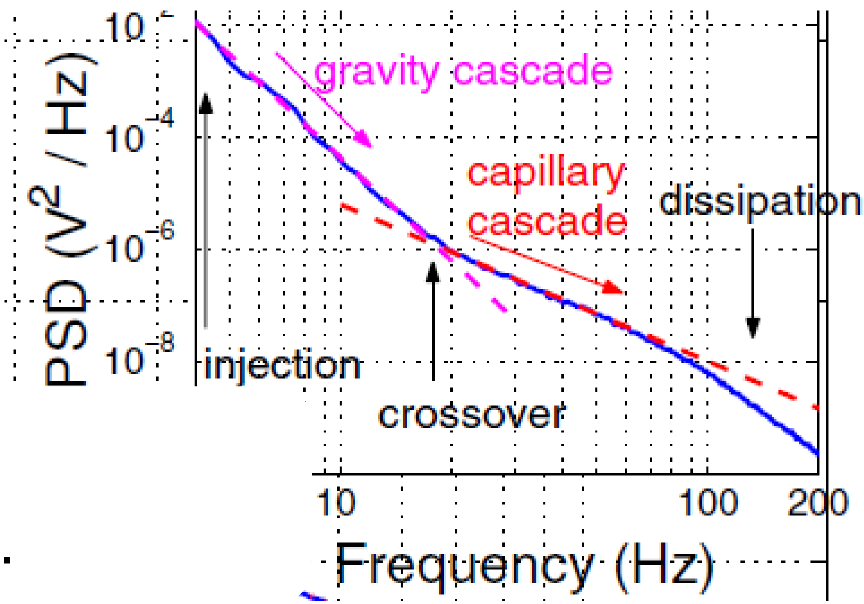
# Laboratory investigation of gravity-capillary wave turbulence




Falcon, Laroche, Fauve **PRL (98)** 2007

- Waves created by paddles moved by electromagnetic shakers.

- Observation of gravity cascade and capillary cascade.




- Local measurement :
capacitive probe
Relation between time and space deduced using dispersion relation.

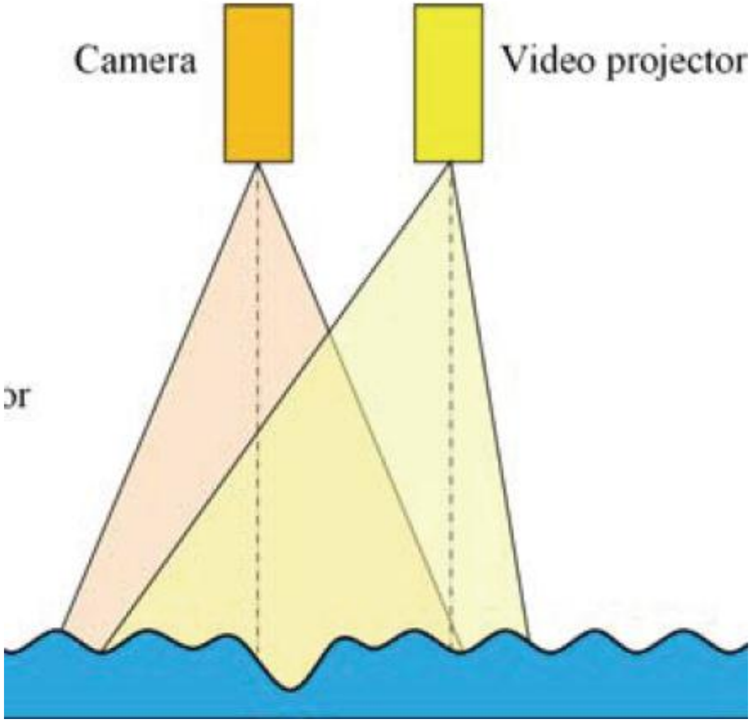
# Spatial temporal measurements of gravity-capillary wave turbulence

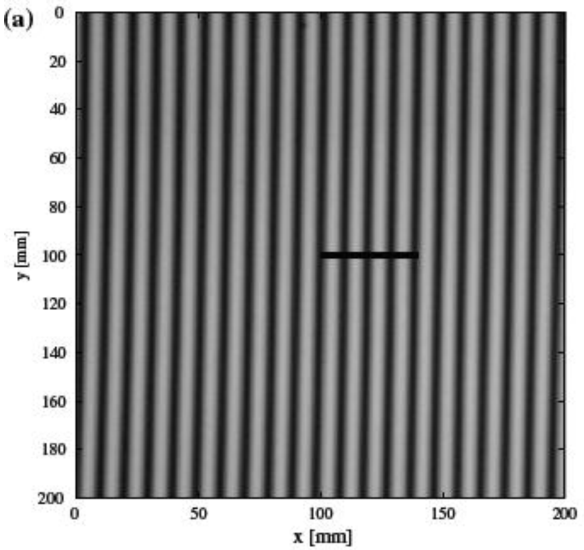
- Spatio-temporal measurement necessary to characterize Wave Turbulence and discuss its relevance.
  => Access to the complete statistics of waves modes as a function of time. Dispersion relation. Evaluation of non-linear transfers ....
- Fourier transform profilometry. Projection of a fringe pattern on an opaque surface. Deformation recorded with a fast camera to reconstruct surface topography as a function of time
- For water, white paint must be added to the liquid.
- Limitations :
  Spatial resolution limited by the quality of projection
  fringe distance ~ 4 mm.
  Does not work for too large slope
  Signal not detected for f >20 Hz

# Fourier transform profilometry

Global measurement of water waves by Fourier transform profilometry

Cobelli, Maurel, Pagneux, Petitjeans **Exp. In Fluids (46)** 2009



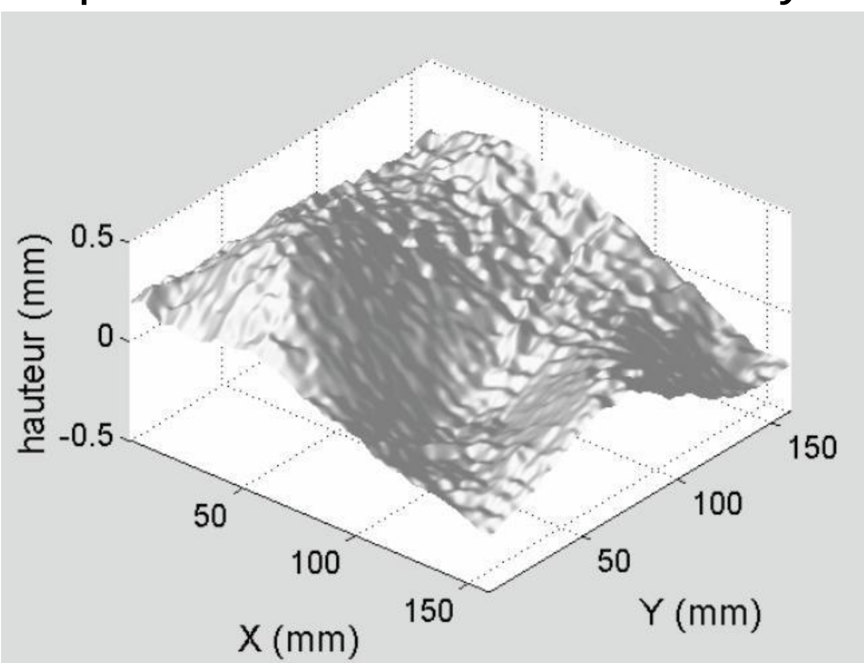



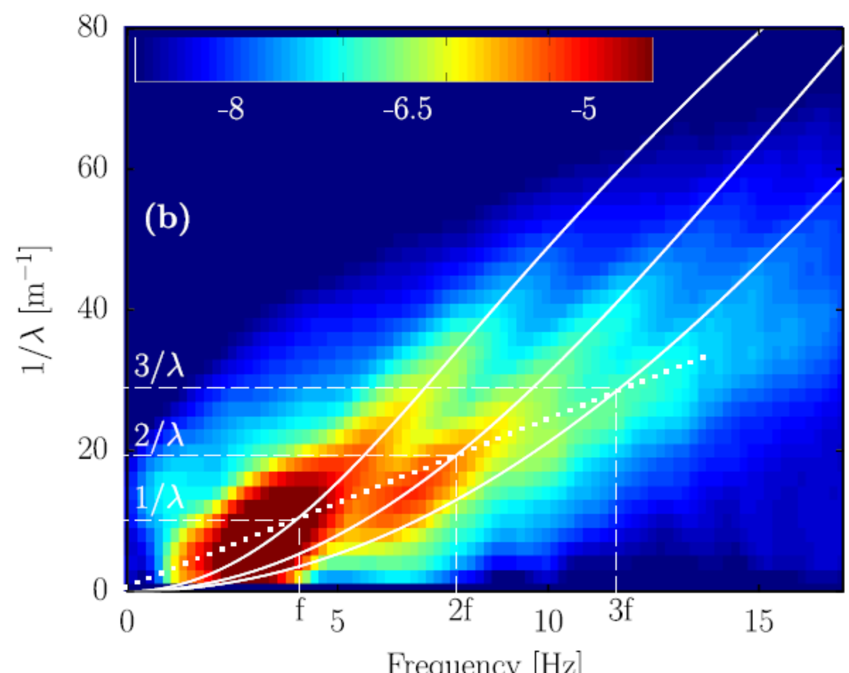
# Spatiotemporal spectrum and non-linear dispersion relation for gravity wave turbulence

- E. Herbert, N. Mordant and E. Falcon, Physical Review Letters 105, 144502 (2010) ***Nonlinear dispersion relation and spatial statistics of wave turbulence on the surface of a fluid***

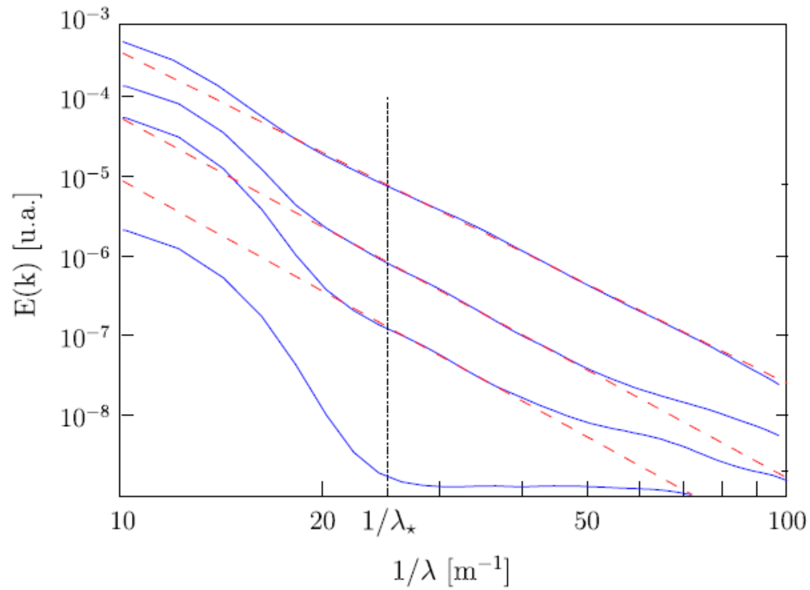
- Relation dispersion given by spatio-temporal spectrum of water elevation $S_h(\omega, k)$

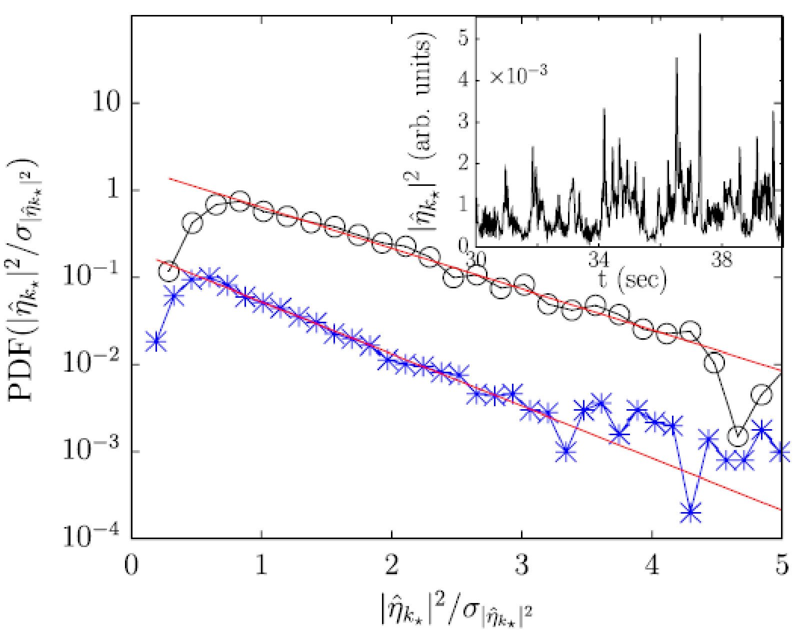
  At high amplitude non linear relation dispersion ( Bound waves) .
  Spatial measurement necessary



# Spatial spectrum and statistics of Fourier modes of gravity Wave turbulence.




- Spatial 1D spectrum (integration over the different directions).
  Power law of the cascade in disagreement with theory

- Recent paper
  Cobelli, Przadka, Petitjeans, Lagubeau, Pagneux, Maurel **PRL (107)** 2011
  Exponent in agreement with WT theory can be found for a narrower bandwidth of excitation. (No explanation given).

- Statistics of Fourier modes. Roughly exponential.

# II-Gravity wave turbulence in large basins

- Large Basins facilities at « l'école Centrale de Nantes. »

Collaboration Turbulon (2012-2015) : MSC, LPS ENS, SPHYNX, SPEC, CEA Saclay
LHEEA, Ecole Centrale Nantes

50*30*5 m

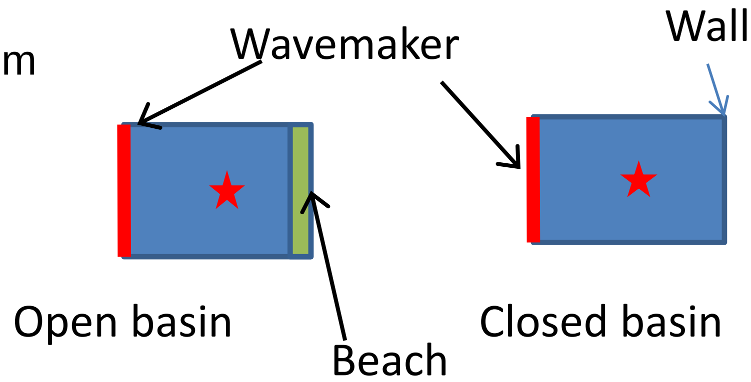
15*10*2m

- Finite size effects,
- Role of boundaries conditions ( purely reflective or absorbing)
- Evolution of slope of gravity wave turbulence cascade as a function of the forcing amplitude.

# Gravity wave turbulence in a large basin : role of boundary conditions

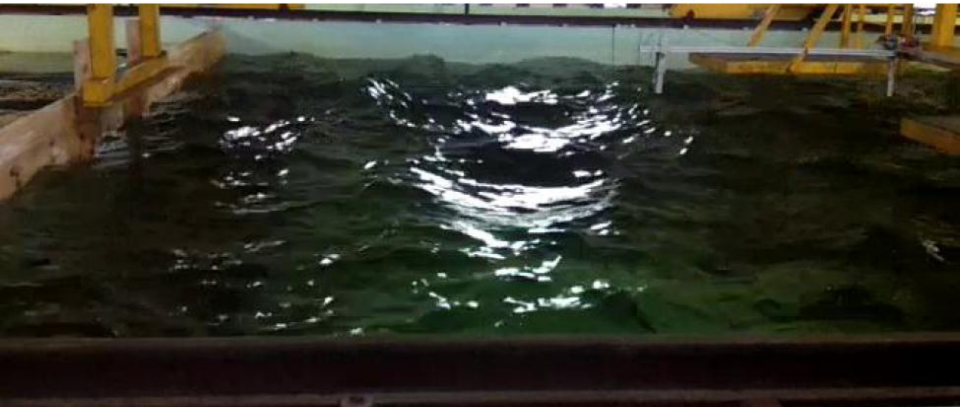
- Medium wave basin  Single flap Dimensions 15x10x2 m

Probes 8 m away from wavemaker
Beach or Wall opposite to the wavemaker
Capacitive Probe: local measurement of
Wave amplitude => Temporal Power Spectra
of wave elevation.




**Beach**

**Wall**



- Wave field structure very different between the two cases.

- **Beach** : Nonlinear propagation of a wave packet. Strong temporal correlation between 2 probes separated from 1 meter. Propagating breakings

- **Wall** : Multiple reflections. Weak correlation.
Splashes breakings.

# Gravity wave turbulence in a large basin : role of boundary conditions

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Probes 8 m away from wavemaker
Beach or Wall opposite to the wavemaker
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Wave amplitude. => Temporal Power Spectra
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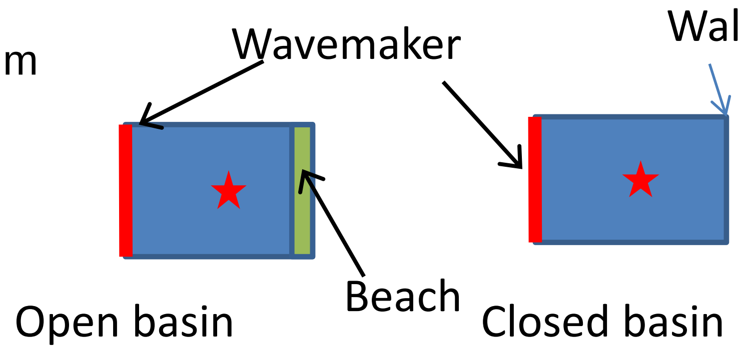
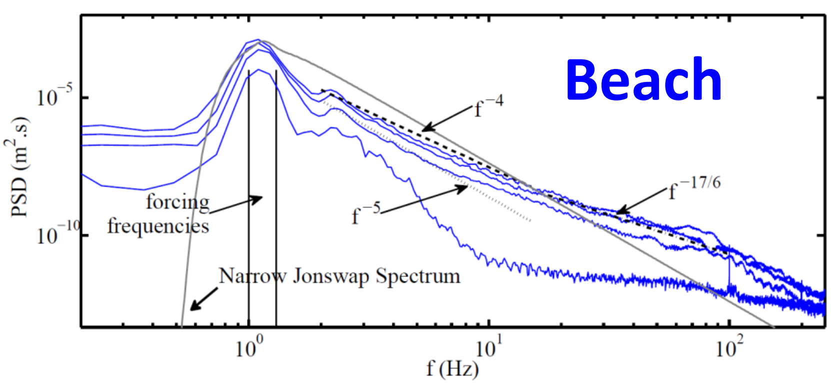
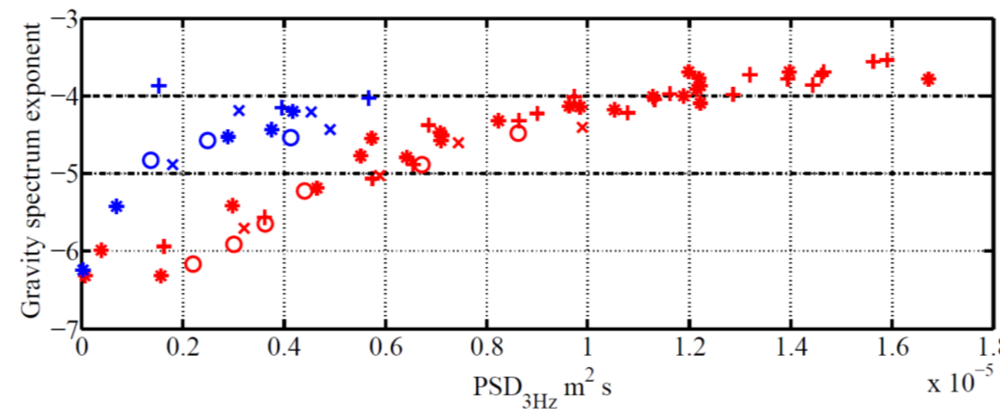
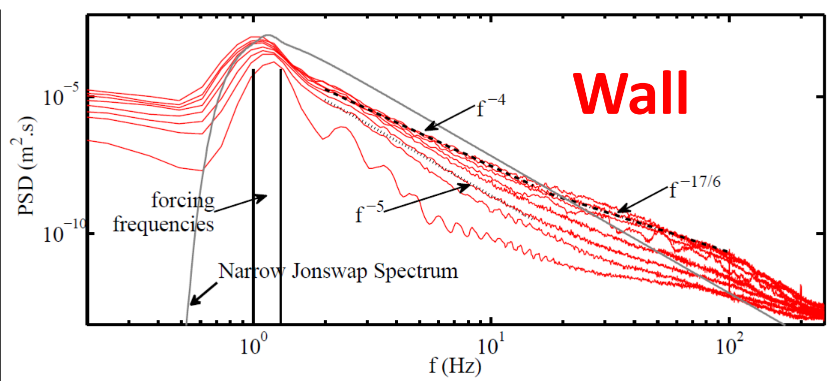



- Growth of gravity spectrum exponent with forcing amplitude.

Agreement with **Nazarenko et al. JFM 2010**
Saturation towards the K.Z value -4.

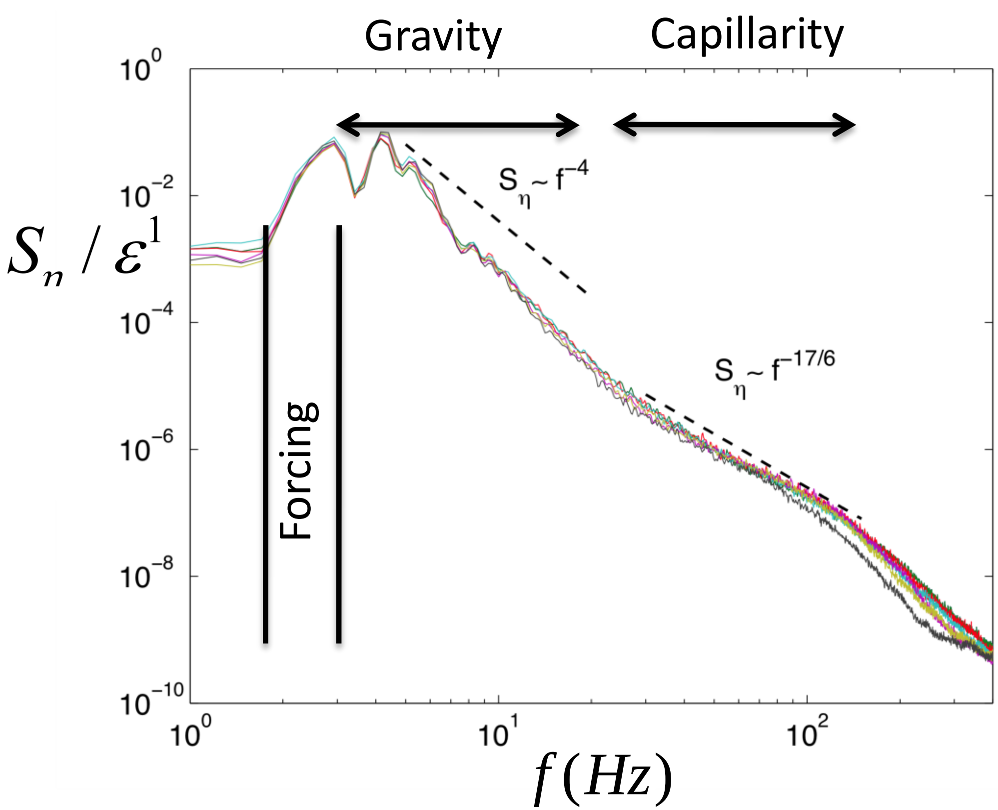
- Saturation is reached for a lower forcing with the **beach**.  More important role of coherent structures in this case ?

# III- Capillary wave turbulence: solving the question on the energy scaling.

3 wave interactions (Zakharov et al 1967)

Direct cascade: $S_\eta(f) \propto \varepsilon^{1/2} f^{-17/6}$

Capillary waves excited by gravity waves relevant case for oceanic waves.




Measurement of $S_\eta(f) \propto \varepsilon^1 f^{-17/6}$

- Agreement on the frequency scaling (Wright et al 1996, Holt & Trinh 1996, Henry et al 2000, Brazhnikov et al 2002, Kolmakov 2009, Falcon et al 2007, 2009, Xia et al 2010)

- **Disagreement on the energy flux scaling**

**Understanding the energy scaling ?**

# Simulation of capillary wave turbulence using solver Gerris

- Work of **Luc Deike** in collaboration with **Daniel Fuster** (IJLRDA/ UPMC)**.**

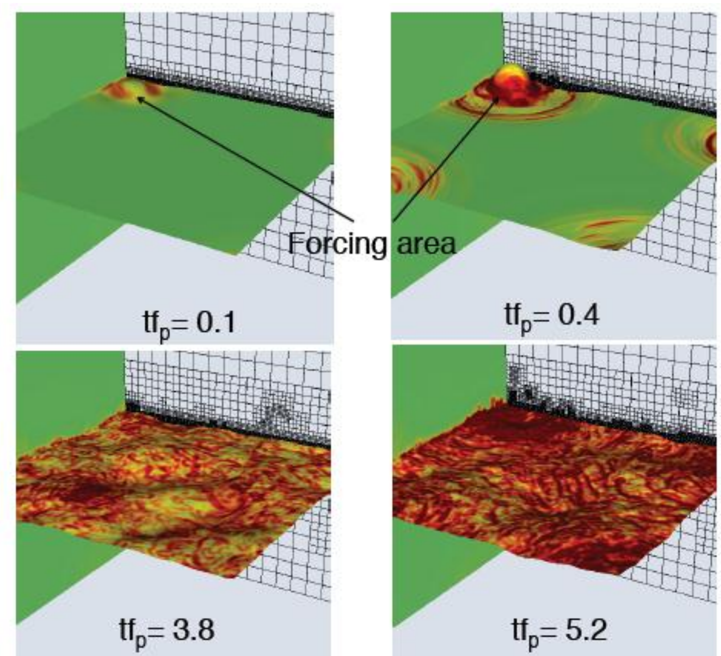
- Open source solver Gerris solves the two-phase Navier-Stokes equations including surface tension. Volume of Fluid method on the interface. Interface resolution 256*256. Forcing at low wave numbers in a small area.

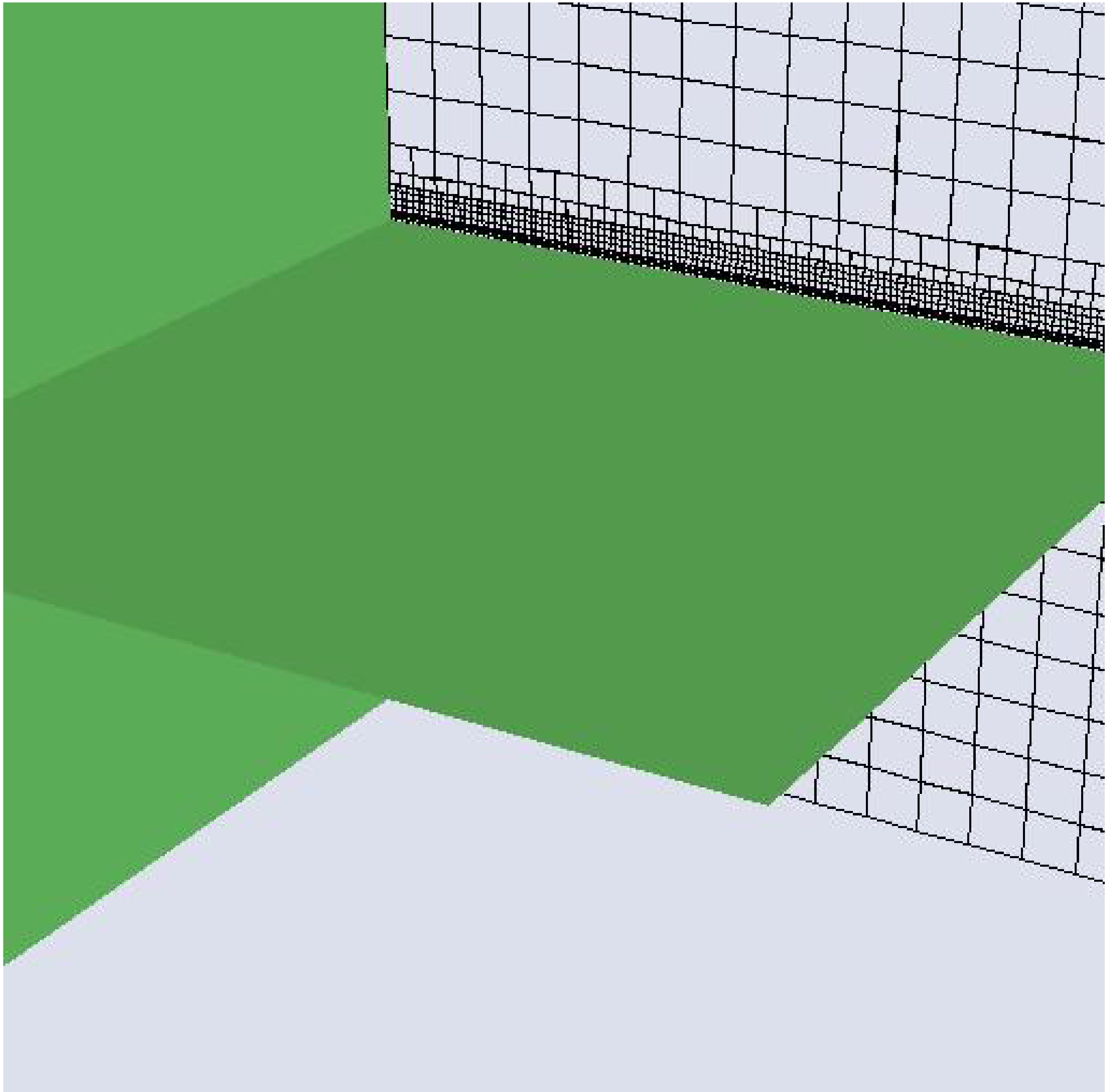
- Possibility to access to wave height and velocity field, modify the viscosity ...

- Study of pure capillary waves in a periodic domain. Demanding on computing time and power.

- After convergence to a stationary state. Wave turbulence regime.

- Energetic balance of Wave Turbulence could be possible to evaluate.



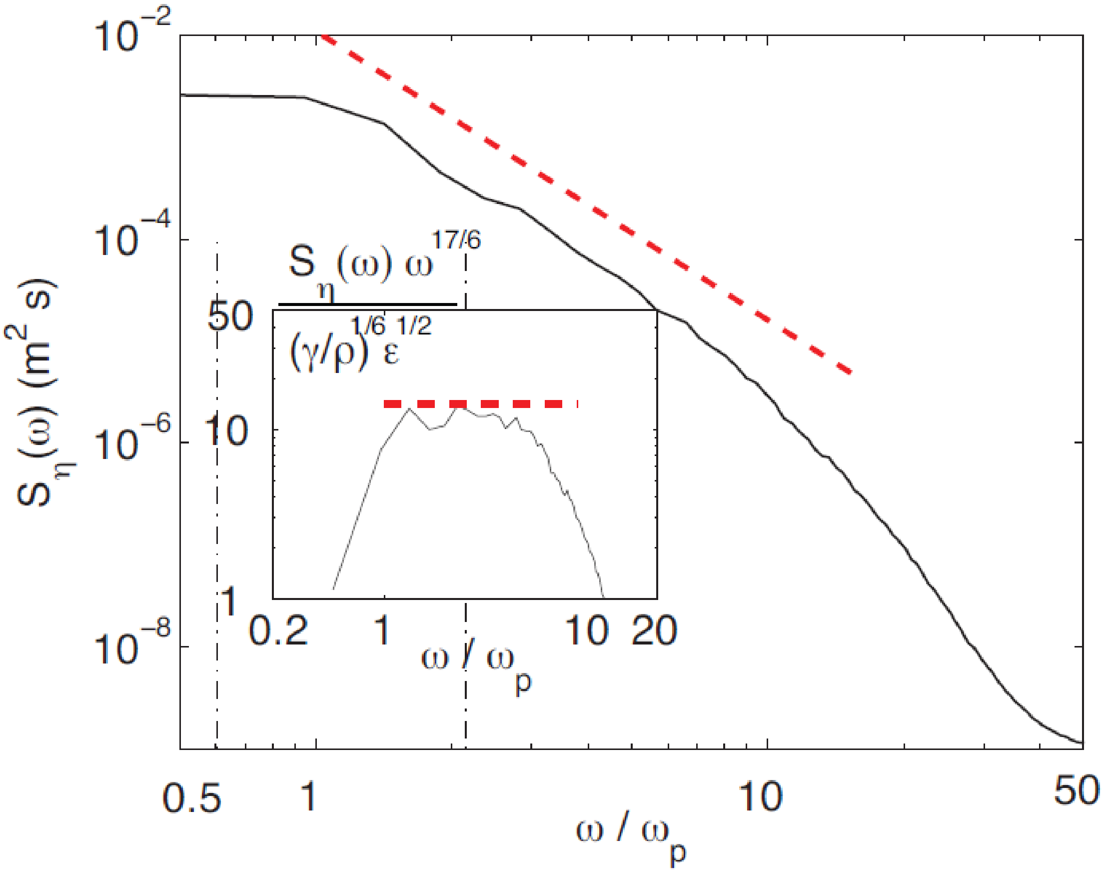


# Gerris Simulation of capillary wave turbulence

- Temporal spectra of wave elevation. Numeric and viscous dissipation explains the end of spectra

- Agreement with Wave Turbulence Theory for pure capillary waves.
Closer to needed hypotheses (weakly non linear).

- Confirmation of theory validity, when hypotheses are verified.




*Deike, Fuster, Berhanu, Falcon submitted to **PRL***

# Decay of capillary wave turbulence

● Luc Deike, Michael Berhanu, and Eric Falcon PRE (85) 2012. ***Decay of Capillary Wave Turbulence***

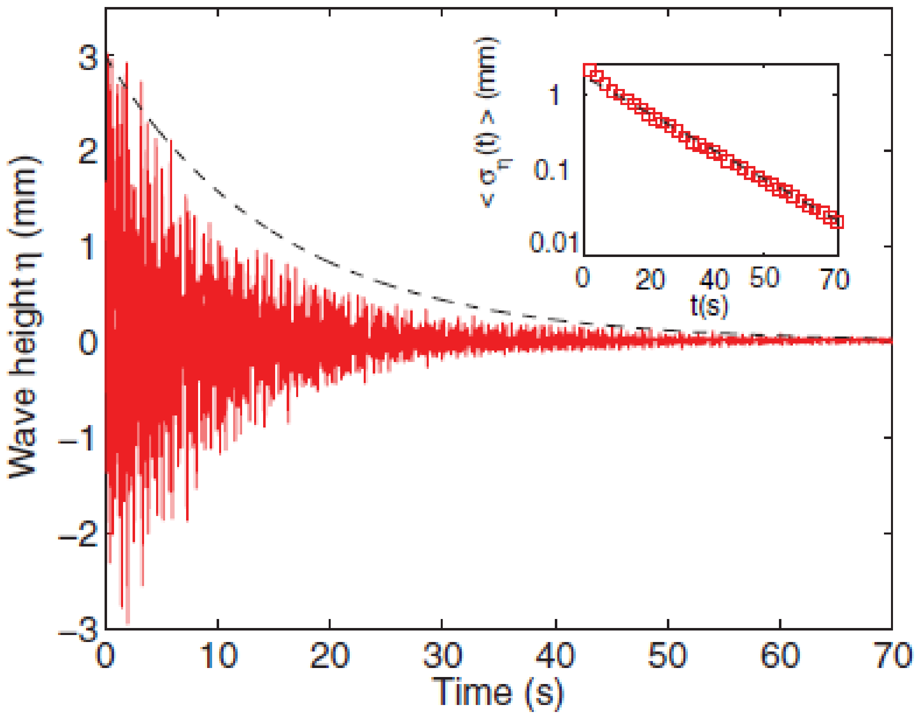
Measurement of decaying Wave turbulence with a capacitive probe (local).

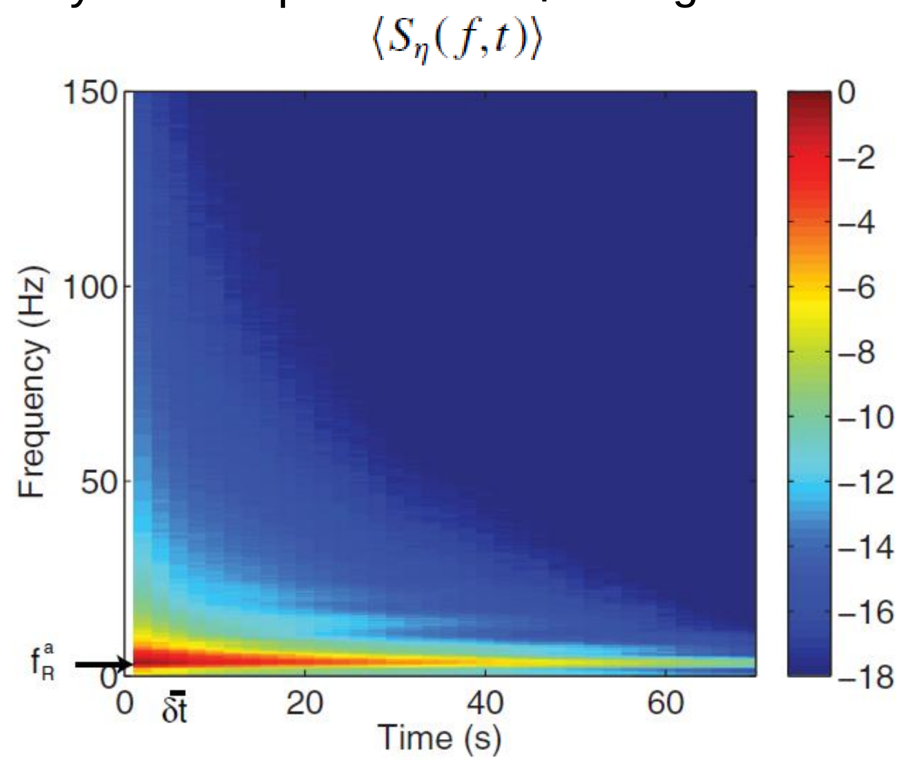
At t=0 s, Wavemaker is stopped

● Exponential decay of Wave height, (linear dissipation due to viscosity).

Not a power law as expected for a pumping due non linear transfers.

● Temporal spectrum at each time step is given by the computation of spectrogram.



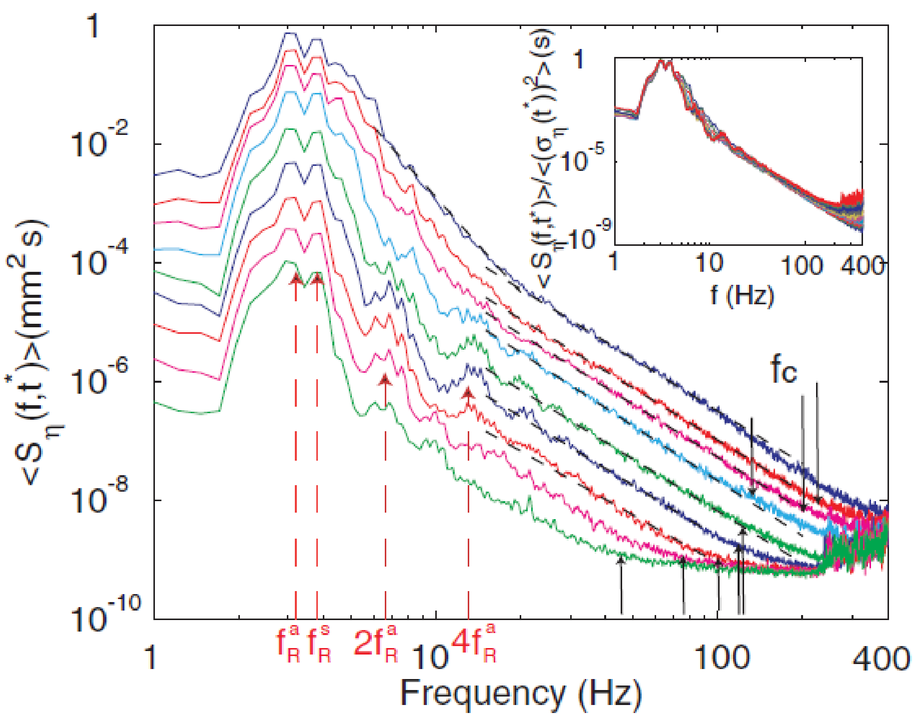
# Decay of capillary wave turbulence

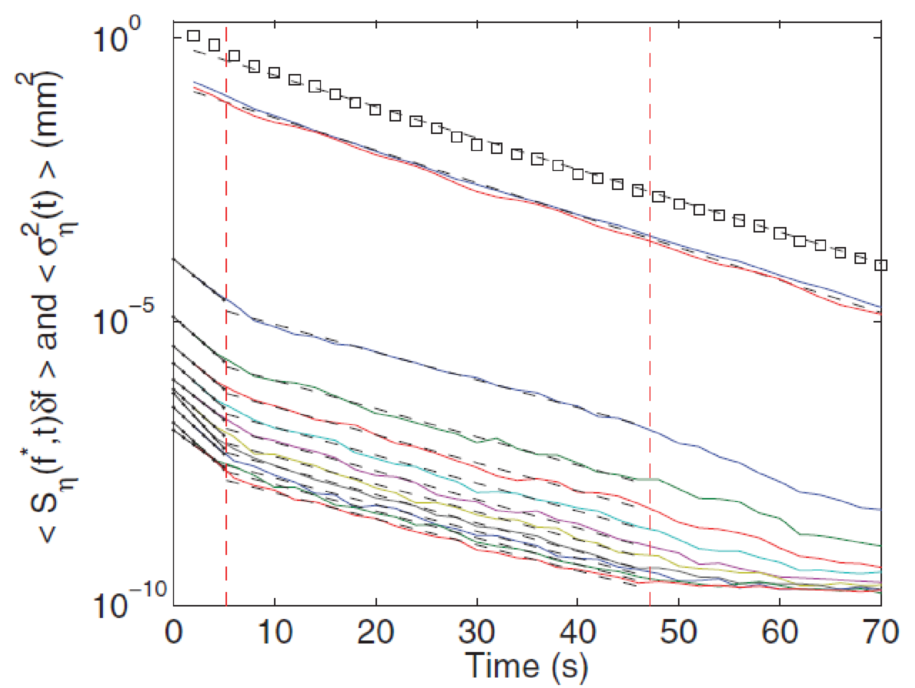
- Self-similar spectrum in the capillary regime. When a cascade of capillary waves can be defined, the exponent is close to the predicted one -17/6.

- Exponential decay of each Fourier mode with the same rate, which is given by the viscous damping of longest wave mode.

- Only a small fraction of wave energy participates to the cascade. During the decay, longest waves modes play the role of an energy source which sustains nonlinear interactions to keep capillary waves in a wave turbulent state.

- Necessity to measure energy flux inside the cascade.



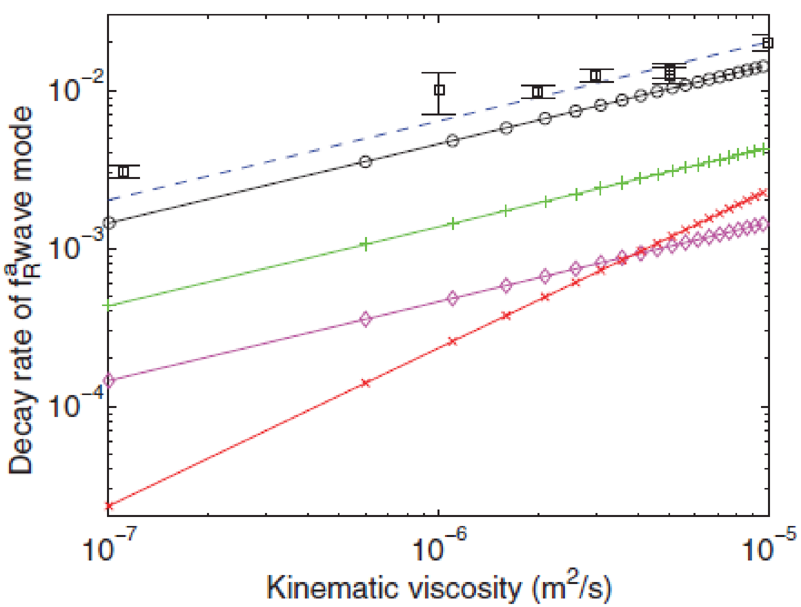
# Surface waves dissipation

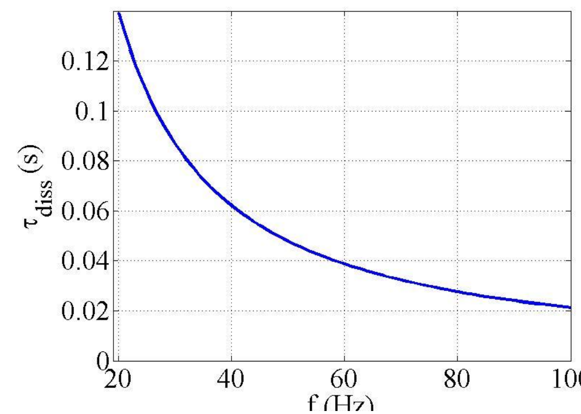
● Study of decaying capillary wave turbulence revealed important role of viscous dissipation at laboratory scales. Experiments with different liquid of different viscosities.

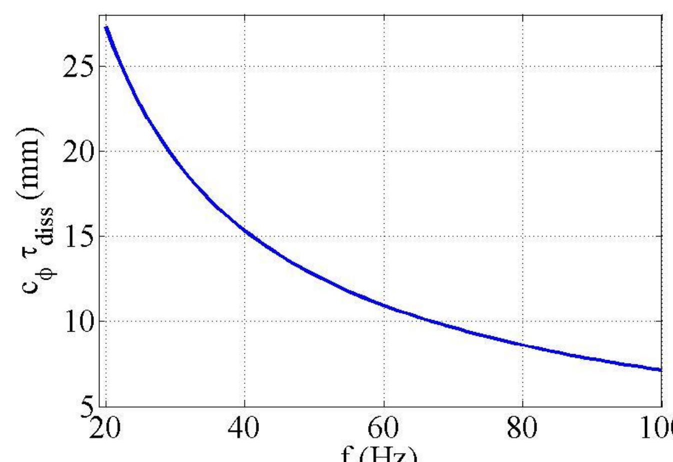
● Dissipation time is given by $\mathcal{T}_S^{-1} \sim (\nu\omega)^{1/2} k$ (saturated free surface) either $\mathcal{T}_\nu^{-1} \sim \nu k^2$

● Consequence: strong dissipation in the capillary scales. Difficulty to obtain homogeneous field of waves.
Scale separation ? Interaction time >> Tdiss ?

● Open question : How a self similar cascade for capillary waves, can be obtained, if viscous dissipation is dominant at the forcing scales ?



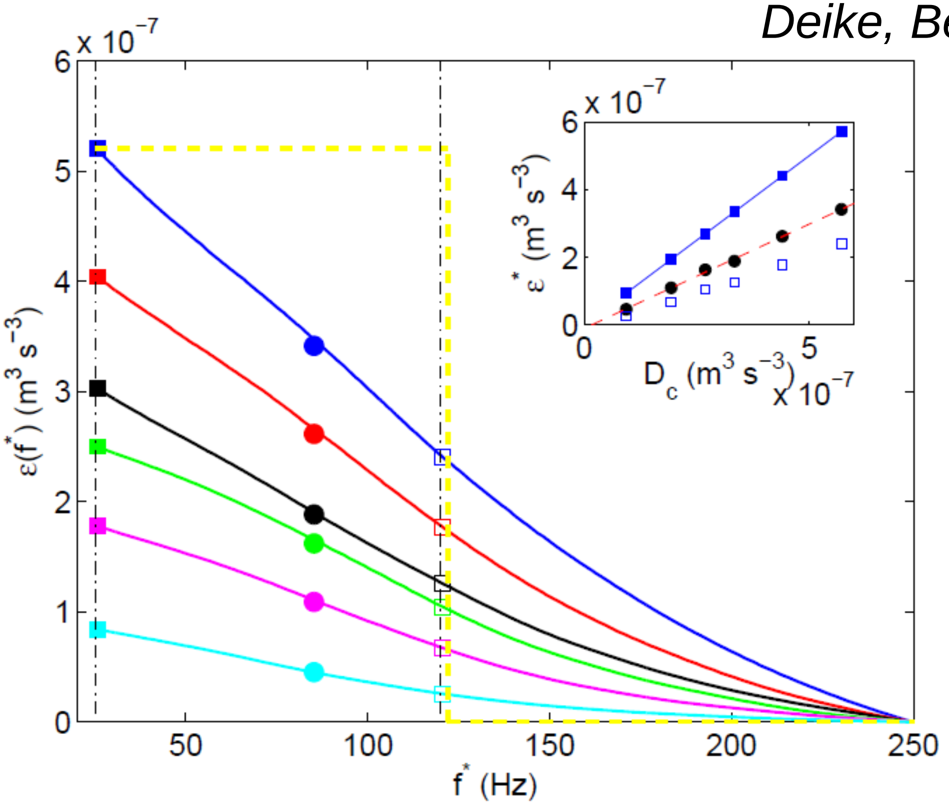



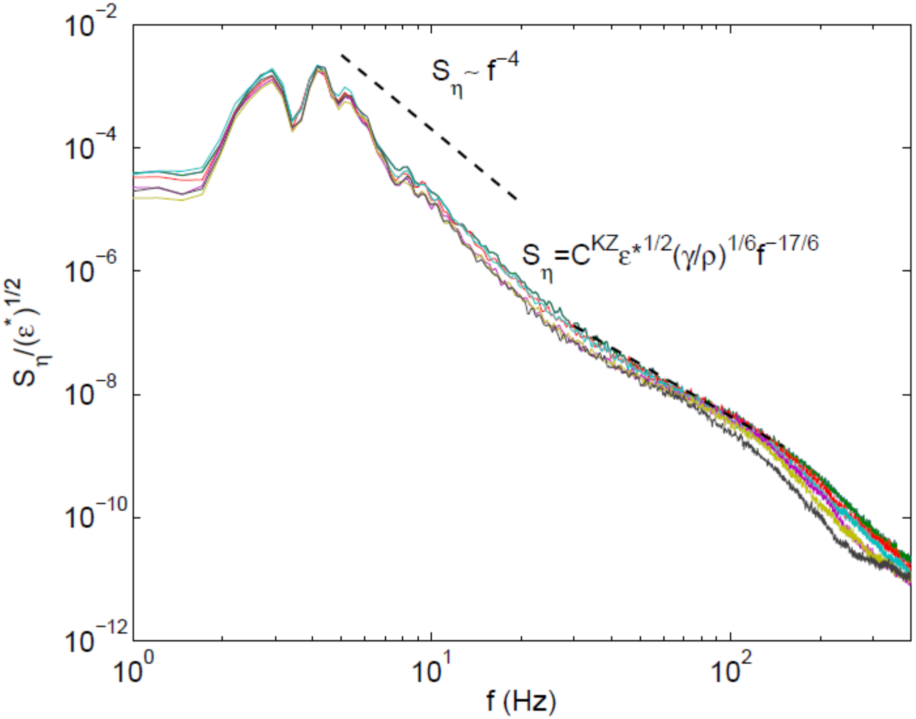
# Energy flux measurement from the dissipated energy

$$D_c = \int_{f_{gc}}^{f_s/2} \frac{\gamma}{\rho} k^2 S_\eta(f)\Gamma(f)df$$

$$\epsilon(f^*) = \int_{f^*}^{f_s/2} D_\eta^c(f)df$$

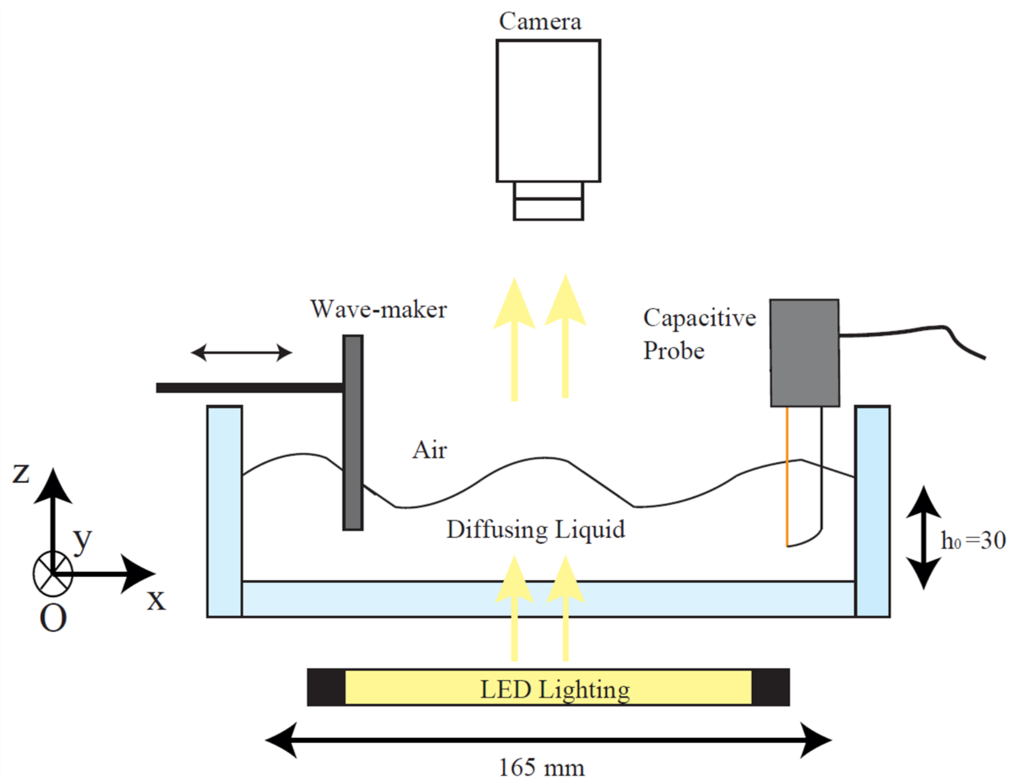
● $\Gamma$ Estimation of viscous dissipative time at the frequency f.
● Energy flux $\varepsilon$ from energy balance.
$\varepsilon$ non conserved through the scales !
● Reasonable rescaling by average $\varepsilon$ at the power ½ as expected theoretically.
*Deike, Berhanu & Falcon PRE **89** (2014)*



# IV-Spatiotemporal measurement of capillary wave turbulence.

- Local temporal measurement not sufficient to describe completely mechanisms at play. Measurement of the 2D shape of the liquid surface as a function of time.

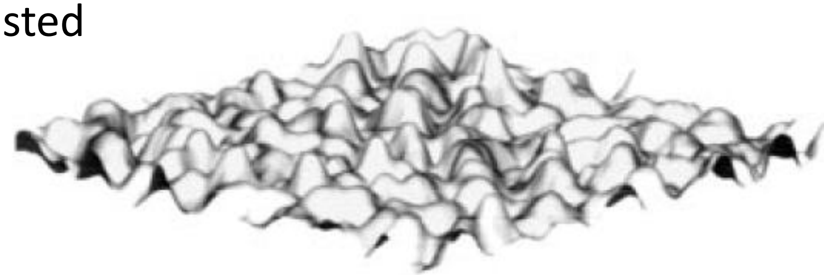


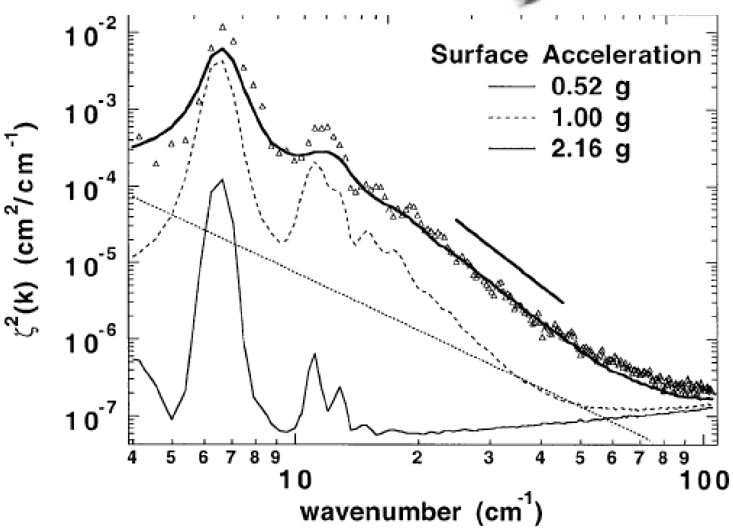

- Michael Berhanu & Eric Falcon PRE 89, 033003 (2013)

***Space-time-resolved capillary wave turbulence***

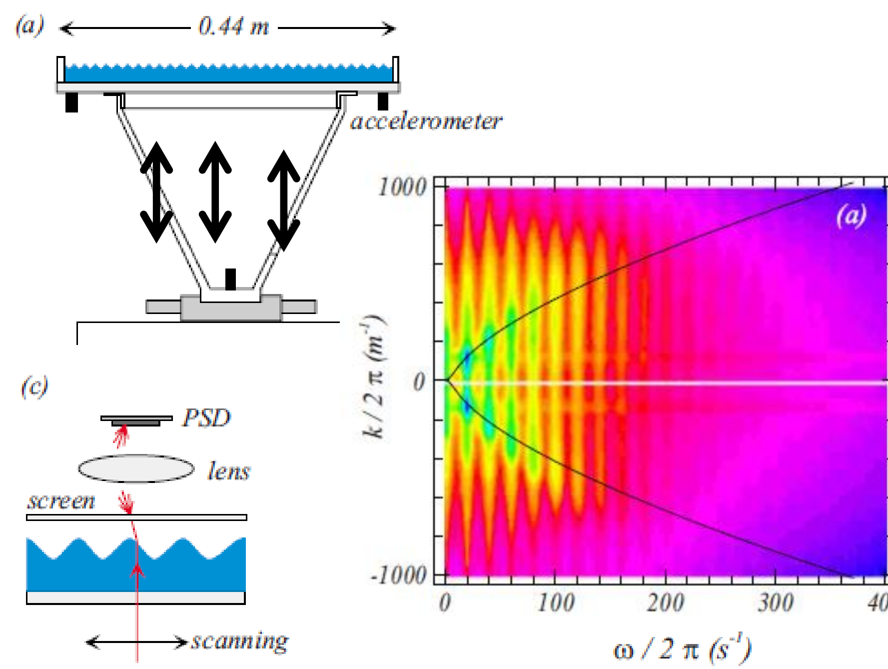
# Spatial investigation of capillary wave turbulence for Faraday excitation.

- «Diffusing Light Photography of Fully Developed Isotropic Ripple Turbulence» Wright, Budakian, Putterman **PRL** (**76**) 1996
- Snapshot of turbulent surface. Parametric excitation at 50 Hz. Spatial spectrum: found slope -4.2 in agreement with theoretical prediction -3.75. Dispersion relation not tested






- «Turbulent parametric surface waves » Snouck, Vestra, van de Water, **Physics of Fluids** (**21**) 2009.
- Spatiotemporal measurement (1D) of height gradient. Forcing at 40 Hz. Approximate dispersion relation. Slopes of spectra in disagreement with theory. Note: Fluid is silicon oil 3 times more viscous than water and surface tension significantly lower.

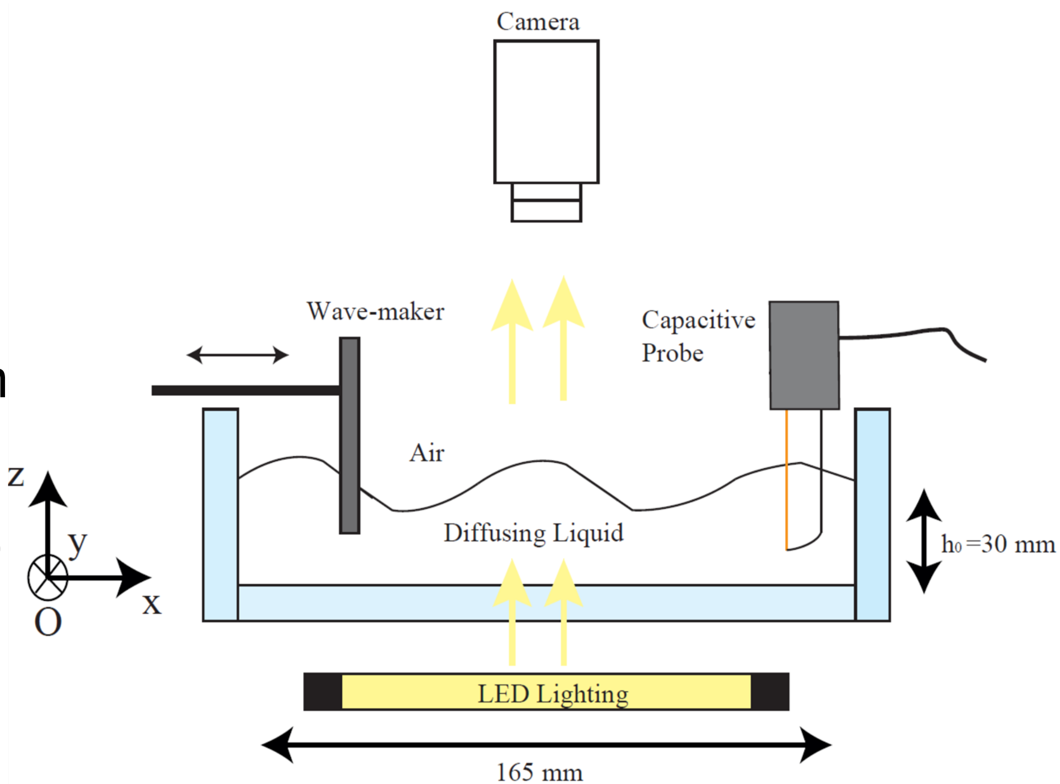


# Experimental device

● Square Plexiglass transparent tank 16.5*16.5 cm
● Diffusing liquid : 1 L water + 8 mL of Intralipid 20 % (suspension of spheres of ~ 1 µm

● Fluid height at rest 30 mm.
Lighting square of LED (Phlox)

● Local measurement with a capacitive probe




● Waves excitation : one paddle moved with electromagnetic shaker.

● Fast camera PCO Edge or Phantom V9

Observation window ~ 1Mpixel 9.6cm*8.9cm

Frame rate : 200 Hz (PCO Edge)
1000 Hz (Phantom V9)

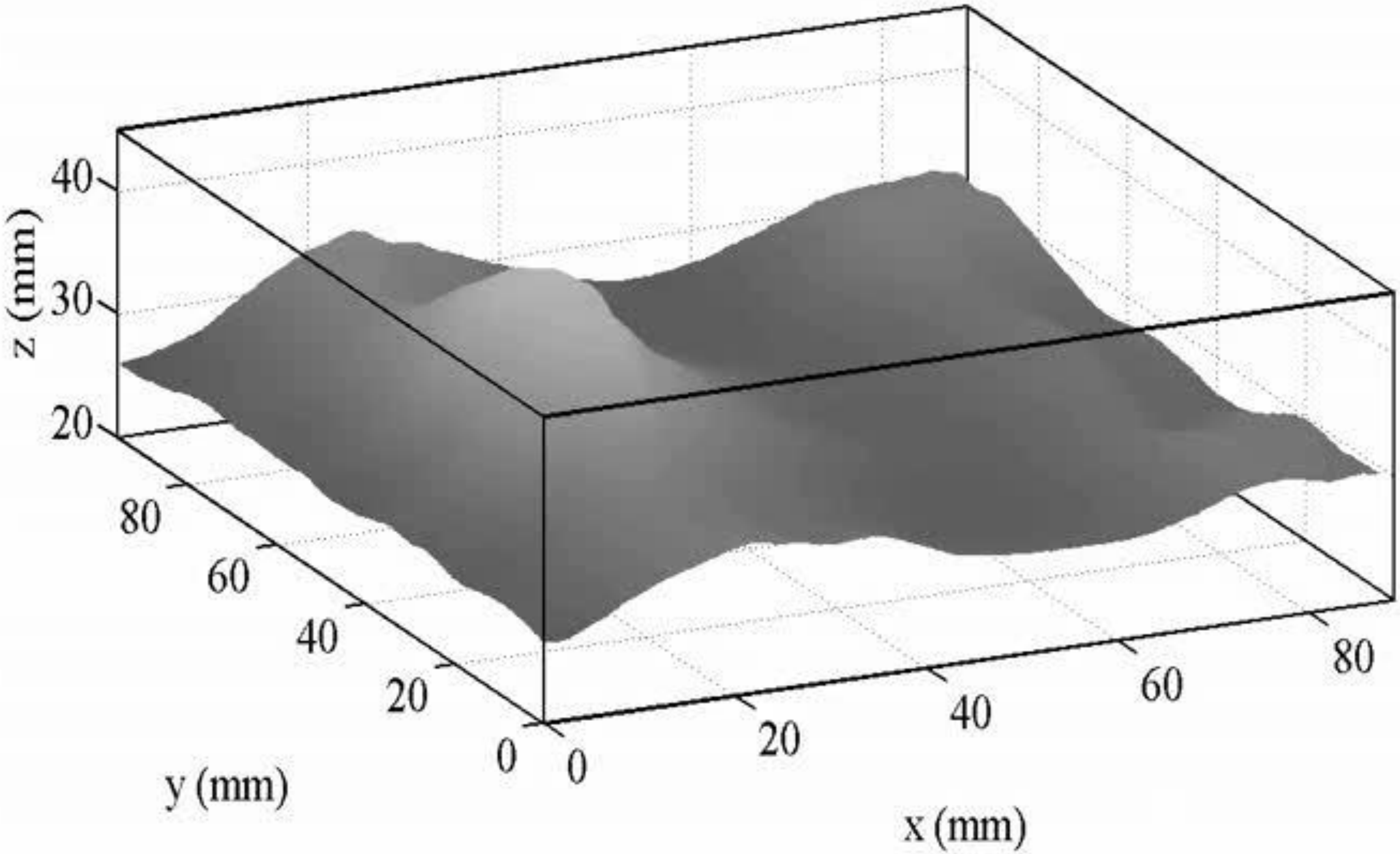
Capillary length ~ 2 mm

Transition between gravity and capillary waves.
$\lambda_c$ ~ 1.7 cm
$f_c$ ~ 15 Hz

# Experimental setup

# Random forcing
# High amplitude

- Forcing: White noise band pass filtered between 2 frequencies 4 Hz and 6 Hz. Container : Square tank $\sigma_h = 3.6\,\mathrm{mm}$ and $\sigma_s = 0.34$ Capillary WT excited through gravity waves.

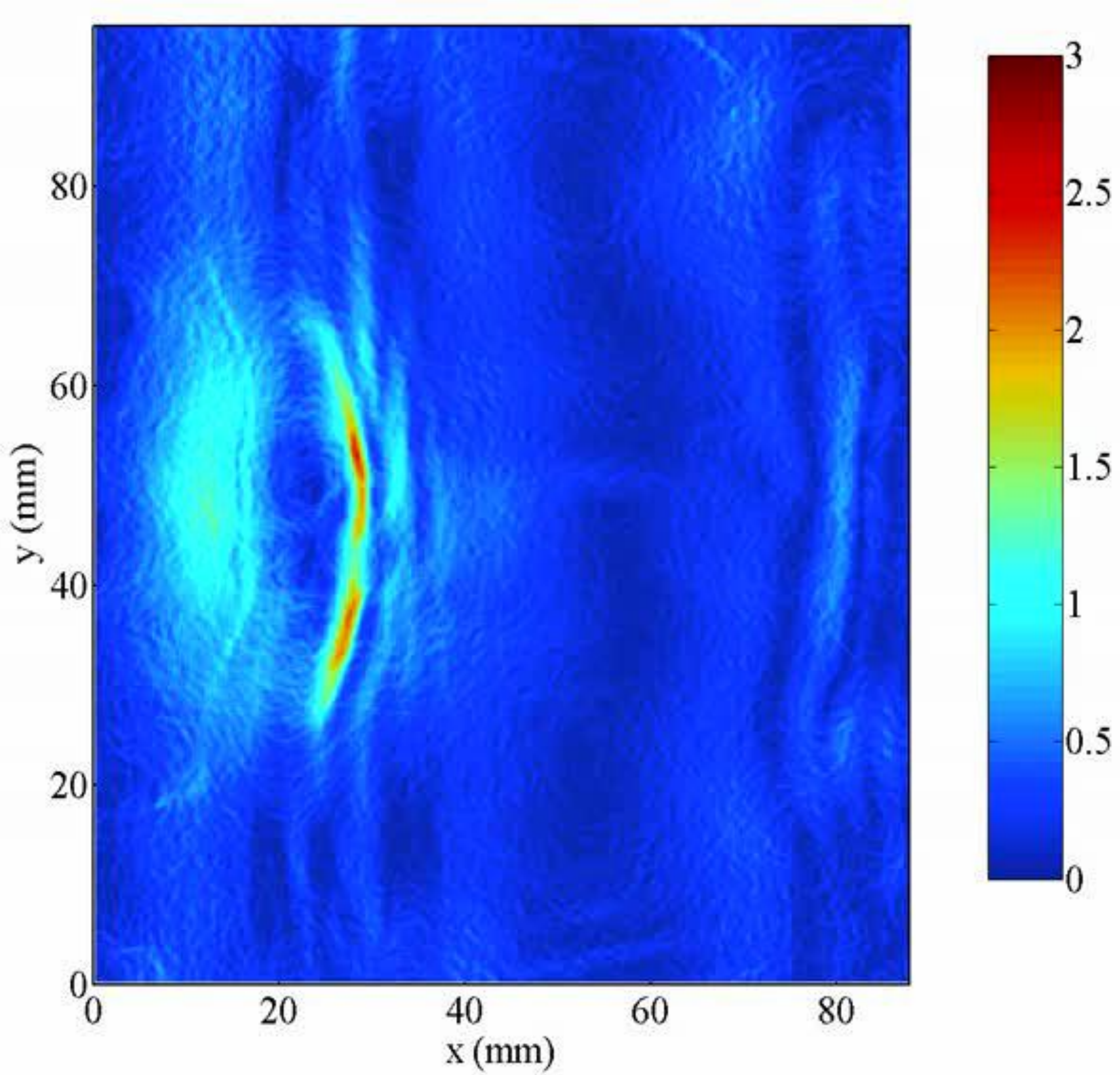


# Random forcing. Higher amplitude/ Steepness

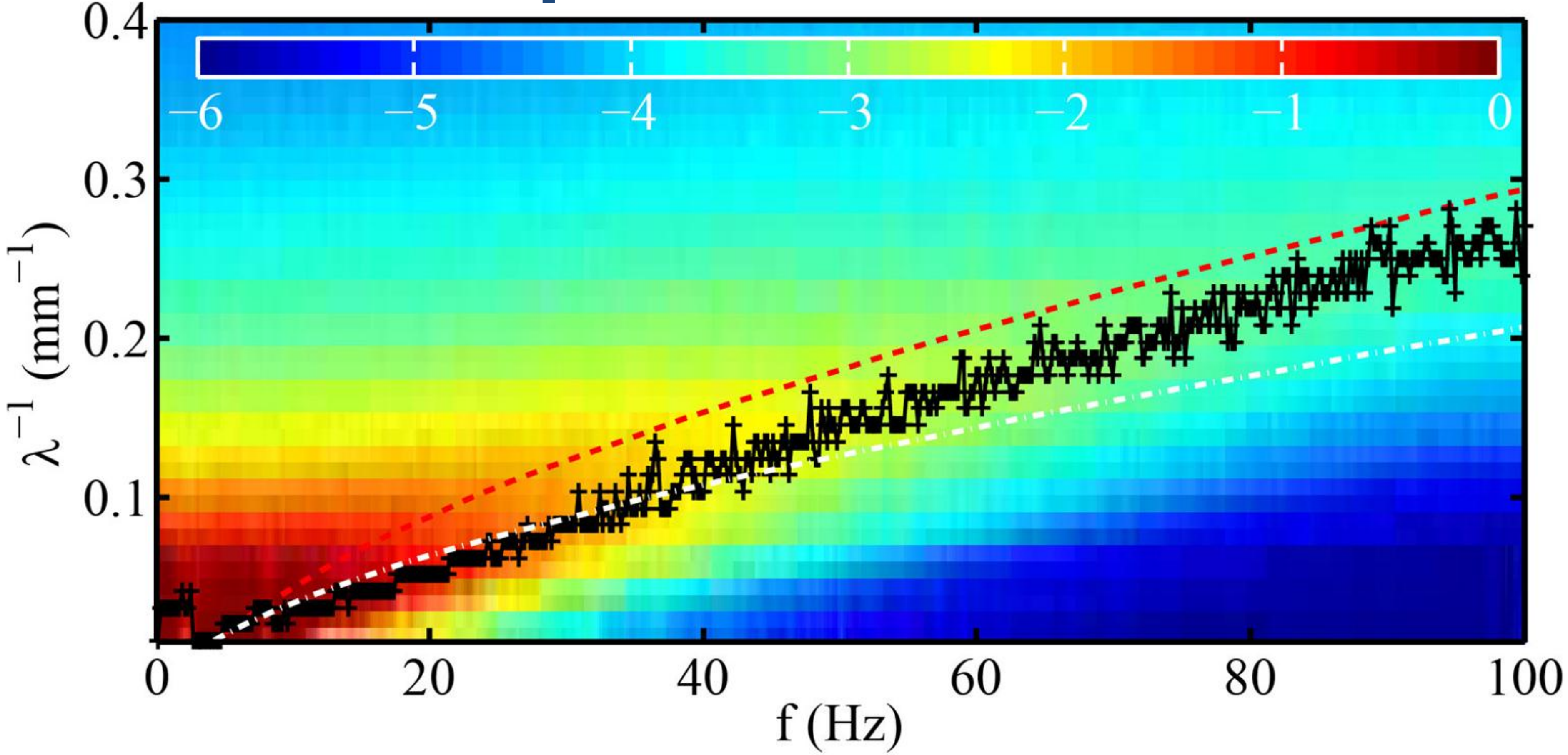
- Forcing: White noise band pass filtered between 2 frequencies 4 Hz and 6 Hz. Container : Square tank Capillary WT excited through gravity waves.

Movie of wave field steepness.

$$s(x,y) = \|\nabla h(x,y)\|$$



# Dispersion relation




- Experimental dispersion relation (**black**) departs from linear one (red).

$$\omega^2 = \left[gk + (\gamma/\rho)k^3\right] \tanh(kh_0)$$

- Non-linear shift estimated with amplitude of wave (white curve).

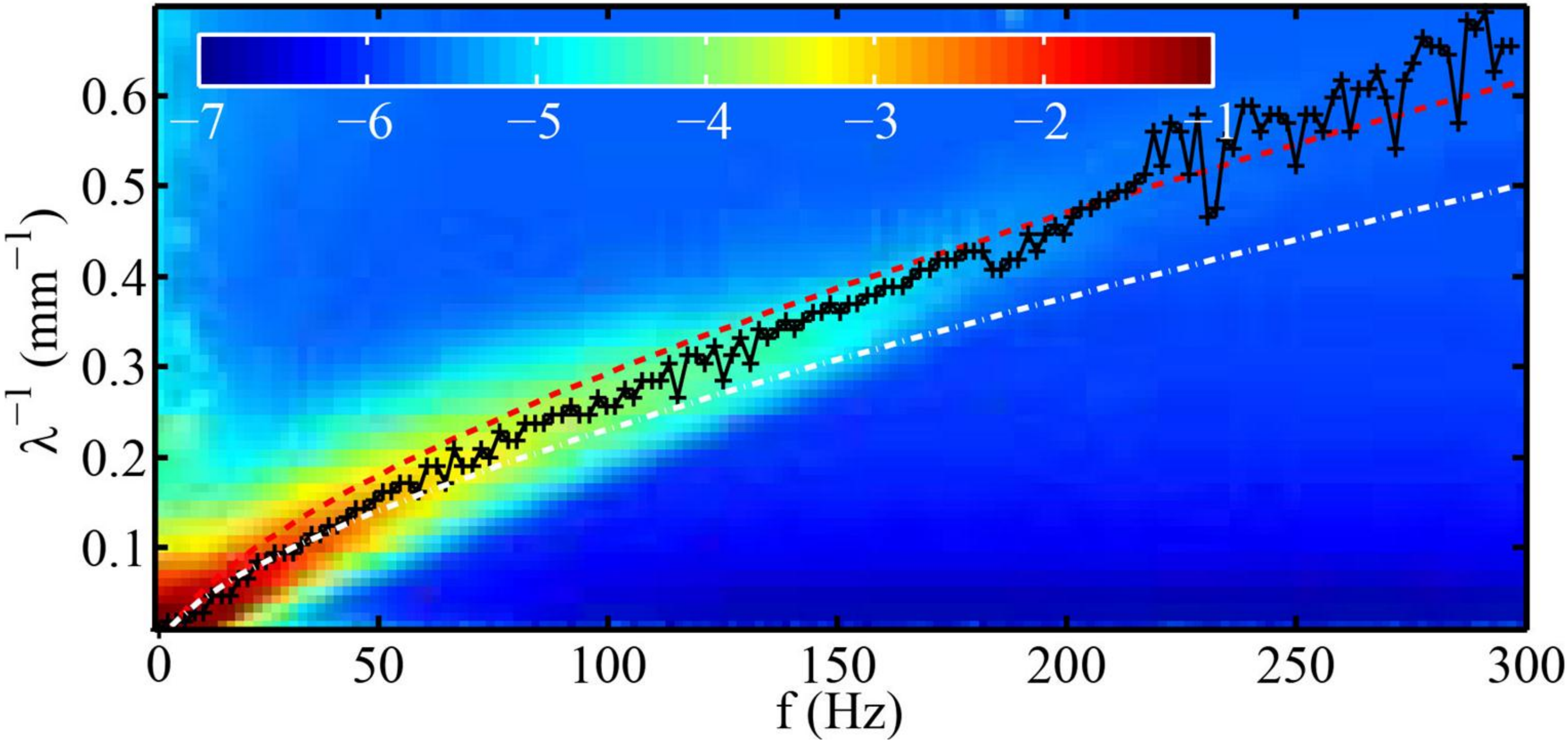
$$\omega^2 = \left(gk\left[1+(ak)^2\right] + \frac{\gamma}{\rho}k^3\left[1+(\frac{ak}{4})^2\right]^{-1/4}\right)\tanh(kh_0)$$

# Experimental Dispersion relation

- **Spatio-temporal Spectrum of water elevation.**

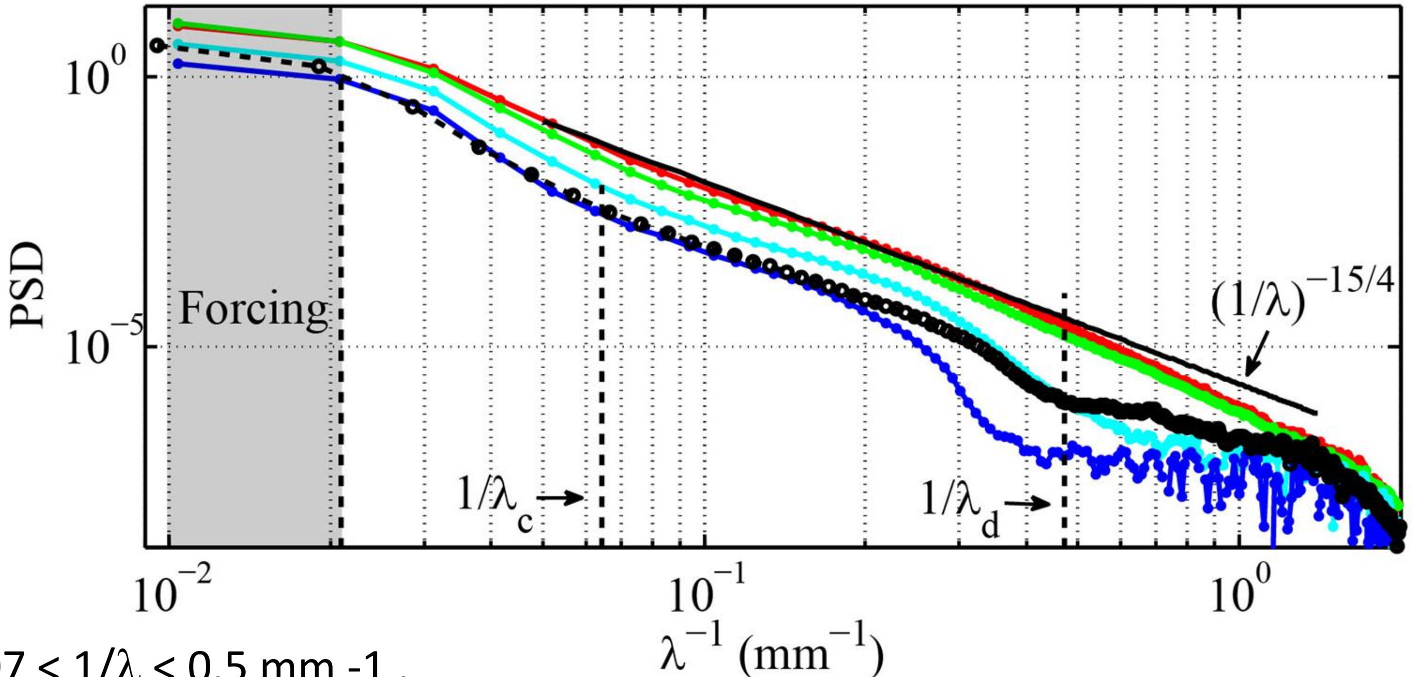
Fourier transform in space and time => Spatiotemporal Spectrum $\mathbf{S_h(\omega, k_x, k_y)}$.
Integration over $\mathbf{k_x}$ et $\mathbf{k_y}$ to obtain spectrum as a function of $||\mathbf{k}||$. ( $1/\lambda = k/(2\pi)$ ).
Color scale $\log_{10}(S_h(\omega, k))$ ( $f = \omega/(2\pi)$ )

- Measurement with a high frame rate (1000 Hz):



# Spatial Spectrum

● Integration over frequencies of **S**h(**ω, k**), to obtain spatial spectrum as a function of ||**k**||



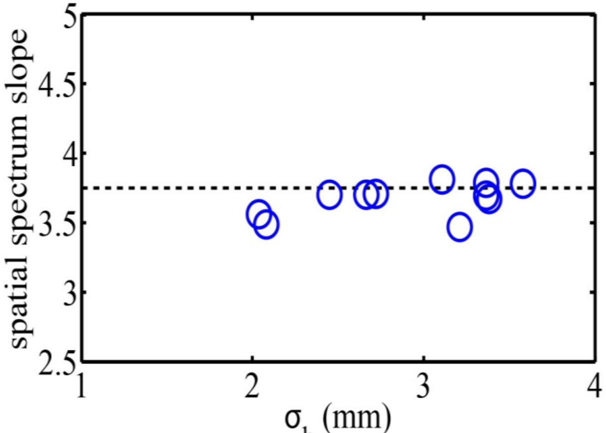

● For 0.07 < 1/λ < 0.5 mm -1 ,

**Power law**, turbulent cascade, in the capillary range

Exponent in agreement with the theoretical prediction for capillary WT.

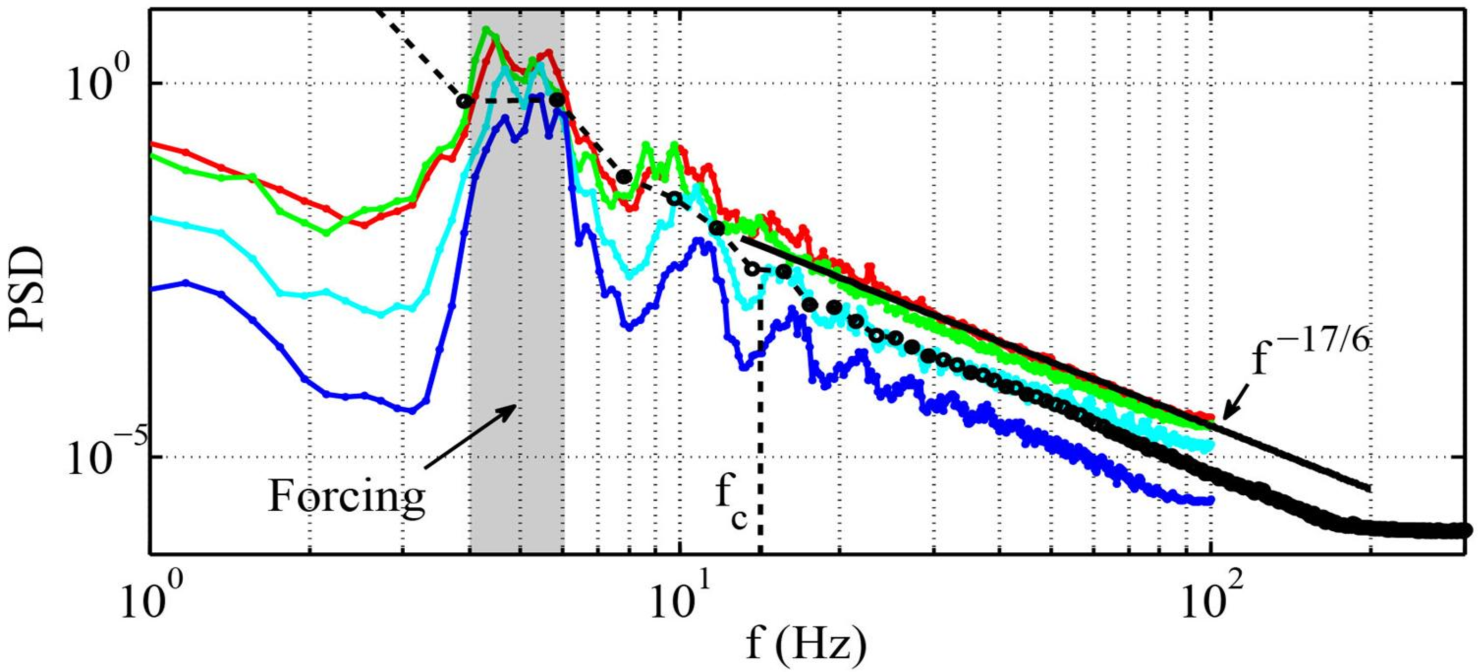
$$S_h^{theo}(k) \sim k^{-15/4}$$

● Increase of forcing amplitude increases size of inertial ranges



# Temporal Spectrum

- Integration over spatial scales of $S_h(\omega, k)$ to obtain temporal spectrum vs $\omega$

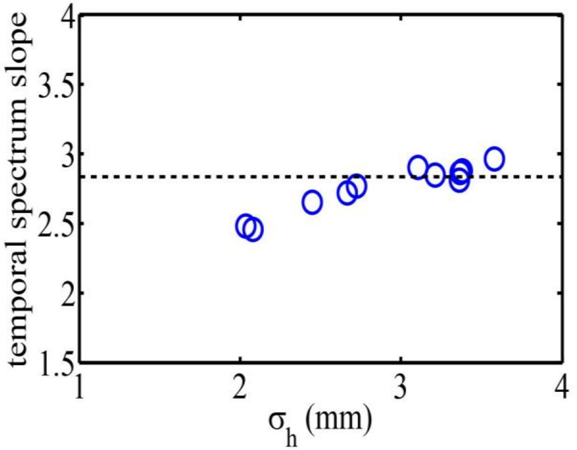



- **Power law** between 20 < f < 200 Hz

  (Black curve frame rate=1khz Otherwise frame rate = 200 Hz.)

- In agreement with WT theory predictions

$$S_h^{theo}(\omega) \sim \omega^{-17/6}$$



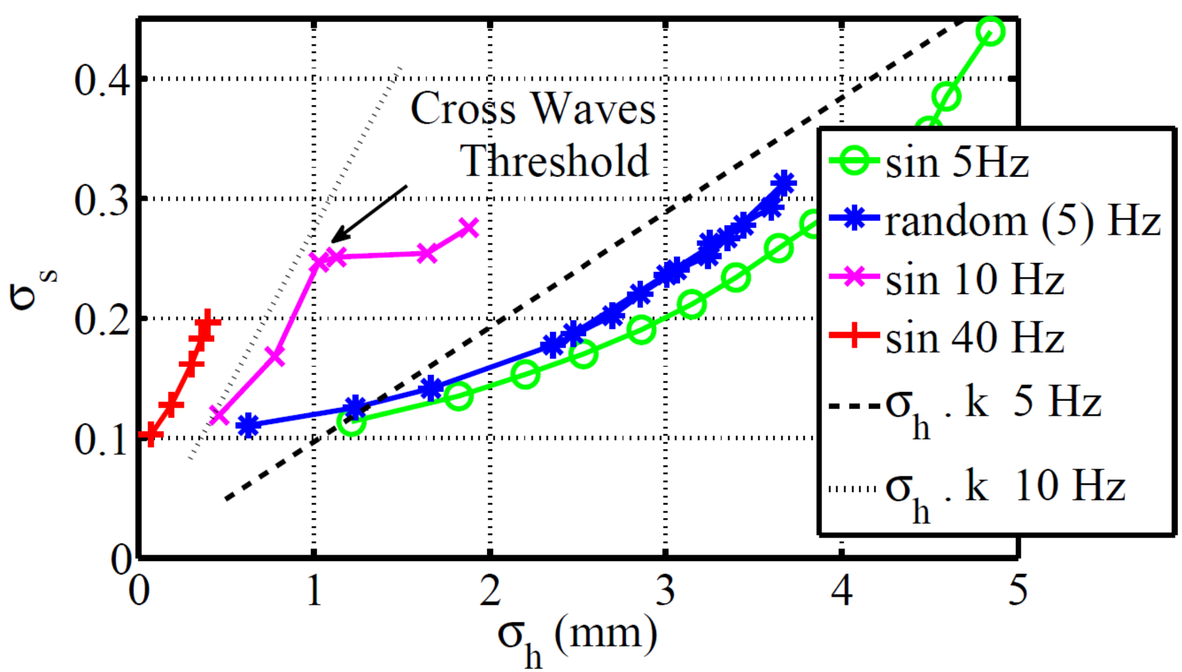

- Memory of forcing disappears when forcing amplitude increases

# Partial conclusion

● "*Diffusing Light Photography*" is used for the first time to characterize spatially and temporally, **Capillary Wave Turbulence**.

● Measurements are in good agreement with theoretical predictions validity of hypotheses used in the analytical derivation (homogeneity, isotropy, weak non linearity).

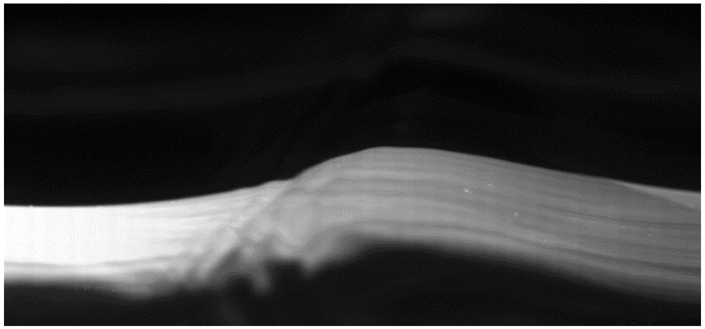
● Range of Wave steepness in Wave Turbulence Regime : from 0.1 to 0.3

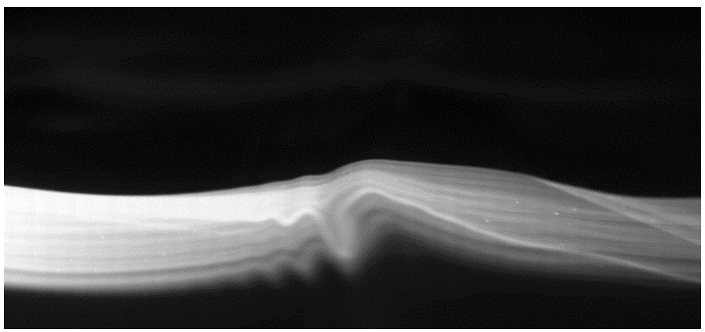


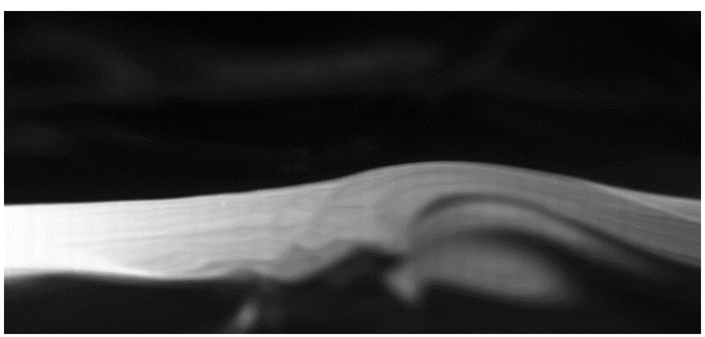

Wave steepness,

non linear parameter expected small.

# Complementary measurement
# 1D spatio-temporal interface acquisition.



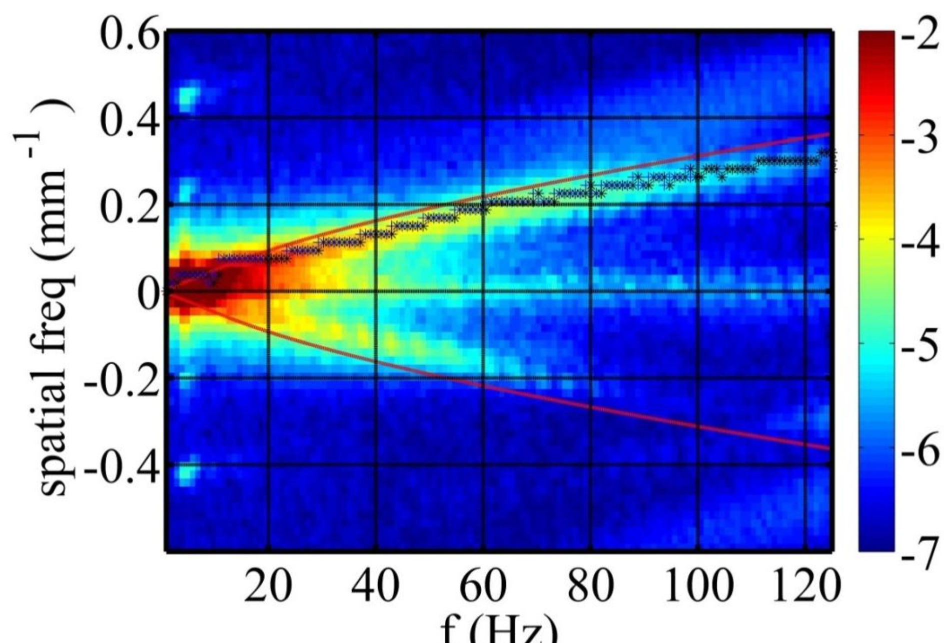




● Complementary measurement to validate previous results, using a laser sheet and a dye Rhodamine 6G.

● Image acquisition with Phantom Camera V9.1, binarisation and interface detection.

● For similar forcing (Random around 5Hz), same spatio-temporal spectra. Frame rate 250 Hz.

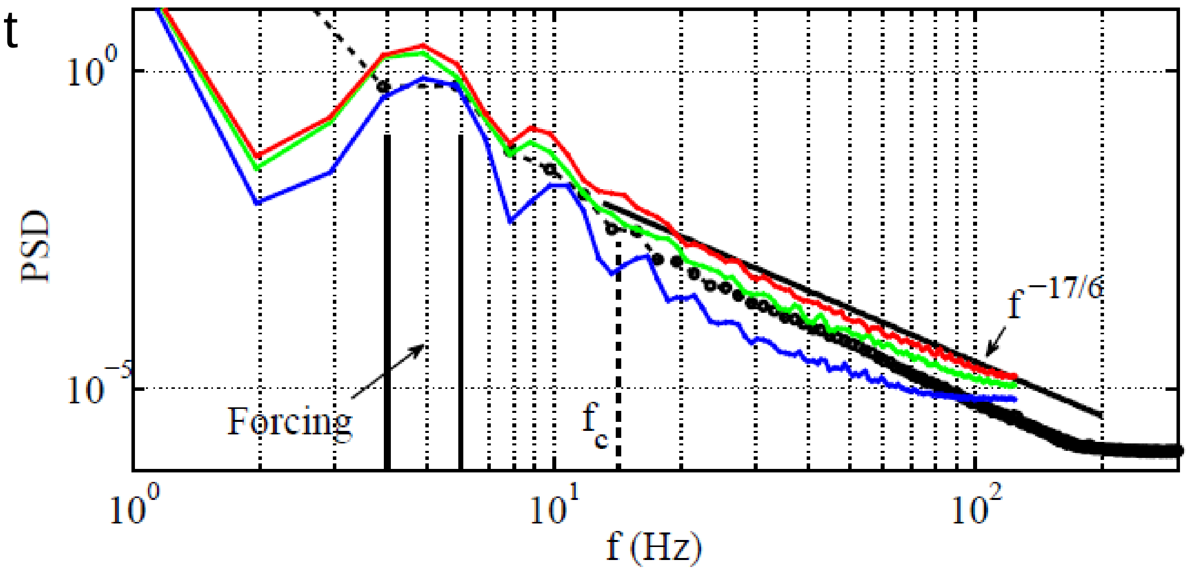


# Wave elevation Spectra
# 1D spatio-temporal interface acquisition.

- Spectra are in agreement with the expected power law, for random between 4 and 5 Hz.

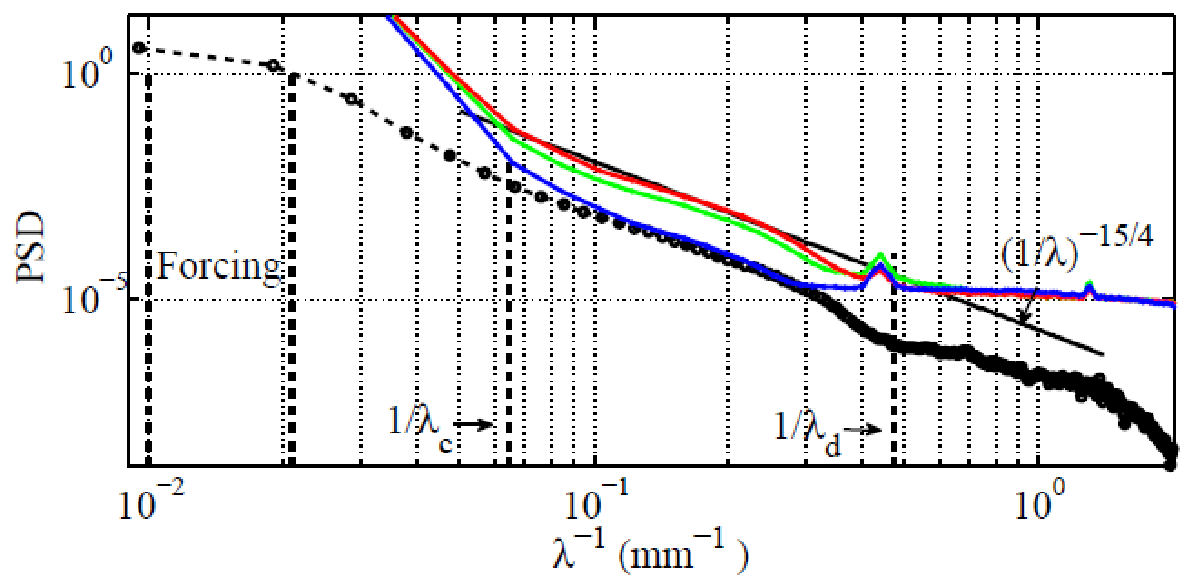
$$S_h^{theo}(\omega) \sim \omega^{-17/6}$$

$$S_h^{theo}(k) \sim k^{-15/4}$$




Black curve :DLP

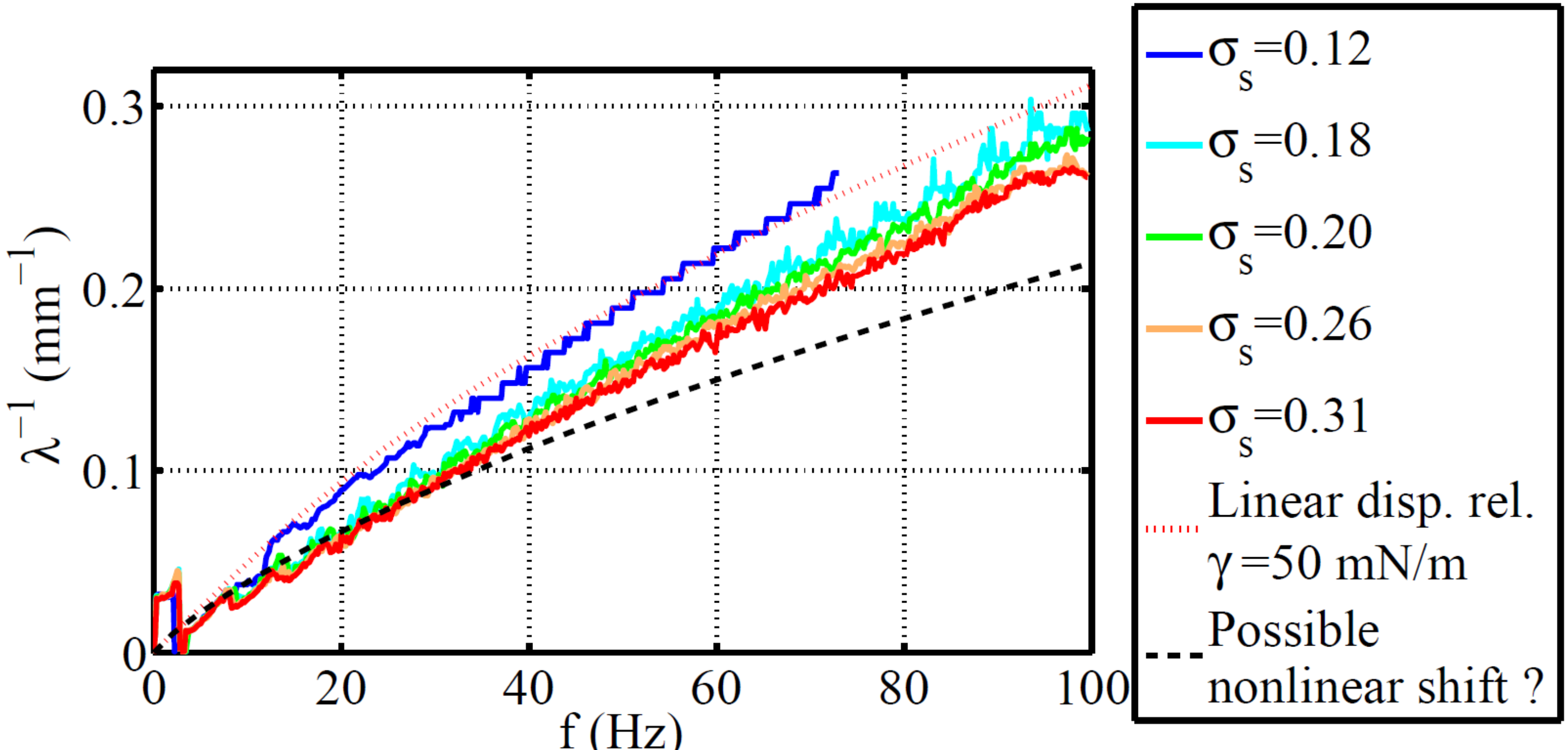
Blue, Green, Red 1D interface detection




- Spatial resolution is lower than with DLP technique. More Noise.

- Frame rate 250 Hz. Calibration of DLP must be improved.

# Nonlinear shift of Dispersion relation using DLP




- Shift of dispersion relation when steepness in increased.

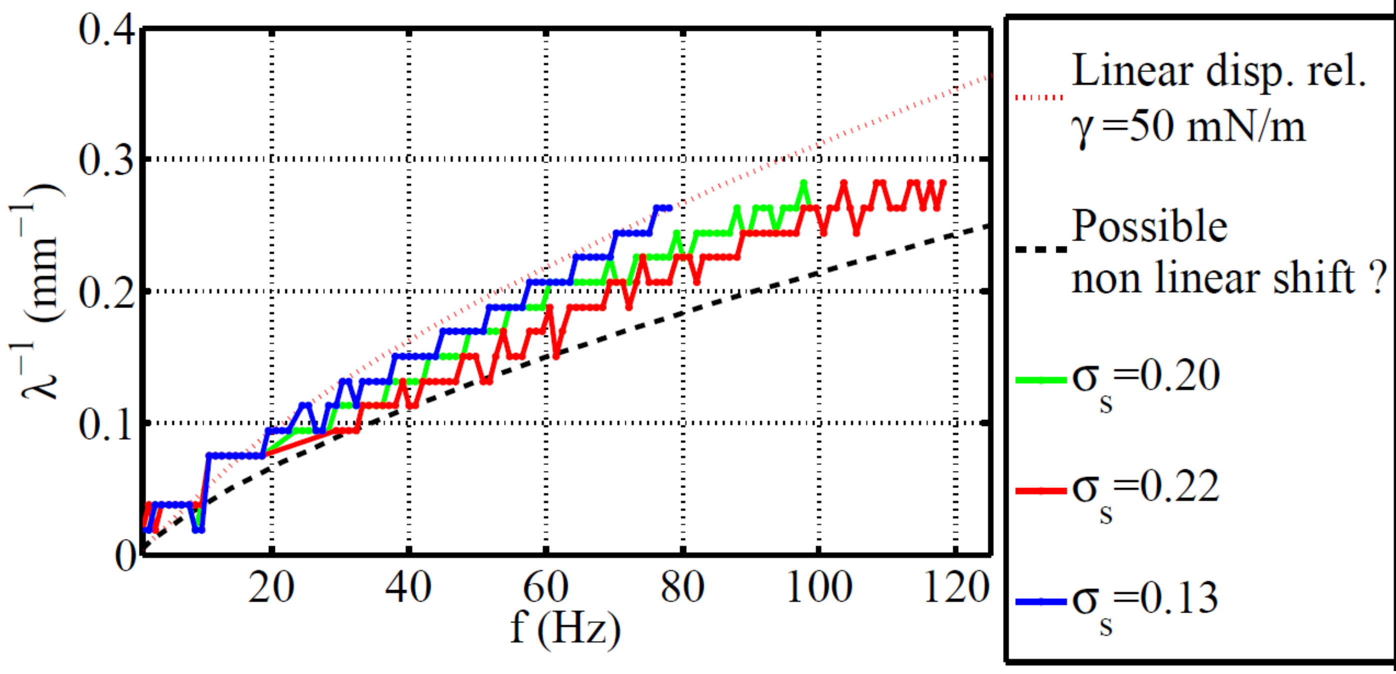
Seems due to high amplitude of gravity wave at the forcing (Stokes waves).

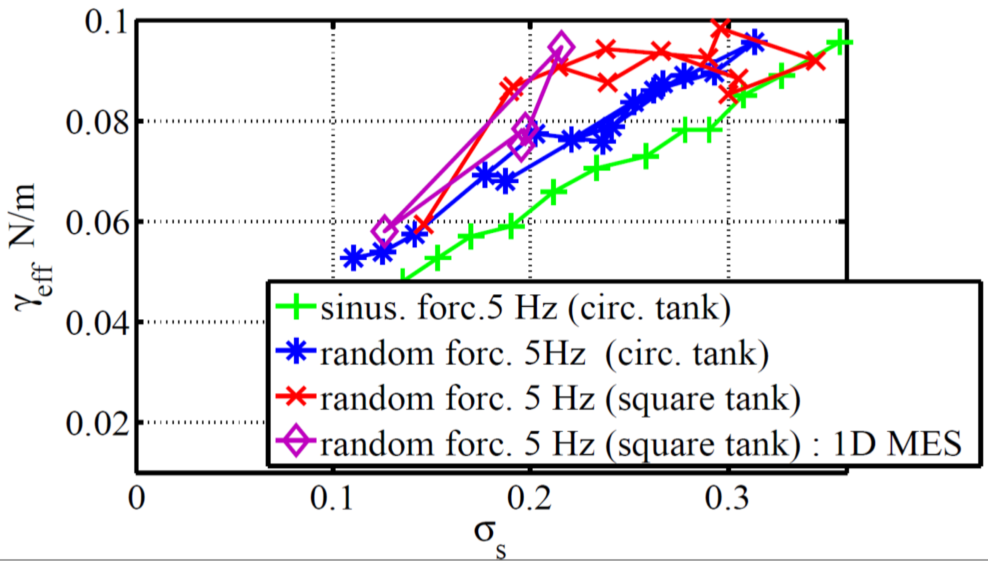
$$\omega^2 = \left( gk\left[1+(ak)^2\right] + \frac{\gamma}{\rho}k^3\left[1+(\frac{ak}{4})^2\right]^{-1/4} \right) \tanh(kh_0)$$

# Nonlinear shift of Dispersion relation

- Confirmed by 1D interface measurement.

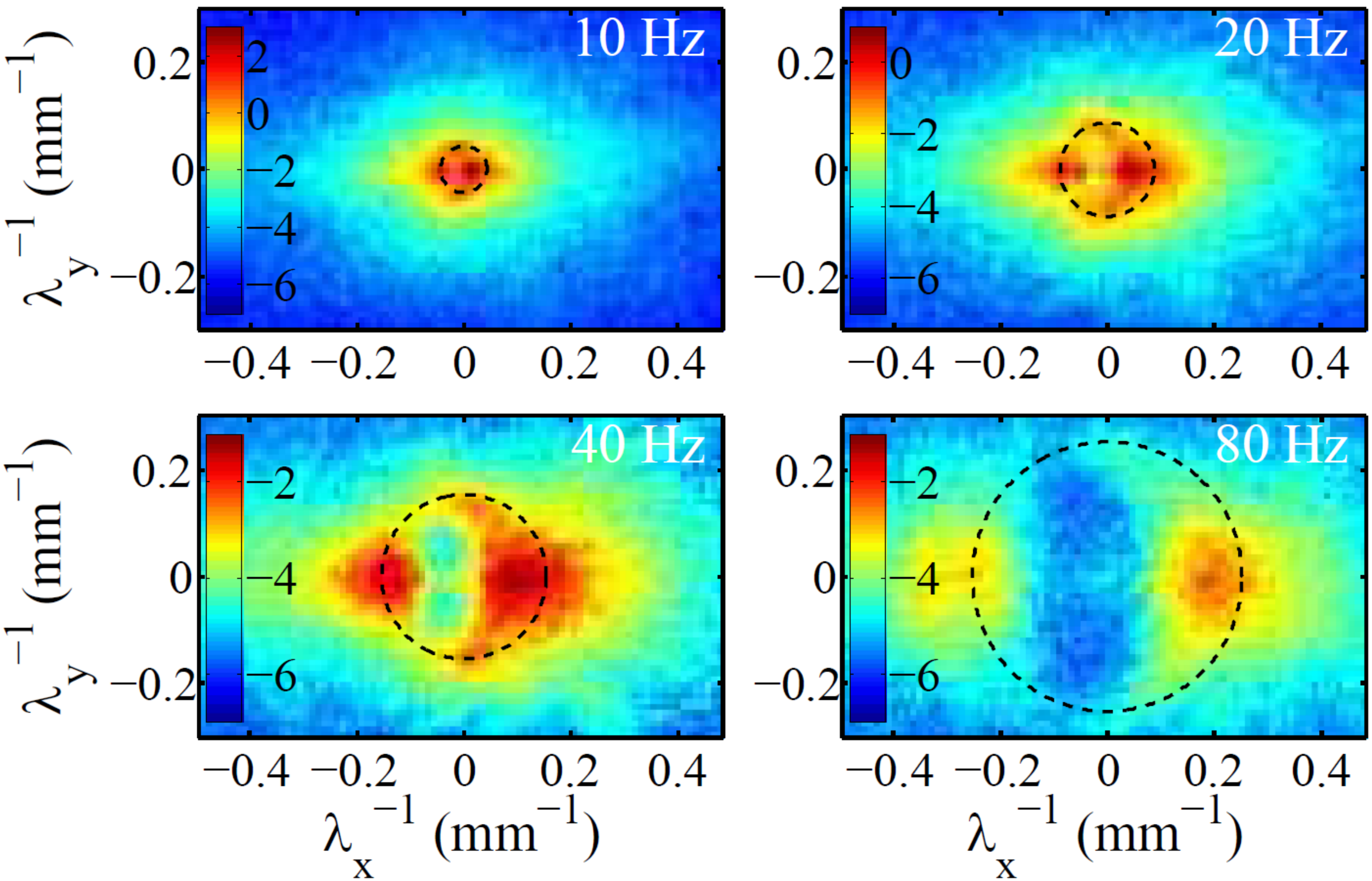
- Could be interpreted as an increased effective surface tension.



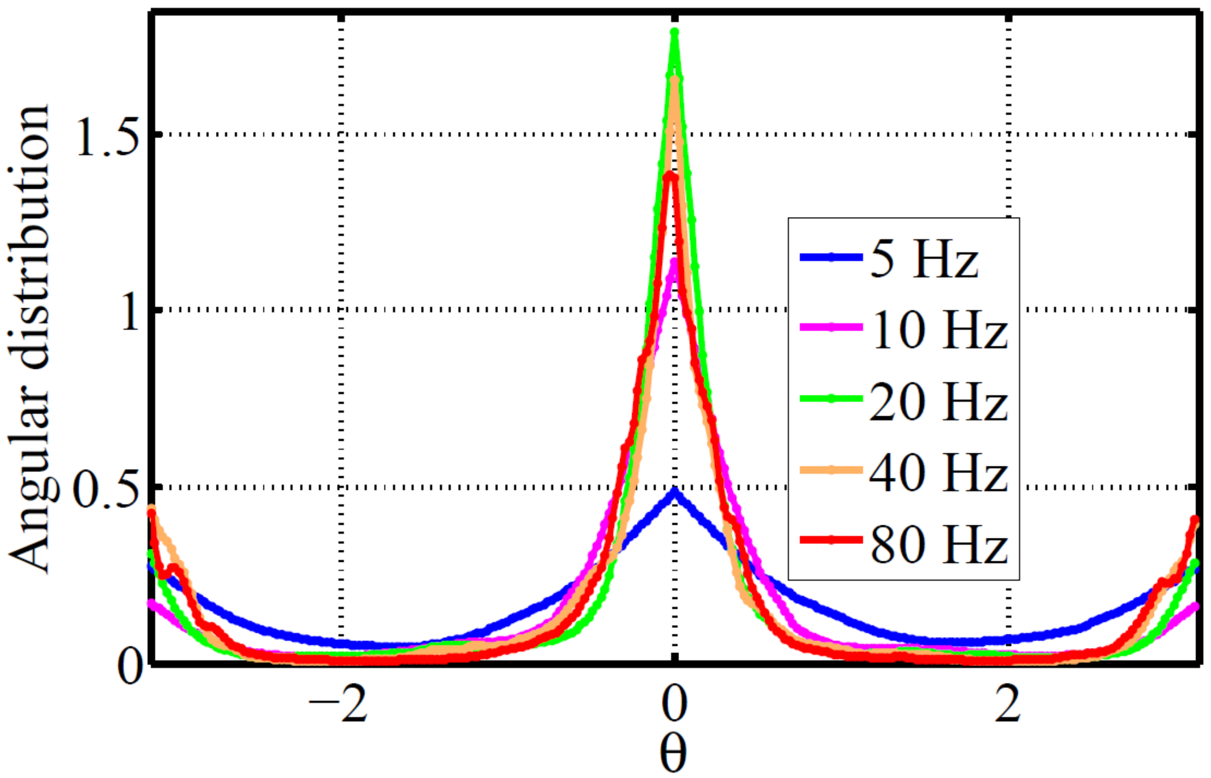



# Angular properties of spectrum. Strong anisotropy

$S_h(\omega, \lambda_x{}^{-1}, \lambda_y{}^{-1})$ $\quad log_{10}$ colorscale



# Angular distribution of spatiotemporal spectrum

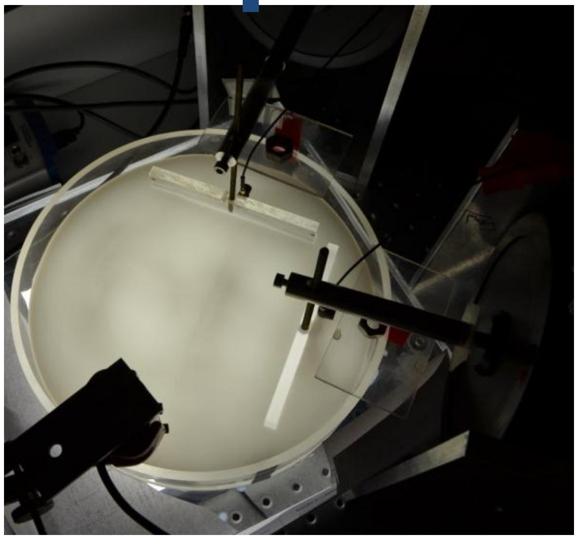



● Anisotropy due to forcing is conserved through the scales.

● Isotropy at small scales is not restored by non-linear interactions.

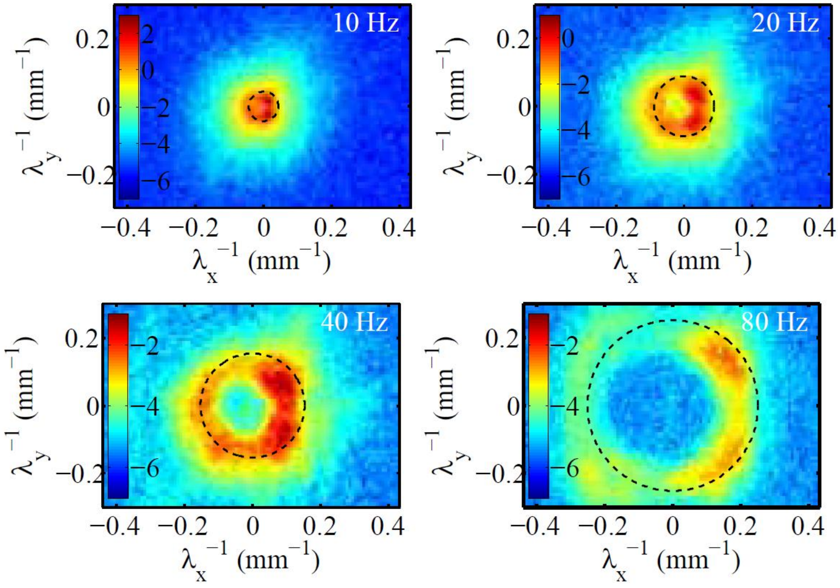
● Compatibility with three waves interactions ? Instability of the K.Z. solution for spectra.

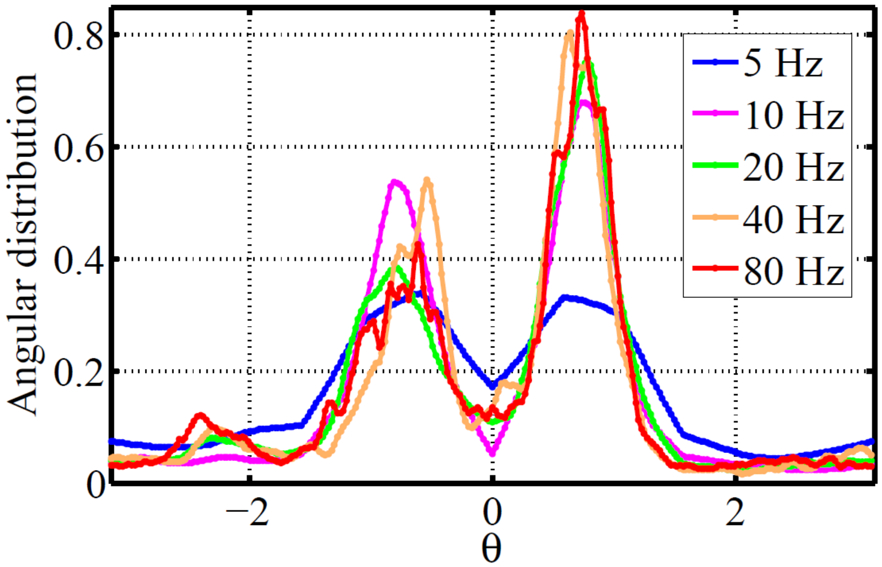
# Angular distribution of spatiotemporal spectrum



- Circular tank,
  2 wave-makers.

Wave field remains anisotropic.
Directions of forcing are dominant.



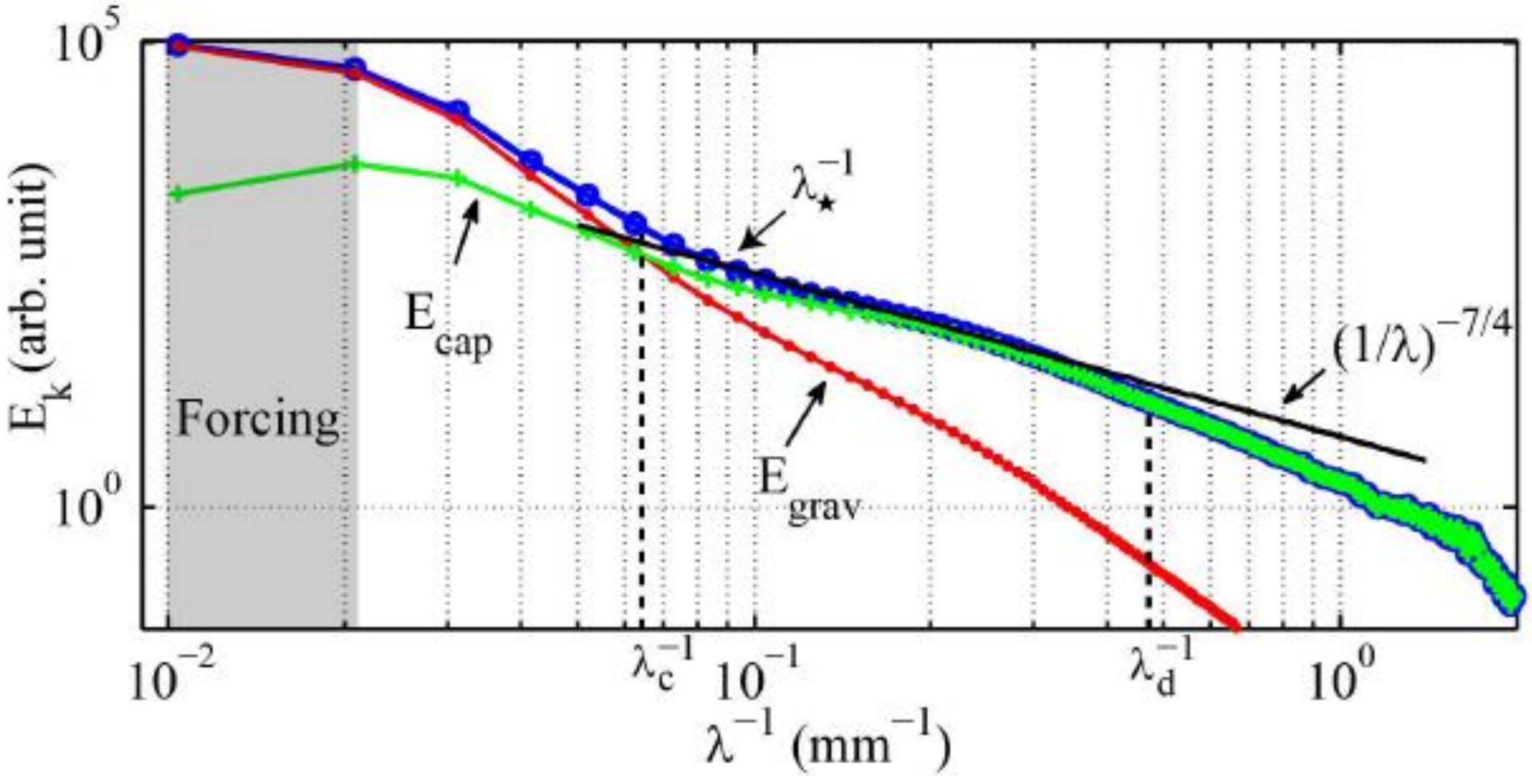



# Energy spectrum

● Rescaling of spatial spectrum to obtain "Energy spectrum".
Sum of gravity potential energy and capillary potential energy.

$$E = L \int \left[\rho\, g\, |\tilde{\eta}_k|^2 + \gamma\, k^2\, |\tilde{\eta}_k|^2\right] \mathrm{d}k$$

● Agreement with WT prediction in the capillary range.



# Bursts of Energy modes

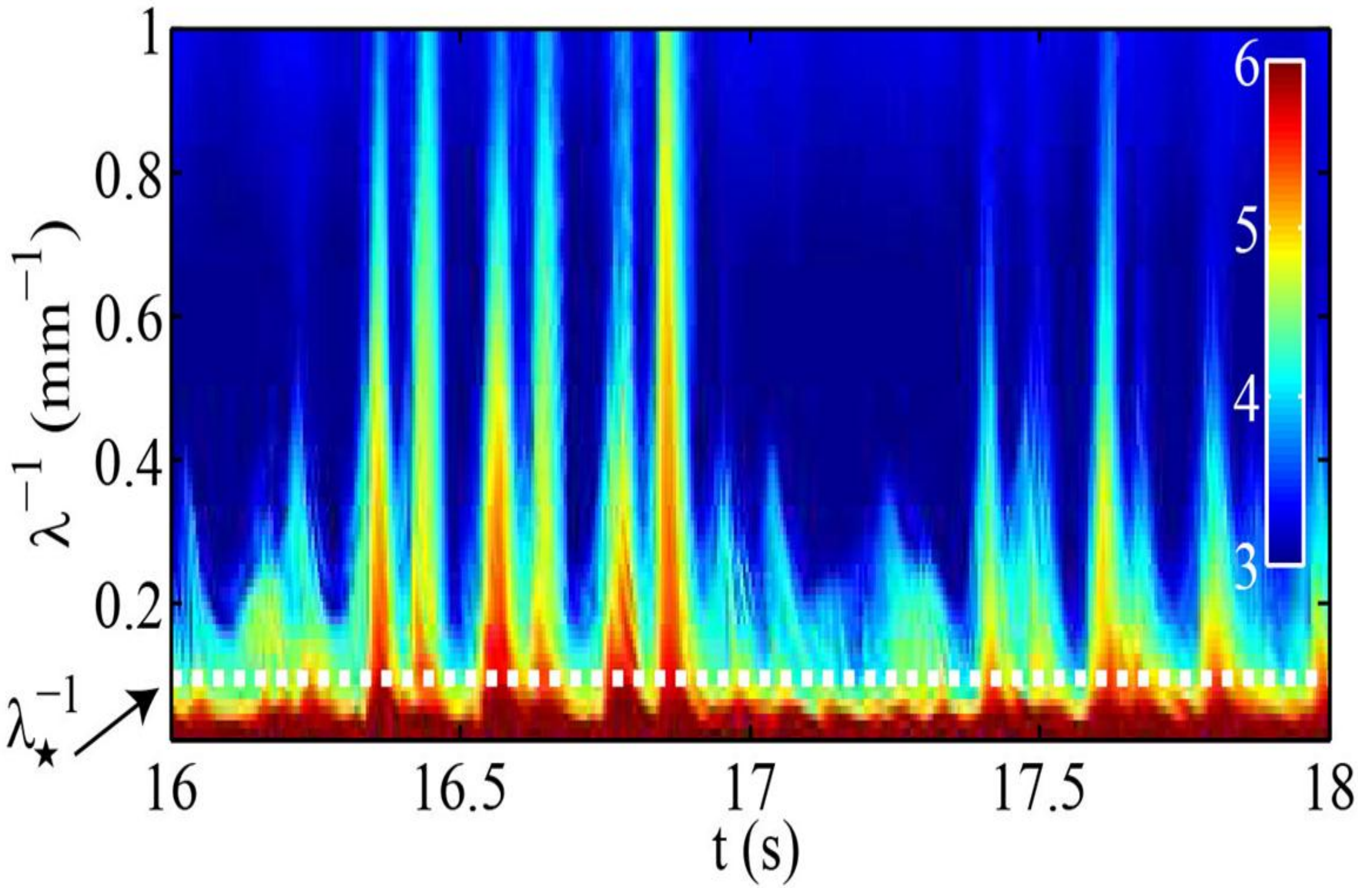
- Performing only spatial Fourier transform: **temporal** behavior of **Spatial spectrum.**

$$E_k(t) = \rho\, g\, |\tilde{\eta}_k|^2 + \gamma\, k^2\, |\tilde{\eta}_k|^2$$

- Evaluation of spreading of energy through the scales as a function of **time**:

bursts

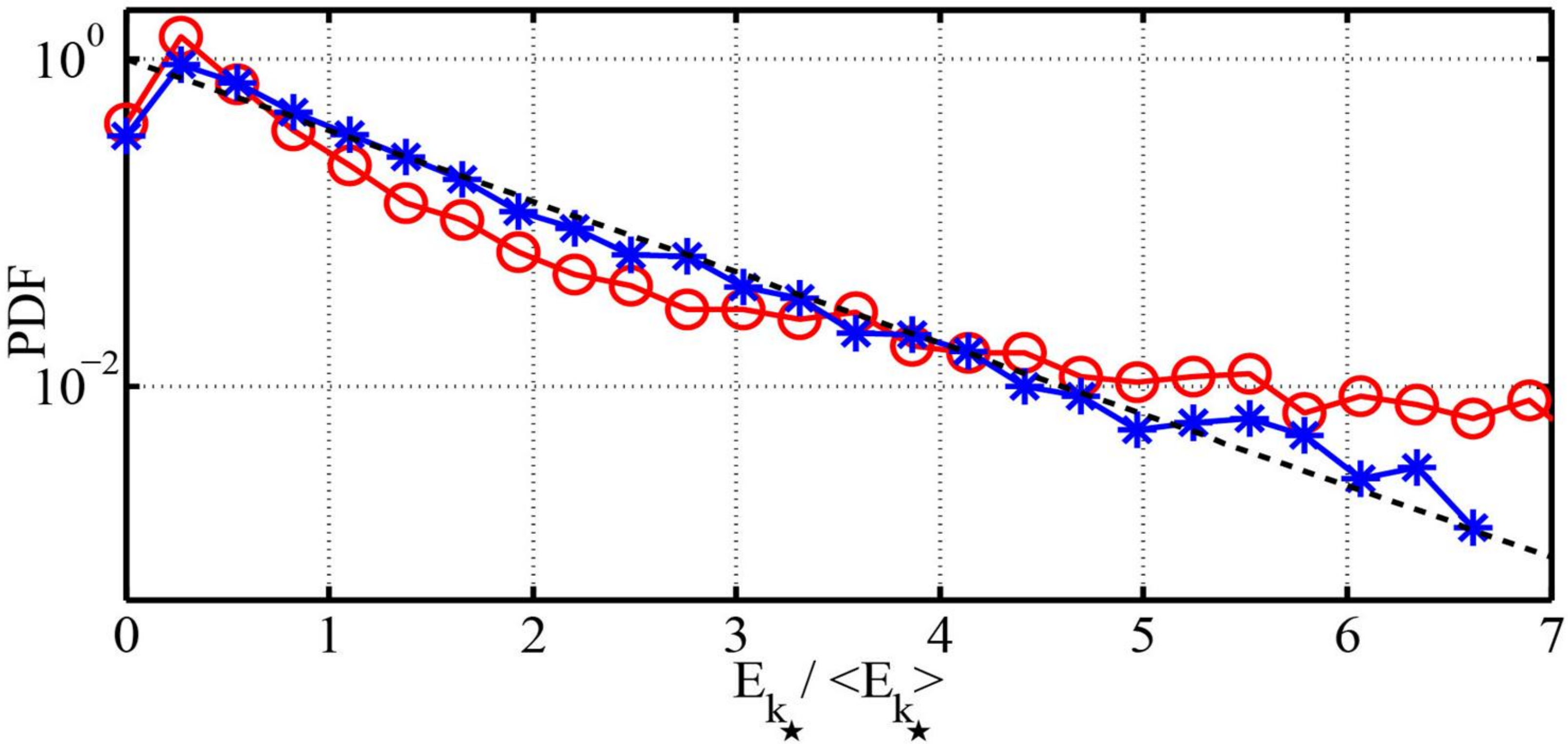
- Color scale in $\log_{10}$ scale of $E(k,t)$



# Distribution of Energy modes

- Temporal statistic of one energy mode at a given scale $\lambda_\star^{-1} = 0.094\,\mathrm{mm}^{-1}$

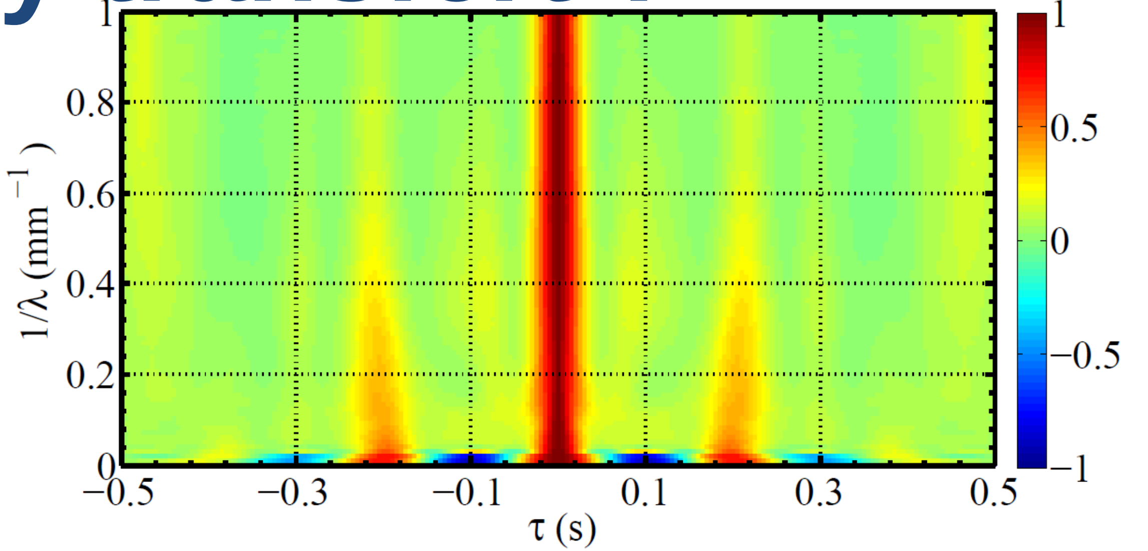
- **Exponential distribution** (expected for Gaussian field of waves) at low amplitude (blue curve).
- Increased probability to observe strong events at higher amplitude, (red curve).



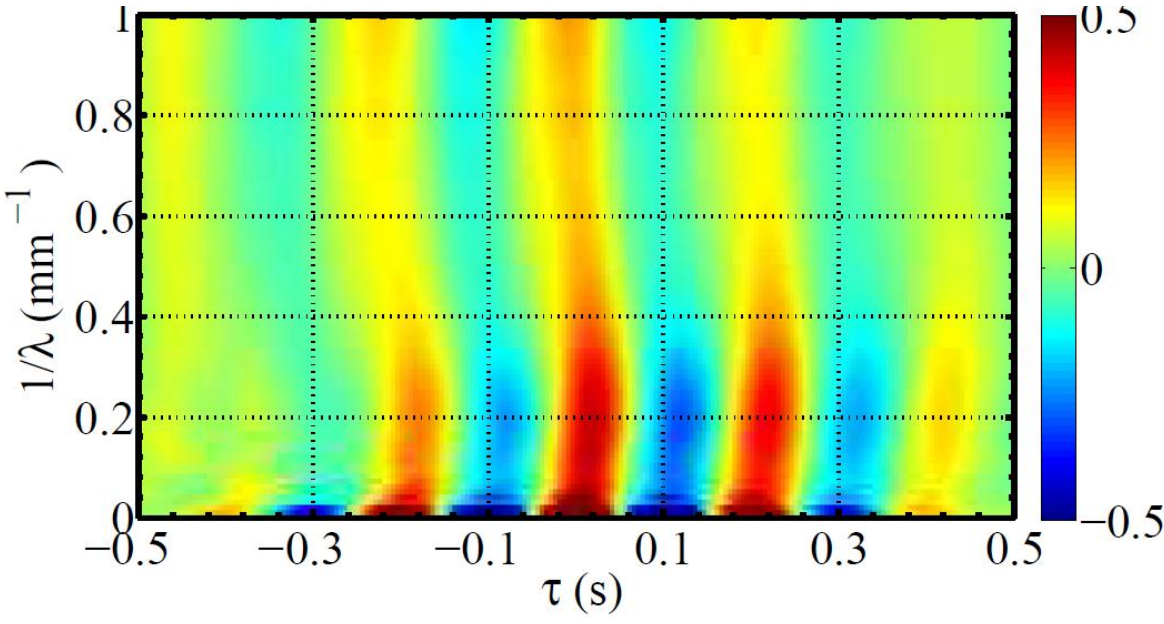
# Time scale associated to energy transfert ?

● Temporal autocorrelation of E(t,k), shows a short correlation time ~ 0.05 s. Not depending on the scale.

System modulated by the forcing (average oscillation at 5 Hz).

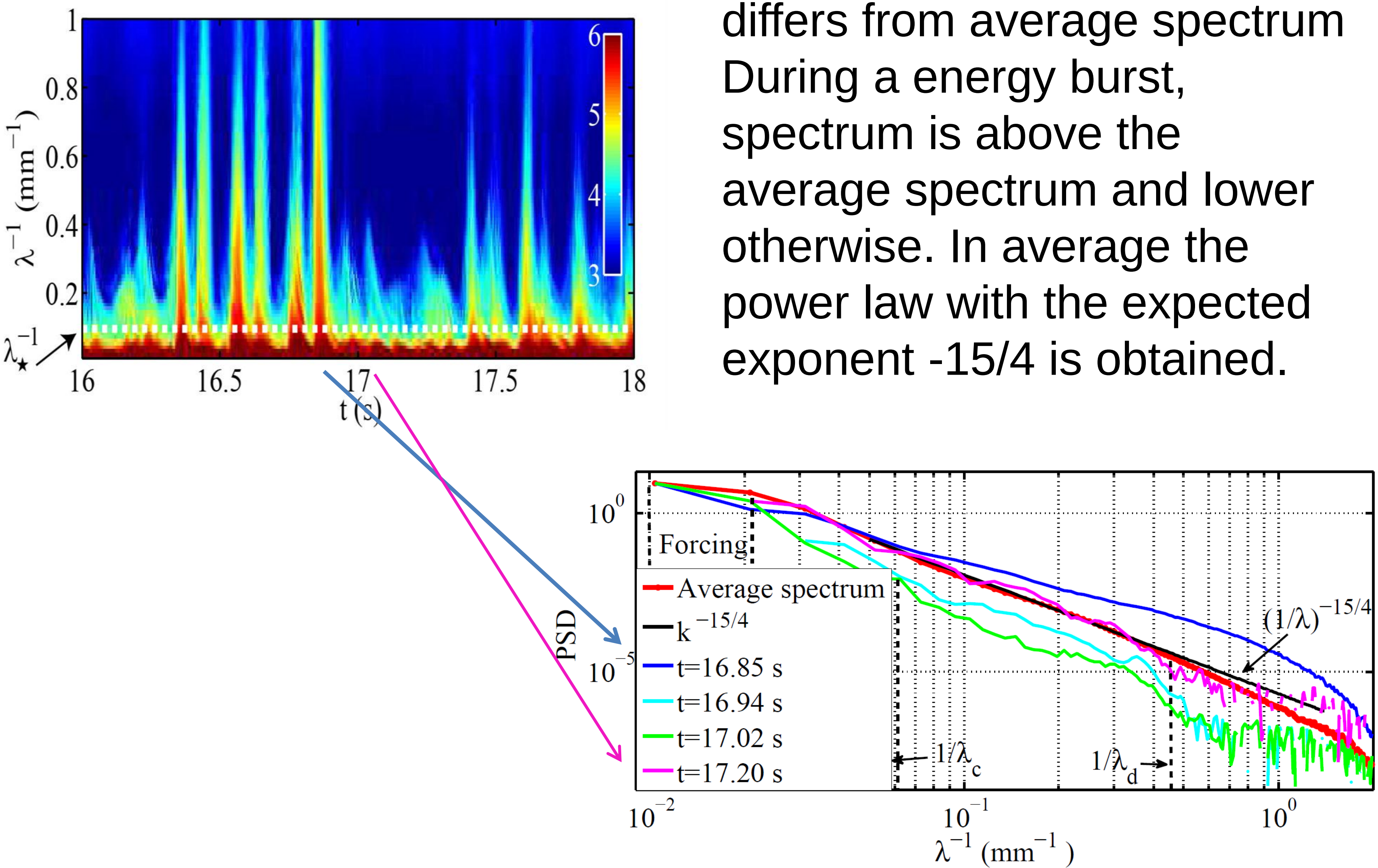



● Temporal crosscorrelation of E(t,kforcing) with E(t,k). Capillary waves are produced with a delay of 0.02 s.
Condition of small interaction time compared to wave period does not seem fulfilled.



# Statistical building of spatial spectrum

- Instantaneous spectrum differs from average spectrum During a energy burst, spectrum is above the average spectrum and lower otherwise. In average the power law with the expected exponent -15/4 is obtained.



# Parasitic capillary waves

● Different interaction mechanism than 3-waves interaction.

Generation of wave train of capillary waves at the crest of steep gravity waves then riding on the forward face of the large wave. Curvature acts as a pressure source.

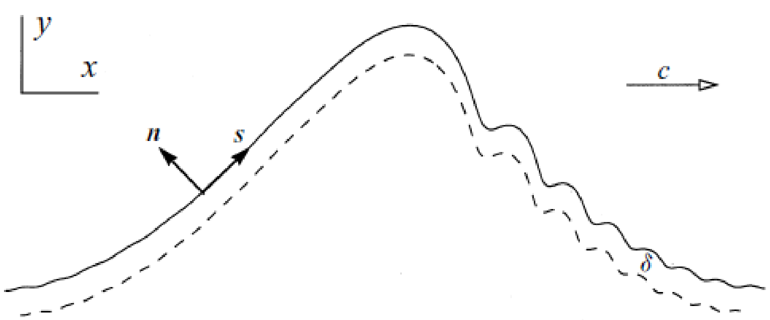
Longuet Higgins JFM 1962 , Fedorov Melville JFM 1998, Fedorov Melville POF 1998

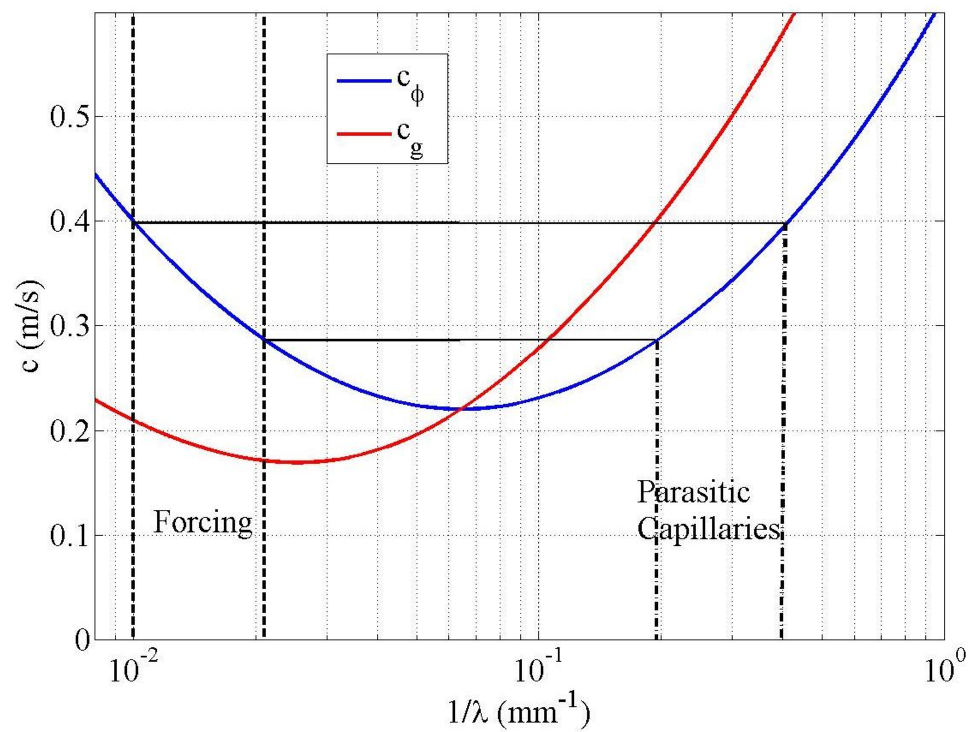
2 D mechanism conserves forcing anisotropy.

Fast pumping of energy, increases dissipation of gravity waves by an order of magnitude.

● Matching of phase speed

between forcing wave and capillary trai



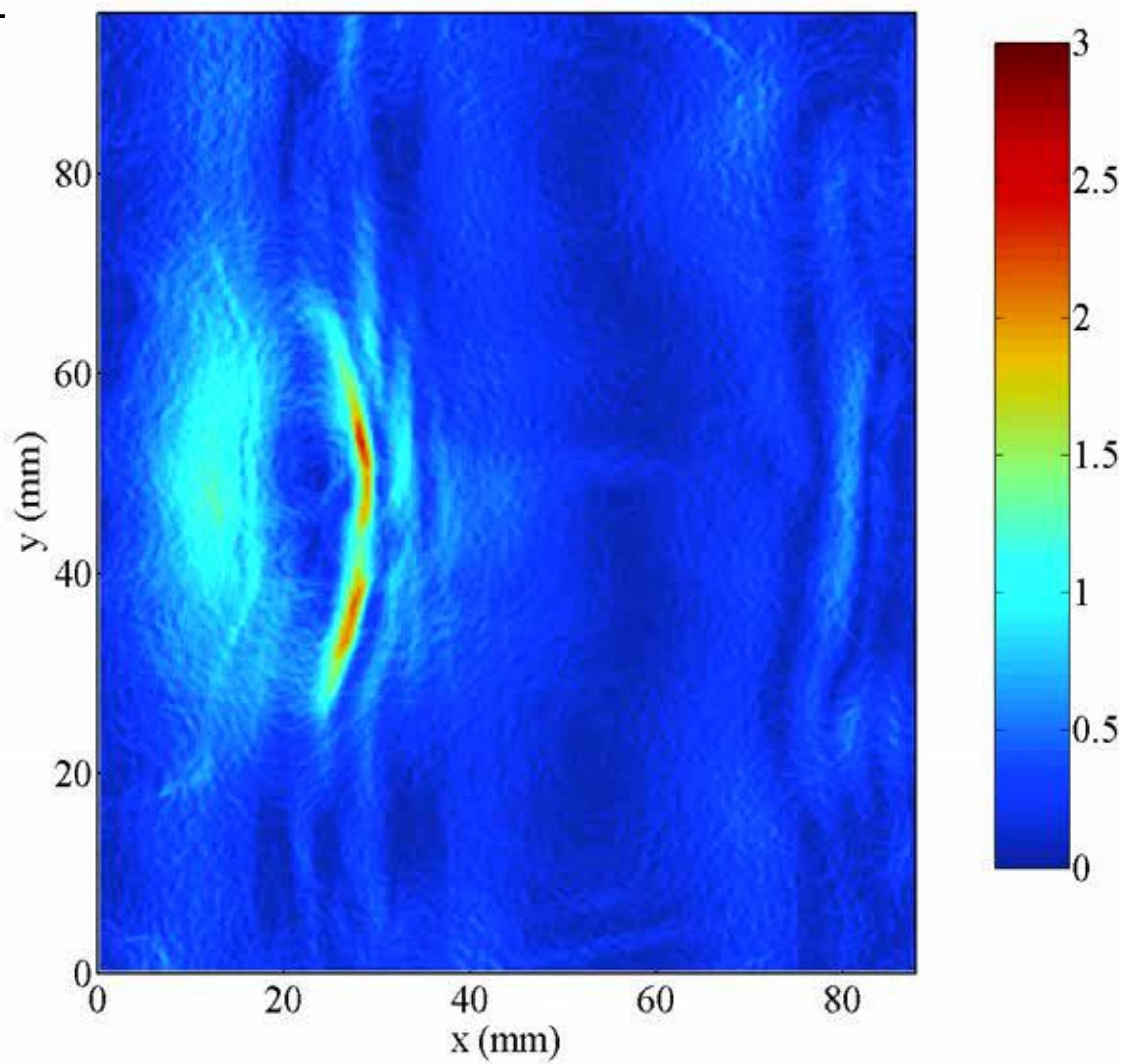




More recent studies associates this with a viscous boundary layer and a vortical region.

# Parasitic capillaries in random forcing ?

- Forcing: White noise band pass filtered between 4 Hz and 6 Hz.
- Criterion for parasitic capillaries gives continuous spectrum between 55 and 155 Hz.
- Wave orbital motion could increases scale range of excited capillaries waves. (Wave flow interaction)
- In our experiment more complex situation (reflection, …)
  But role of non-resonant interactions must be characterized.

$\sigma_h = 3.6\,\mathrm{mm}$ and $\sigma_s = 0.34$



# Conclusion

- **Gravity wave turbulence**: growth of the $\omega$-slope with wave amplitude, saturation towards the K.Z. value. But this behavior could be produced with a one dimensional field of non-linear waves (role of coherent structures).

- **Capillary wave turbulence:** Spectra of wave elevation in agreement with K.Z. spectra Validity of analytical derivation (simulations).

- But **experimentally** important role of viscous dissipation in Capillary W.T.
Capillary waves seem to be created at the crest of steep gravity waves.
Parasitic capillary waves. Non-local interaction in **k** space.
Compatibility with 3-Waves interaction ?

# Perspectives

- Spatial correlations to identify interaction mechanisms (local or non local in **k** space).
  How to justify observed power law ? Scaling of energy flux through the scales.