The Role of Kelvin Waves in Superfluid Turbulence

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New Challenges in Turbulence Research III
Les Houches, 21 March 2014

Outline

I. Introduction
   - Classical vs. Superfluid Turbulence

II. Theoretical Setup for Kelvin Waves
   - Hamiltonian structure, wave action representation, weak nonlinear expansion, canonical tranformation

III. Wave Turbulence Theory
   - Derivation of the kinetic wave equation, Kolmogorov-Zakharov solutions, nonlocality analysis, theoretical controversies

IV. Kelvin Waves in Numerical Simulations and Experiments

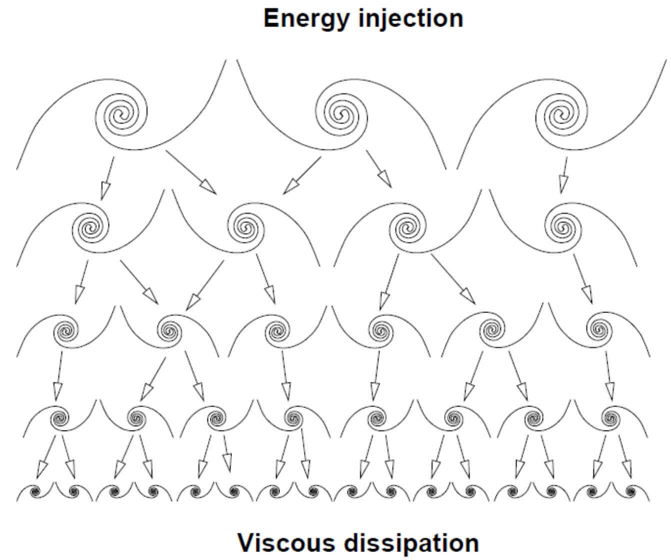
V. Conclusions

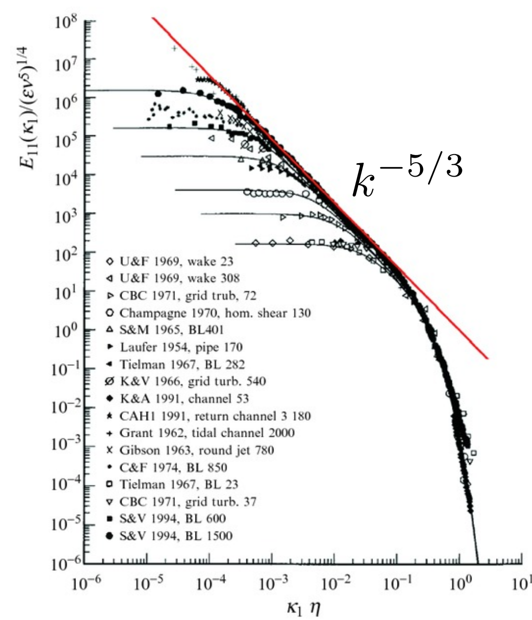
3D Navier-Stokes Turbulence

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P + \nu \Delta \mathbf{v}$$

- Energy $\frac{\mathbf{v}^2}{2}$ injection at large scales
- Energy dissipated at small scales for high Reynolds number $\mathrm{Re} = VL/\nu$
- Concept of inertial range if injection and dissipation scales are well separated in scale space
- Conservative energy transfer by nonlinear term of Navier-Stokes equations to small scales
- Local energy transfer (cascade) characterized by constant energy flux ϵ

Richardson's cascade picture





Kolmogorov 1941 Theory

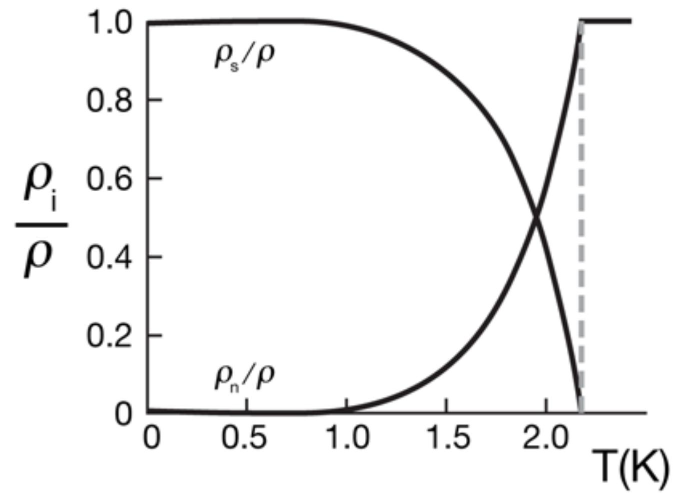
As $\mathrm{Re} \to \infty$, the inertial range statistics depend only on scale k and energy flux ϵ

$$E_k = C\,\epsilon^{2/3}\,k^{-5/3}$$

Kolmogorov energy spectrum

Liquid Helium and Superfluidity

- Superfluidity is a state of matter in which the fluid behaviour is like a zero viscosity fluid
- Related to Bose-Einstein Condensation in Bose gases
- Superfluidity is observed in both helium-3 and helium-4
- Occurs in helium-4 below the lambda transition point: $T_\lambda = 2.17\ \mathrm{K}$



At $T < 1\ \mathrm{K}$ helium-4 is more than 99% superfluid

Two-fluid model:

- Below T_λ helium-4 is composed of two coexisting fluids:

  Normal (viscous) component with density ρ_n

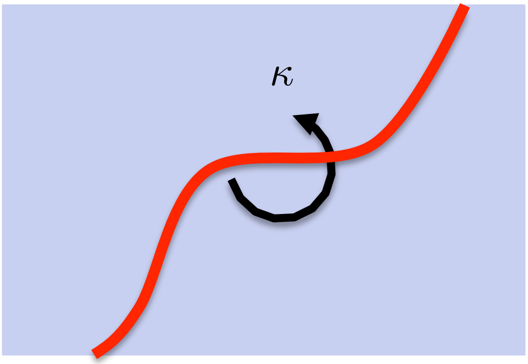
  Superfluid (inviscid) component with density ρ_s

- The two components coexist together to give total fluid density of $\rho = \rho_n + \rho_s$
- The relative densities of the two components is temperature dependent

Properties of the Superfluid Component

Hydrodynamical properties

- A completely inviscid fluid (no viscosity)
- Superfluid flow is irrotational flow $\omega = \nabla \times \mathbf{v} = 0$



What happens if we externally induce vorticity?

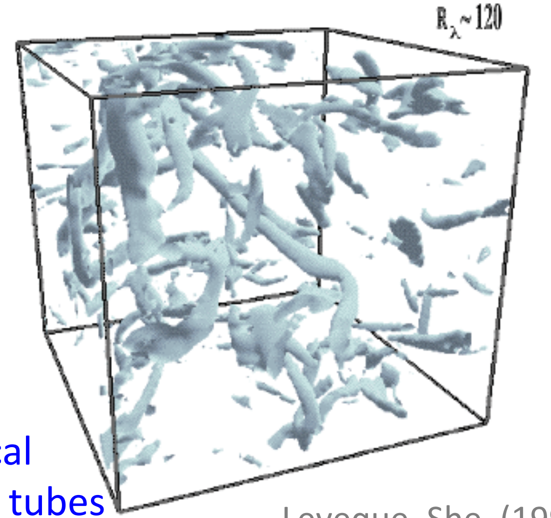
- Vorticity appears through the creation of quasi-1D topological defects (density of the fluid vanishes)

These are known as quantized vortices

- Every quantized vortex line is identical
- Vortex core is incredibly small: $\xi \sim 10^{-8}$ cm (for helium-4)
- Circulation is discretized in units of $\kappa = h/m$
- Normal and superfluid components interact through quantized vortex lines by mutual friction

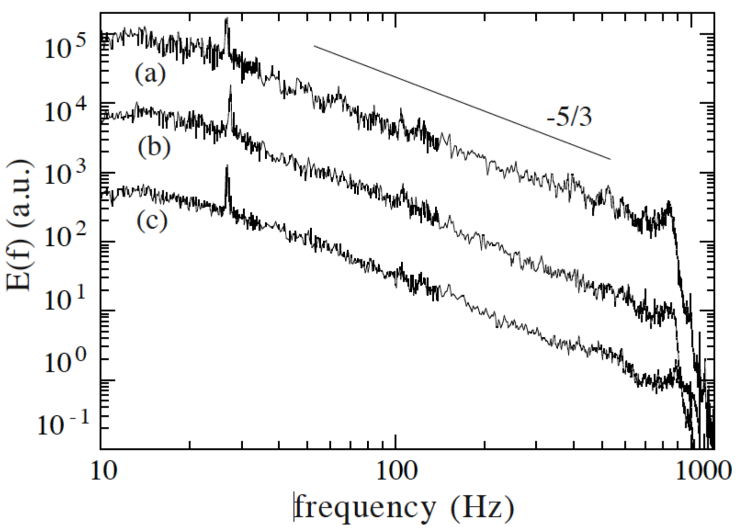
Superfluid Turbulence (Large Scales)

- Superfluid turbulence can defined as the study of the chaotic motion of the superflow induced by a tangle of quantized vortices
- Energy can be injected into the superfluid through classical means (mixing, rotation) or through quantum methods (counterflow)
- Polarization of vortex bundles mimic classical eddies
- Richardson cascade of vortex bundles
- Kolmogorov picture also observed in superfluid

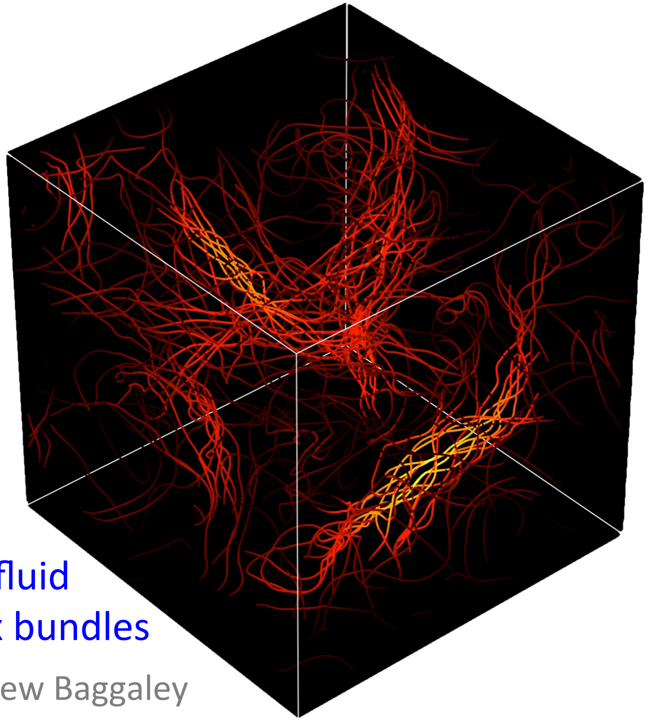


Classical vortex tubes

Leveque, She, (1993)



Mauer, Tabeling, *Europhys. Lett*. 43, 29, (1998)

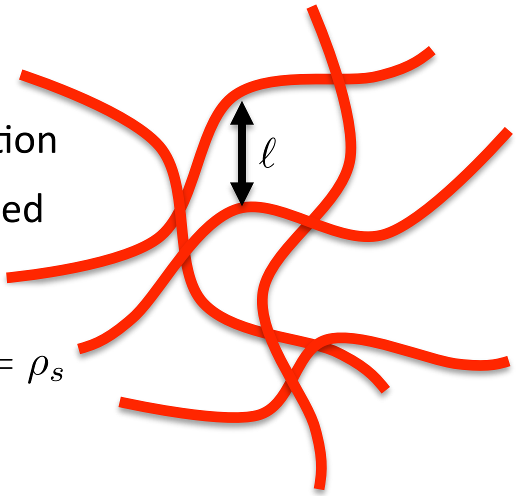


Superfluid vortex bundles

Courtesy of Andrew Baggaley

Superfluid Turbulence (Small Scales)

How is energy dissipated in superfluid turbulence?



Finite (positive) temperature superfluid turbulence

- Interlocking of normal and superfluid components via mutual friction
- Energy in the superfluid is transferred to normal fluid and dissipated

Zero temperature superfluid turbulence

- Liquid helium solely consists of superfluid component $\rho_n = 0$, $\rho = \rho_s$
- No normal fluid to dissipate energy
- Phonons are suggested to be responsible for energy dissipation

How does energy reach phonons?

- The cascade of polarized vortex bundles breaks down at scales $< \ell$
- Phonon scale is much smaller than the typical mean inter-vortex distance ℓ
- There must exist another mechanism that allows energy to reach such small scales

Hypothesis

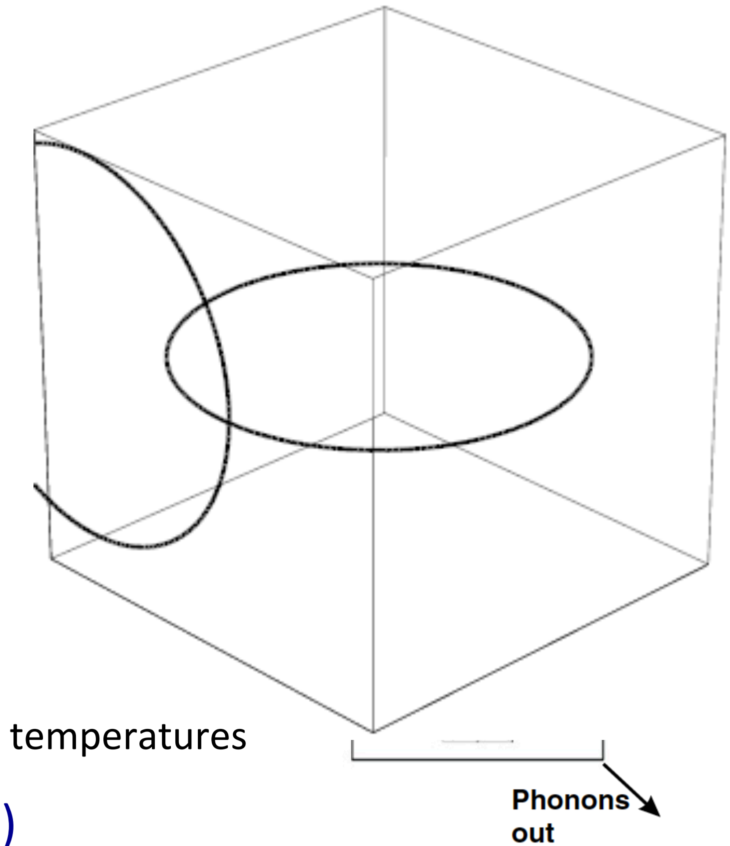
Energy transfer to propagating Kelvin waves

What are Kelvin Waves?

- Natural perturbations that occur on quantized vortex lines
- Kelvin waves get excited by vortex reconnections
- Reconnection events excite Kelvin waves at scales close to the inter-vortex distance ℓ
- In the long Kelvin wave limit $\mathbf{k}\xi \ll 1$, Kelvin waves propagate with dispersion relation

$$\omega_{\mathbf{k}} = \frac{\kappa \mathbf{k}^2}{4\pi}\left[\ln\left(\frac{1}{\mathbf{k}\xi}\right) - \gamma - \frac{3}{2}\right]$$

- $\gamma \approx 0.577\ldots$ is the Euler-Mascheroni constant
- Kelvin waves are damped by mutual friction at finite temperatures



Kelvin wave cascade picture (zero temperature)

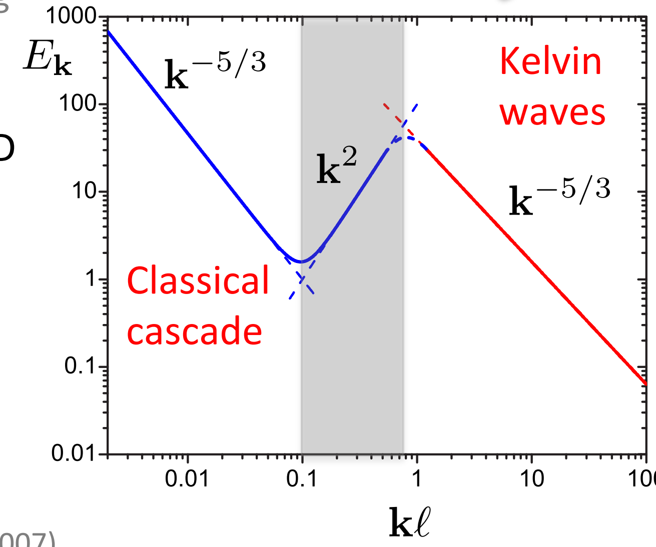
- Kelvin waves of similar scale interact exciting smaller scale Kelvin waves
- Continues until the frequency is sufficiently high enough to excite phonons in the superfluid

Small scale superfluid turbulence characterized by a Kelvin wave cascade

Superfluid Turbulence Crossover

large scales | Kolmogorov cascade | ? | Kelvin wave cascade | small scales

scale of inter-vortex spacing



What lies in between?

- Crossover region where energy is converted from a 3D energy cascade to that of a 1D Kelvin wave cascade
- Energy transfer by Kelvin waves is not as efficient as the classical picture
- Energy must stagnate?

1. Thermalization (bottleneck effect)

L'vov, Nazarenko, Rudenko, Phys. Rev. B, **76**, 024520, (2007)

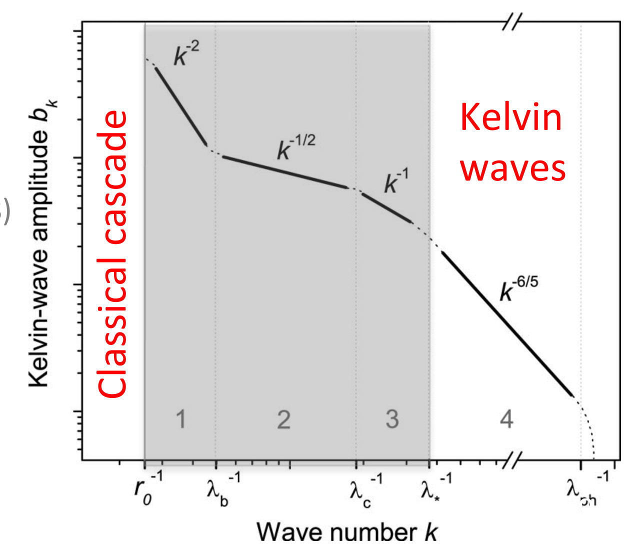
- Energy stagnates and thermalizes until of sufficient intensity to excite Kelvin waves (bottleneck)

2. Hierarchy of reconnection mechanisms

Kozik, Svisuntov, Phys. Rev. B, **77**, 060502, (2008)

- No stagnation! Energy reaches Kelvin waves via a series of different vortex reconnection mechanisms

  1. Bundle-bundle reconnections
  2. Inter-bundle reconnections
  3. Self reconnections



Models of Zero-Temperature Superfluid Turbulence I

The Gross-Pitaevskii equation

$$i\dot{\Psi} = -\nabla^2\Psi + \Psi\,|\Psi|^2$$

Madelung transformation

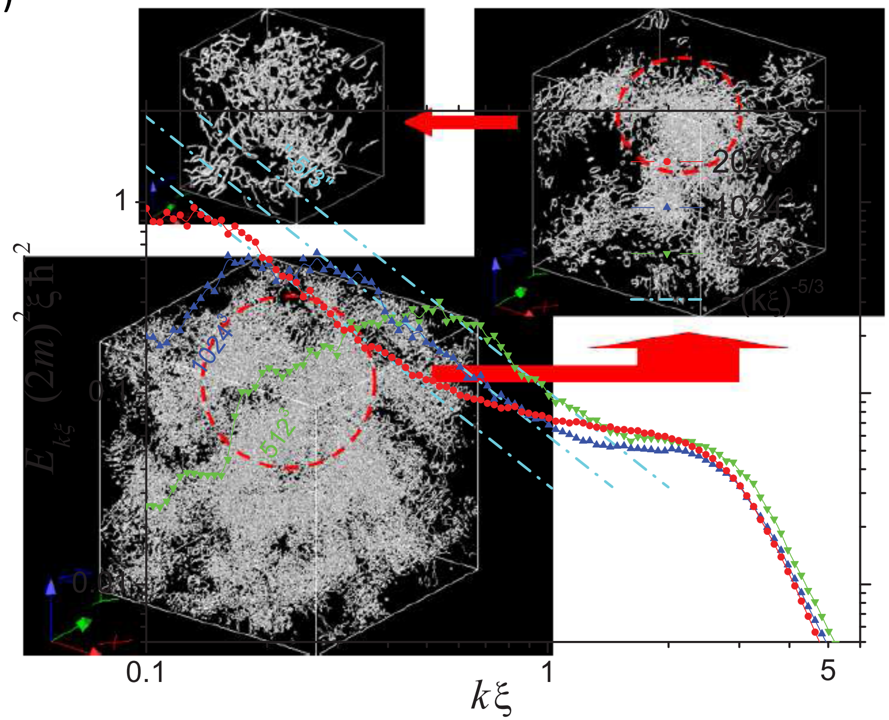
$$\Psi = \rho\, e^{i\theta}$$

where $\rho = |\Psi|$ and $\mathbf{v} = \nabla\theta$

- A model for weakly interacting Bose gases (Bose-Einstein Condensation)
- Hamiltonian dynamics (energy conservation)
- Equivalent to time-dependent Bernoulli equation for irrotational compressible flow with additional *quantum pressure* term

Properties of Gross-Pitaveskii turbulence

- Quantized vortex lines with well-defined vortex core structure
- Vortex reconnections
- Phonons emission
- Observed Kolmogorov spectrum in incompressible energy spectrum at large scales

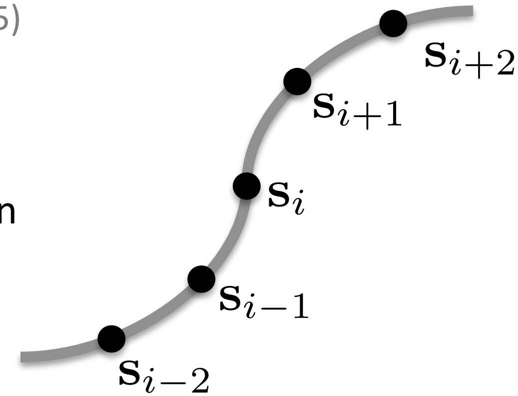


Sasa *et al.*, *Phys. Rev. B*, **84**, 054525, (2011)

Models of Zero-Temperature Superfluid Turbulence II

The vortex filament model

Schwarz, *Phys. Rev. B*, **31**, 5782, (1985)



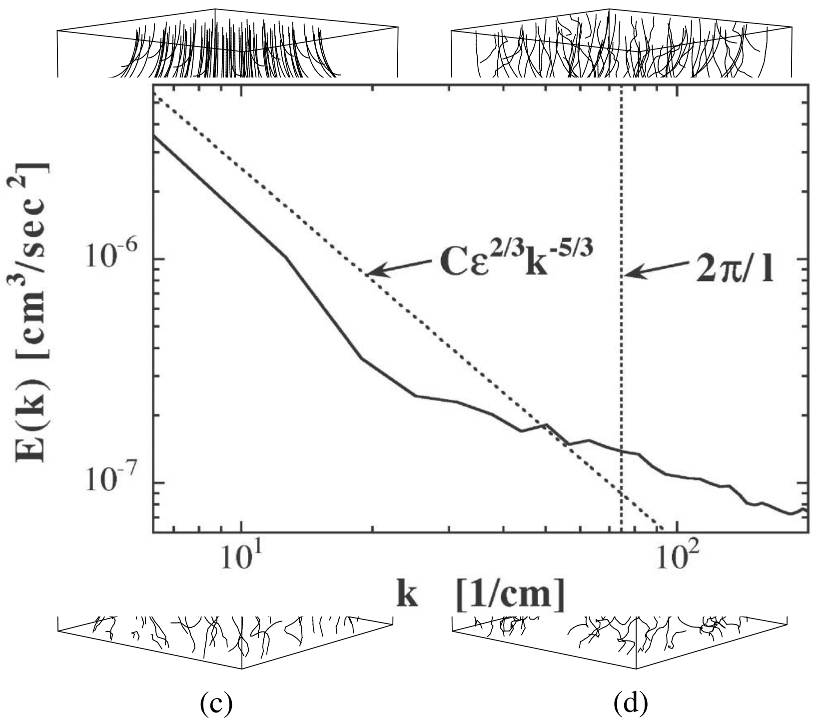
- Approximate quantized vortex lines as 1D space curves
- Velocity field is constructed through the Biot-Savart equation

$$\mathbf{v}(\mathbf{r}) = \frac{\kappa}{4\pi} \oint_{\mathcal{L}} \frac{\mathbf{s}-\mathbf{r}}{|\mathbf{s}-\mathbf{r}|^3} \times \mathrm{d}\mathbf{s}$$

- Quantized vortex lines are advected by the the self-induced velocity field $\dot{\mathbf{s}} = \mathbf{v}$

Comparison to Gross-Pitaevskii equation

- No vortex core structure
- No nature vortex reconnection mechanism
- Incompressible model (no phonons)
- Observed Kolmogorov spectrum at large scales



Araki, Tsubota, Nemirovskii, Phys. Rev. Lett., **89**, 145301, (2002)

Summary I

Classical (normal fluid) turbulence

- Vorticity is distributed across a wide range of scales in for form of eddies
- Kolmogorov energy spectrum up to viscous dissipation scale

Superfluid turbulence

- Vorticity is confined to extremely thin identical vortex lines of fixed circulation
- Polarized vortex bundles act as large scale (classical) eddies
- Observed Kolmogorov energy spectrum up to the mean inter-vortex distance
- Phonons are responsible for energy dissipation at zero temperature
- Kelvin waves transfer energy from the inter-vortex distance to phonons

Kelvin wave dynamics play an essential role in zero temperature superfluid turbulence

II. Theoretical setup of Kelvin Waves

Wave Turbulence Theory

Definition

Wave turbulence is the non-equilibrium statistical description of weakly interacting dispersive waves

Non-equilibrium

- Forcing and dissipation are required to sustain turbulent state

Statistical

- Many degrees of freedom

Weakly interacting

- Weak nonlinear coupling is required

Dispersive

- Non-dispersive waves are tougher to deal with theoretically

Applications

- Water (gravity, capillary)
- Plasmas (Alfvén, drift, sound)
- Optics, BECs, superfluids (phonons, Kelvin)
- Geophysical (Rossby, inertial)
- Vibrating plates (elastic waves)

Books

1. Zakharov, L'vov, Falkovich, *Kolmogorov Spectra of Turbulence I: Wave Turbulence*, Springer, (1992)
2. Nazarenko, *Wave Turbulence*, Springer, (2011)

Starting Point

Many wave systems possess a Hamiltonian structure described by natural canonical variables

- Consider the Hamiltonian $\mathcal{H}$ with canonical coordinates $q(\mathbf{x},t)$ and momenta $p(\mathbf{x},t)$

$$\frac{\partial q}{\partial t} = \frac{\delta \mathcal{H}}{\delta p}, \qquad \frac{\partial p}{\partial t} = -\frac{\delta \mathcal{H}}{\delta q}$$

- Equivalently express Hamilton's equations in terms of a single canonical complex variable: $a(\mathbf{x},t) = \dfrac{\lambda q + ip/\lambda}{\sqrt{2}}$

Then Hamilton's equations are equivalent to: $i\dfrac{\partial a}{\partial t} = \dfrac{\delta \mathcal{H}}{\delta a^*}$

Fourier Representation

- A natural representation is in Fourier space

$$a(\mathbf{x},t) = \sum_{\mathbf{k}} a_{\mathbf{k}}(t)\, \exp\left(i\mathbf{k}\cdot\mathbf{x}\right)$$

- Evolution of each modes is given by

$$i\frac{\partial a_{\mathbf{k}}}{\partial t} = \frac{\delta \mathcal{H}}{\delta a_{\mathbf{k}}^*}$$

Hamiltonian Structure

Wave turbulence theory relies on weak nonlinearity to provide a small parameter ϵ to enable an expansion of the Hamiltonian in powers of the wave action variable $a_{\mathbf{k}}$

- Naïvely, we can assume that the wave amplitudes are small: $|a_{\mathbf{k}}| \sim \epsilon \ll 1$
- Expand $\mathcal{H}$ in powers of ϵ

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{int} \qquad \mathcal{H}_{int} = \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5 + \cdots$$

$\mathcal{H}_n$ describes n-wave processes

- Weak nonlinearity implies $\mathcal{H}_2 \gg \mathcal{H}_3 \gg \mathcal{H}_4 \gg \cdots$
- $\mathcal{H}_2 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^*$ correspond to linear energy of linear propagating waves
- Nonlinearities manifest themselves in terms of higher order powers of $a_{\mathbf{k}}$

e.g. 3-wave interactions:

$$\mathcal{H}_3 = \sum_{1,2,3} V^1_{2,3} \left(a_1 a_2^* a_3^* + a_1^* a_2 a_3 \right) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3)$$

Interaction coefficient: $V^1_{2,3} = V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$

Notation: $a_1 = a_{\mathbf{k_1}}$

Wave Amplitude Evolution Equation

- Explicitly, the mode evolution equation is

$$i\frac{\partial a_{\mathbf{k}}}{\partial t} = \omega_{\mathbf{k}} a_{\mathbf{k}} + \frac{\delta \mathcal{H}_{int}}{\delta a^*_{\mathbf{k}}}$$

- For linear dynamics, we observe that the complex mode $a_{\mathbf{k}}$ rotates by the linear frequency

- Nonlinear mode coupling (wave mixing) is characterized by the interaction Hamiltonian $\mathcal{H}_{int}$

- Many interesting cases possess scale invariance which is exploited by WT theory

$$\omega(\lambda \mathbf{k}) = \lambda^{\alpha} \omega(\mathbf{k})$$

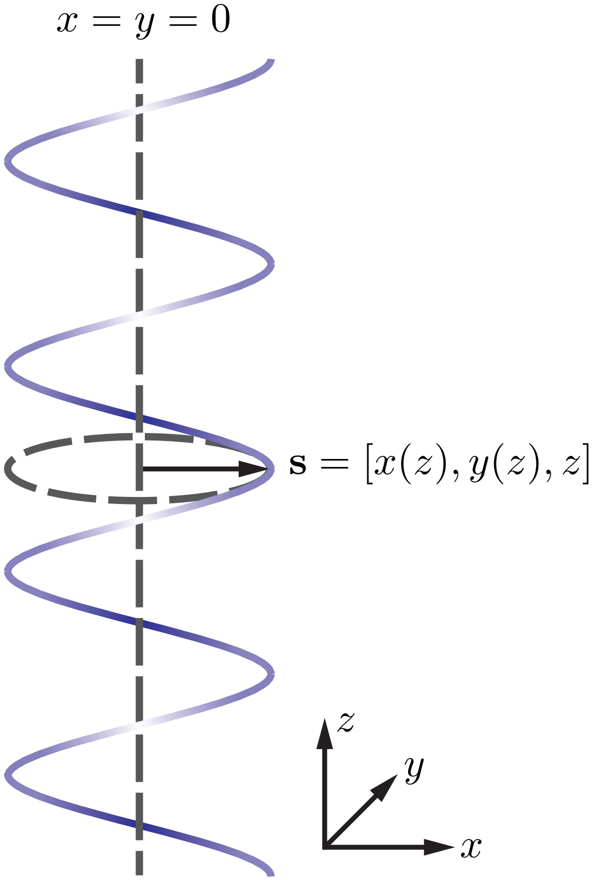
$$V(\lambda \mathbf{k}_1, \lambda \mathbf{k}_2, \lambda \mathbf{k}_3) = \lambda^{\beta} V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$$

Wave Turbulence Theory for Kelvin Waves

Wave turbulence theory can only be applied to Kelvin waves in the most idealized setups

- The most natural setup is studying Kelvin waves on 1D quantized vortex lines
- We consider the vortex filament model description:

$$\dot{\mathbf{s}} = \frac{\kappa}{4\pi} \oint_{\mathcal{L}} \frac{\mathbf{r} - \mathbf{s}}{|\mathbf{r} - \mathbf{s}|^3} \times \mathrm{d}\mathbf{r}$$



Theoretical setup

- Consider a single, quantized vortex line along $x = y = 0$ and periodic in z
- Parametrize 2D perturbations by $\mathbf{s} = [x(z), y(z), z]$
- Assume that $x(\cdot)$ and $y(\cdot)$ remain single valued (no self crossings)
- Define a complex canonical variable: $a(z) = x(z) + iy(z)$

Hamiltonian Description of Kelvin Waves

- The idealized setup just described can be written in terms of a Hamiltonian for $a(z)$

$$i\kappa\frac{\partial a}{\partial t} = \frac{\delta\mathcal{H}}{\delta a^*} \quad \text{with} \quad \mathcal{H} = \frac{\kappa^2}{4\pi}\int \frac{1 + \mathrm{Re}\left[a'^*(z_1)a'(z_2)\right]}{\sqrt{(z_1 - z_2)^2 + |a(z_1) - a(z_2)|^2}}\,\mathrm{d}z_1\mathrm{d}z_2$$

Svistunov, *Phys. Rev. B*, **52**, 3647, (1995)

- Observe that the Hamiltonian is divergent as $|z_1 - z_2| \to 0$
- Regularization of integral by introducing cutoff $|z_1 - z_2| = \xi$

Interpret ξ as vortex core radius

Additional conserved quantity

$$\mathcal{N} = \int |a(z)|^2\,\mathrm{d}z$$

Wave action

Weak nonlinearity expansion

- The small parameter in this problem is the Kelvin wave steepness $\epsilon = \dfrac{|a(z_1) - a(z_2)|}{|z_1 - z_2|} \ll 1$

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \mathcal{H}_6 + \cdots$$

- Only even powers of a appear in the Hamiltonian (associated to wave action conservation)

Dual Cascade Scenario

In systems with more than one invariant, we can get more than one cascade

- In the weak nonlinear regime the energy can be approximated by the linear energy:

$$\mathcal{H} \approx \mathcal{H}_2$$

- As $\mathcal{H}$ is conserved, then in the weakly nonlinear limit $\mathcal{H}_2$ will also be conserved at leading order

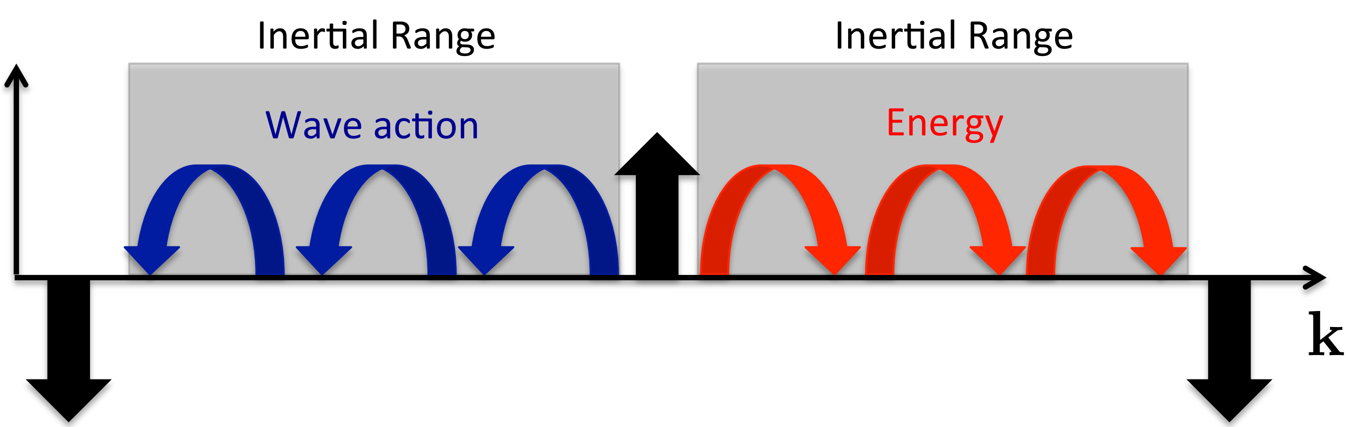
- By assuming linear energy conservation, we have two sign-definite quadratic invariants:

$$\mathcal{H}_2 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}} a^*_{\mathbf{k}} \qquad \mathcal{N} = \sum_{\mathbf{k}} a_{\mathbf{k}} a^*_{\mathbf{k}}$$

- Expect two constant flux cascades (c.f. two-dimensional turbulence)

Fjørtoft Argument

Fjørtoft, *Tellus*, **5**, 225, (1953)



Hamiltonian Fourier Representation

- Natural Fourier mode representation $a = \kappa^{-1/2} \sum_{\mathbf{k}} a_{\mathbf{k}}(t) \exp(i\,\mathbf{k}\,z)$
- We consider the first three Hamiltonian terms: $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4 + \mathcal{H}_6$

$$\mathcal{H} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^* + \frac{1}{4} \sum_{1,2,3,4} T_{3,4}^{1,2}\; a_1 a_2 a_3^* a_4^*\, \delta_{3,4}^{1,2} + \frac{1}{36} \sum_{1,2,3,4,5,6} W_{4,5,6}^{1,2,3}\; a_1 a_2 a_3 a_4^* a_5^* a_6^*\, \delta_{4,5,6}^{1,2,3}$$

$$\delta_{3,4}^{1,2} = \delta\left(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4\right) \qquad T_{3,4}^{1,2} = T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) \qquad a_1 = a_{\mathbf{k}_1}$$

Interaction coefficient expressions

$$\omega_{\mathbf{k}} = \frac{\kappa \mathbf{k}^2}{4\pi}\left[\ln\left(\frac{1}{\mathbf{k}\xi}\right) - \frac{3}{2}\right],$$

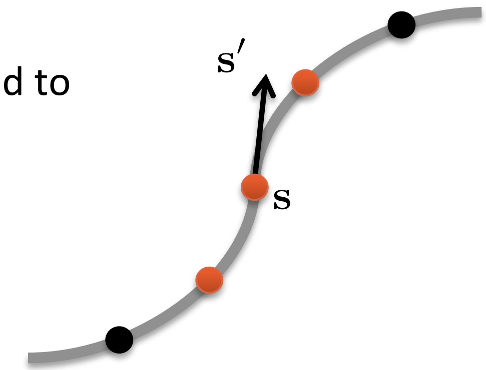
$$\omega_{\mathbf{k}} = \frac{\Lambda}{4\pi}\mathbf{k}^2 - \frac{\kappa}{4\pi}\mathbf{k}^2 \ln(\mathbf{k}\,\ell_{eff}), \qquad \Lambda = \ln\left(\ell_{eff}/\tilde{\xi}\right) \gg 1$$

$$T_{3,4}^{1,2} = -\frac{\Lambda}{4\pi}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4 - \frac{1}{16\pi}\left[5\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4 + \mathcal{F}_{3,4}^{1,2}\right]$$

$$W_{4,5,6}^{1,2,3} = \frac{9\Lambda}{8\pi\kappa}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5\mathbf{k}_6 + \frac{9}{32\pi\kappa}\left[7\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5\mathbf{k}_6 + \mathcal{G}_{4,5,6}^{1,2,3}\right]$$

- We separate the logarithmic divergent terms by introducing an effective scale ℓ_{eff}
- $\mathcal{F}_{3,4}^{1,2}$ and $\mathcal{G}_{4,5,6}^{1,2,3}$ are terms containing many logarithmic contributions

Local Induction Approximation



- At leading approximation in Λ the Hamiltonian can be simplified to

$$\mathcal{H} = \frac{\kappa^2 \Lambda}{2\pi} \int \sqrt{1 + |a'(z)|^2} \, \mathrm{d}z$$

- This is equivalent to

$$\dot{\mathbf{s}} = \frac{\kappa \Lambda}{4\pi} \, \mathbf{s}' \times \mathbf{s}''$$

The Local Induction Approximation

- Evolution of each vortex line element is determined only by the adjacent elements

- The LIA can be mapped to the one-dimensional nonlinear Schrödinger equation by the Hasimoto transformation

LIA corresponds to integrable dynamics

- Expect leading order LIA terms to not contribute to nonlinear Kelvin wave dynamics

Subleading $\mathcal{O}\left(\Lambda^0\right)$ terms in the interaction coefficients are essential for describing nonlinear Kelvin wave dynamics

Nonlinear Wave Resonance Condition

Consider the Kelvin wave Hamiltonian up to the leading nonlinear term: $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_4$

Mode evolution equation

$$i\frac{\partial a_{\mathbf{k}}}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*_{\mathbf{k}}} = \omega_k a_{\mathbf{k}} + \frac{1}{2}\sum_{1,2,3} T^{1,2}_{3,\mathbf{k}}\ a_1 a_2 a^*_3\ \delta^{1,2}_{3,\mathbf{k}}$$

- Fast linear oscillation by frequency $\omega_{\mathbf{k}}$
- Change into the rotating coordinate frame $b_{\mathbf{k}} = a_{\mathbf{k}}\ \exp\left(i\,\omega_{\mathbf{k}}\,t\right)$

$$i\frac{\partial b_{\mathbf{k}}}{\partial t} = \frac{1}{2}\sum_{1,2,3} T^{1,2}_{3,\mathbf{k}}\ b_1 b_2 b^*_3\ \delta^{1,2}_{3,\mathbf{k}}\ \exp\left(-i\,\omega^{1,2}_{3,\mathbf{k}}\,t\right)$$

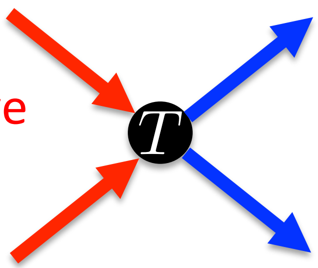
- Fast oscillating sum in time
- Main contribution in nonlinear term will be when $\omega^{1,2}_{3,\mathbf{k}} \equiv \omega_1 + \omega_2 - \omega_3 - \omega_{\mathbf{k}} = 0$
- Nonlinear interactions are dominated by waves satisfying

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}$$

$$\omega_1 + \omega_2 = \omega_3 + \omega_{\mathbf{k}}$$

Four-wave resonance condition

Four-wave process



Non-Resonant Four-Wave Interactions

What Kelvin waves satisfy the four-wave resonance condition?

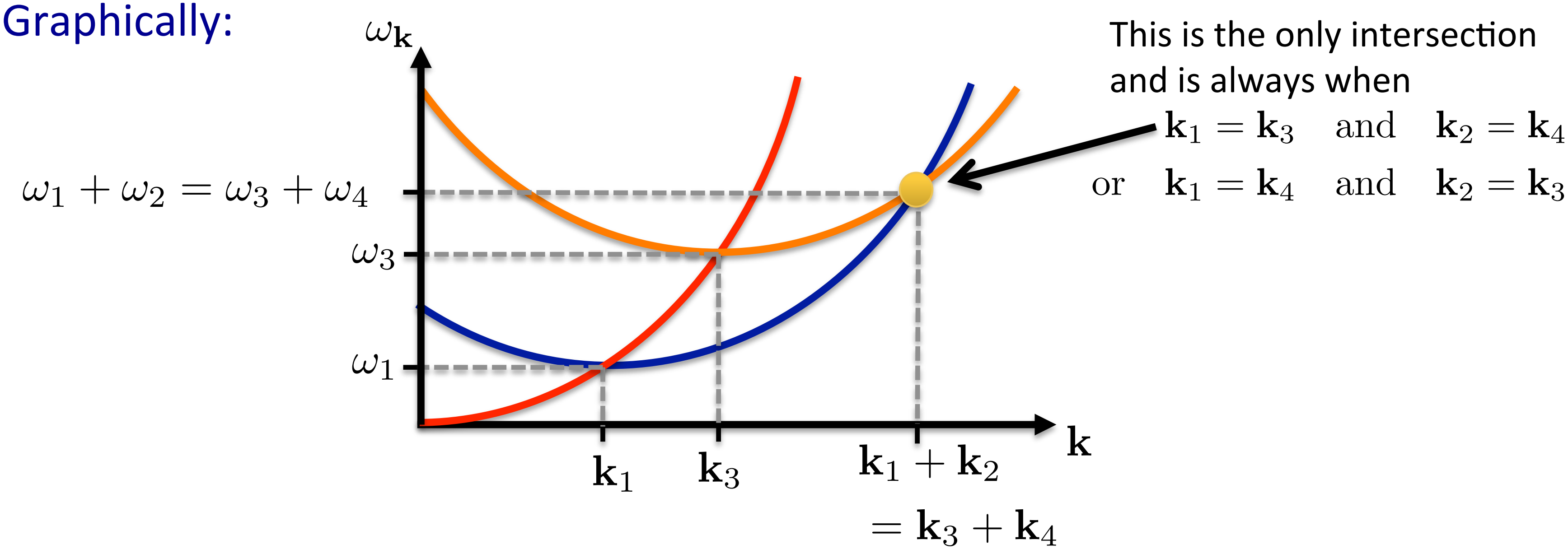
$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4, \quad \omega_1 + \omega_2 = \omega_3 + \omega_4$$

- Kelvin waves have a dispersion relation $\omega_{\mathbf{k}} \propto \mathbf{k}^2 \log(\mathbf{k}\xi)$

For one-dimensional systems:

Dispersion relations $\omega_{\mathbf{k}} \propto \mathbf{k}^{\alpha}$ with $\alpha \geq 2$ only yield trivial four-wave resonances

Graphically:



Trivial resonances do not transfer energy at leading order

Canonical Transformation

- If there are no non-trivial four-wave resonances we must consider the next order $\mathcal{H}_6$
- However, we would also like to eliminate the non-resonant four-wave term by changing coordinates

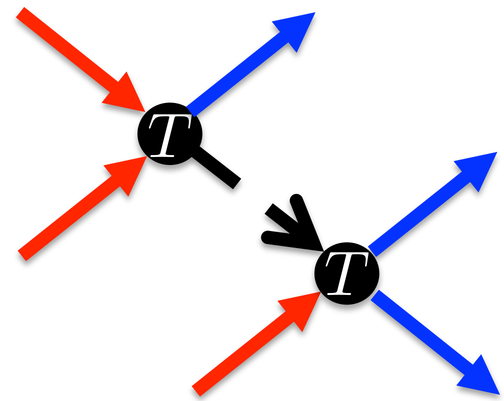
Canonical transformation

$$a_{\mathbf{k}}(t) = c_{\mathbf{k}}(t) + \mathcal{O}\left(c_{\mathbf{k}}^3\right)$$

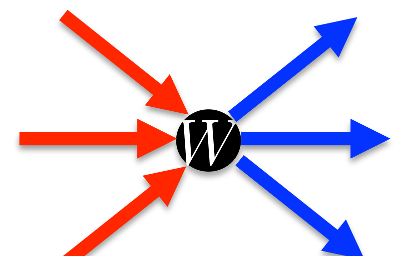
- We can choose the structure of the CT to eliminate the non-resonant four-wave term $\mathcal{H}_4$
- The penalty is in the introduction of a six-wave correction

$$\mathcal{H} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}} a_{\mathbf{k}}^* + \frac{1}{4} \sum_{1,2,3,4} W_{3,4}^{1,2} a_1 a_2 a_3^* a_4^* \delta_{3,4}^{1,2} + \frac{1}{36} \sum_{1,2,3,4,5,6} \tilde{W}_{4,5,6}^{1,2,3} a_1 a_2 a_3 a_4^* a_5^* a_6^* \delta_{4,5,6}^{1,2,3}$$

$$\tilde{W} = W + \frac{T \circ T}{\omega}$$



Six-wave processes



The non-resonant four-wave terms play an important role in the next order contribution

Six-wave resonance condition

$$\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = \mathbf{k}_4 + \mathbf{k}_5 + \mathbf{k}_6$$

$$\omega_1 + \omega_2 + \omega_3 = \omega_4 + \omega_5 + \omega_6$$

Explicit Expression of the Six-Wave Kernel

To understand Kelvin wave interactions it is essential to determine the six-wave interaction coefficient exactly

- Recall, that the Hamiltonian interaction coefficients were separated in logarithmic divergent $\mathcal{O}(\Lambda)$ terms and order one contributions:

$$\omega = \omega^\Lambda + \omega^1, \quad T = T^\Lambda + T^1, \quad W = W^\Lambda + W^1$$

- Using the expressions obtained from the canonical transformation and making an expansion in Λ

Six-wave interaction coefficient expression of $\mathcal{H}_6$

$$\tilde{W} = W^\Lambda + \frac{T^\Lambda \circ T^\Lambda}{\omega^\Lambda} + W^1 + \frac{T^1 \circ T^\Lambda}{\omega^\Lambda} + \frac{T^\Lambda \circ T^1}{\omega^\Lambda} + \frac{T^\Lambda \circ T^\Lambda}{(\omega^\Lambda)^2}\omega^1 + \mathcal{O}\left(\Lambda^{-1}\right)$$

Divergent terms that correspond to LIA cancel each other $= 0$

Resulting leading order terms describing Kelvin wave interactions

- $\tilde{W}^{1,2,3}_{4,5,6}$ consists of over 20,000 terms, but we know that $\tilde{W}^{1,2,3}_{4,5,6} \propto \mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5\mathbf{k}_6$

JL, L'vov, Nazarenko, Rudenko, *Phys. Rev. B*, **81**, 104526, (2010)

Summary II

Formulated the Kelvin wave setup

- Single periodic vortex line modelled by the Biot-Savart equation

Hamiltonian representation

- Dual invariant dynamics – energy and wave action conservation
- Computed the explicit expressions for the interaction coefficients

Four-wave resonances are absent for the Kelvin wave problem

- One-dimensionality and the structure of the linear frequency

Canonical Transformation

- Removed non-resonant four-wave terms
- Leading resonant interactions are six-wave terms

Six-wave interaction coefficient

- Performed a Λ^{-1}-expansion
- Leading (divergent) Λ contribution vanish through integrability of LIA
- Next $\mathcal{O}\left(\Lambda^0\right)$ terms are the relevant ones for Kelvin wave interactions

III. Wave Turbulence Theory

What Do We Want From a Statistical Description?

- Ideally, we would like the full joint PDF $\mathbb{P}(a_{\mathbf{k}_1}(t_1), a_{\mathbf{k}_2}(t_2), \dots)$
- Usually, we start with single time correlator functions:

$$C_2(\mathbf{k}_1, \mathbf{k}_2) = \left\langle a_{\mathbf{k}_1} a^*_{\mathbf{k}_2} \right\rangle$$
$$C_4(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) = \left\langle a_{\mathbf{k}_1} a_{\mathbf{k}_2} a^*_{\mathbf{k}_3} a^*_{\mathbf{k}_4} \right\rangle$$
$$\vdots \qquad\qquad \vdots$$

- Want to determine the evolution equations for the correlator functions

Closure problem

- Dynamics of correlators depend on higher order correlators

Wave turbulence theory provides a way to complete closure

Typical assumptions

- Ergodicity (ensemble averages can be replace by temporal averages)
- Random phases and random amplitudes
- Scale invariance
- Weak nonlinearity

Wave Turbulence Strategy

Of particular interest is the second order correlator function $\left\langle c_{\mathbf{k}} c_{\mathbf{k}_1}^* \right\rangle = n_{\mathbf{k}} \delta\left(\mathbf{k} - \mathbf{k}_1\right)$

- $n_{\mathbf{k}}$ is known as the wave action density

Procedure

- Consider an ϵ-expansion of the wave action variable: $c_{\mathbf{k}}(T) = c_{\mathbf{k}}^{(0)} + \epsilon^4 c_{\mathbf{k}}^{(1)} + \epsilon^8 c_{\mathbf{k}}^{(2)} + \cdots$
- Solve each level using the amplitude equation

$$i \frac{\partial c_{\mathbf{k}}}{\partial t} = \omega_{\mathbf{k}} c_{\mathbf{k}} + \frac{\epsilon^4}{12} \sum_{1,2,3,4,5} \tilde{W}_{4,5,\mathbf{k}}^{1,2,3} \, c_1 c_2 c_3 c_4^* c_5^* \, \delta_{4,5,\mathbf{k}}^{1,2,3}$$

- Substitute into $n_{\mathbf{k}} = \langle c_{\mathbf{k}} c_{\mathbf{k}}^* \rangle$ and perform averaging using the random phase and amplitude approximation
- Non-zero contribution from tenth order correlator that is represented as products of second order correlators (quasi-normal approximation)
- Take infinite box limit followed by the weak nonlinear limit (equivalently long nonlinear time limit) to obtain kinetic equation

Six-Wave Kinetic Equation for Kelvin-Waves

Collision integral

- The final expression of the kinetic equation is

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = \frac{\epsilon^8 \pi}{6} \int \left| \tilde{W}^{1,2,3}_{4,5,\mathbf{k}} \right|^2 \delta^{1,2,3}_{4,5,\mathbf{k}} \, \delta\left(\omega^{1,2,3}_{4,5,\mathbf{k}}\right) n_1 n_2 n_3 n_4 n_5 n_{\mathbf{k}}$$
$$\times \left[\frac{1}{n_{\mathbf{k}}} + \frac{1}{n_5} + \frac{1}{n_6} - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3} \right] \mathrm{d}\mathbf{k}_1 \, \mathrm{d}\mathbf{k}_2 \, \mathrm{d}\mathbf{k}_3 \, \mathrm{d}\mathbf{k}_4 \, \mathrm{d}\mathbf{k}_5$$

Nonlinear evolution equation for the wave action density

Kozik & Svistunov, *Phys. Rev. Lett*, **92**, 035301, (2004)

- Valid for weak nonlinearity and in the inertial range of scales
- We observe that the nonlinear evolution timescale $T_{nl} = \frac{n_{\mathbf{k}}}{(\partial n_{\mathbf{k}}/\partial t)} \sim \frac{1}{\epsilon^8}$
- The integral on the right-hand side is known at the *collision integral*
- Energy spectrum is recovered from $E_{\mathbf{k}} = \omega_{\mathbf{k}} n_{\mathbf{k}}$

The Stationary Kolmogorov-Zakharov Spectrum

Interested in stationary power-law solutions $n_{\mathbf{k}} = C\, k^{-x}$ of the kinetic equation where $k = |\mathbf{k}|$

- Solutions in which the five-dimensional collision integral is zero: $\dfrac{\partial n_{\mathbf{k}}}{\partial t} = 0$

Zakharov Transform

- Coordinate transformation that wraps six sub-domains onto one
- Integrand will now be exactly zero over the whole new domain for stationary solutions

$$\mathbf{k}_1 = \frac{\mathbf{k}^2}{\tilde{\mathbf{k}}_1}, \quad \mathbf{k}_2 = \frac{\mathbf{k}\tilde{\mathbf{k}}_2}{\tilde{\mathbf{k}}_1}, \quad \mathbf{k}_3 = \frac{\mathbf{k}\tilde{\mathbf{k}}_4}{\tilde{\mathbf{k}}_1}, \quad \mathbf{k}_4 = \frac{\mathbf{k}\tilde{\mathbf{k}}_4}{\tilde{\mathbf{k}}_1} \quad \text{and} \quad \mathbf{k}_5 = \frac{\mathbf{k}\tilde{\mathbf{k}}_5}{\tilde{\mathbf{k}}_1},$$

- Collision integral becomes with $y = 5x - 15$

$$0 = \frac{C^5 \epsilon^8 \pi}{6} \int |\tilde{W}^{1,2,3}_{4,5,\mathbf{k}}|^2 \delta^{1,2,3}_{4,5,\mathbf{k}}\ \delta(\omega^{1,2,3}_{4,5,\mathbf{k}})\, |\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5\mathbf{k}|^{-x} \{k_1^x + k_2^x + k_3^x - k_4^x - k_5^x - k^x\}$$
$$\times \left[\left(\frac{k_1}{k}\right)^y + \left(\frac{k_2}{k}\right)^y + \left(\frac{k_3}{k}\right)^y - \left(\frac{k_4}{k}\right)^y - \left(\frac{k_5}{k}\right)^y - 1\right] \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2\, \mathrm{d}\mathbf{k}_3\, \mathrm{d}\mathbf{k}_4\, \mathrm{d}\mathbf{k}_5$$

Thermodynamic solutions (zero flux)

$x = 0 \quad n_{\mathbf{k}} \propto k^0$ Equipartition of wave action

$x = 2 \quad n_{\mathbf{k}} \propto k^{-2}$ Equipartition of energy

Kolmogorov-Zakharov solutions (constant flux)

$y = 0 \quad n_{\mathbf{k}} \propto k^{-3}$ Constant wave action flux

Kozik-Svistunov energy spectrum

$y = 2 \quad n_{\mathbf{k}} \propto k^{-17/5}$ Constant energy flux

$E_{\mathbf{k}} = C_{KS}\, \kappa^{7/5} \Lambda\, \epsilon^{1/5} k^{-7/5}$

Locality of Kolmogorov-Zakharov Spectra

The Zakharov transform can only be applied if it does not result in divergences

- i.e. that the collision integral is convergent upon the solution
- Usually divergences will appear when either one or several of the integration variables go to 0 or ∞

Check for convergence when $\mathbf{k}_2 \to 0$

- First we must parameterize the resonant condition $\delta^{1,2,3}_{4,5,6}\,\delta(\omega^{1,2,3}_{4,5,6})$
- In this limit, the integration over $\mathbf{k}_2$ can be factorized out of the collision integral

$$\frac{\partial n_{\mathbf{k}}}{\partial t} \propto \int_0^{k^*} k_2^2\, n_2\, \mathrm{d}k_2$$

Divergent for $n_2 \propto k_2^{-x}$ with $x > 3$

- The direct energy Kozik-Svistunov spectrum $n_{\mathbf{k}} \propto k^{-17/5}$

Kozik-Svistunov spectrum makes collision integral divergent

JL *et al.*, *Phys. Rev. B*, **81**, 104526, (2010)

What does this mean?

- There is a nonlocal (in scale space) energy transfer from Kelvin waves
- The wave turbulence prediction (under the assumptions of locality) of $n_{\mathbf{k}} \propto k^{-17/5}$ is irrelevant and unphysical

The Nonlocal Kelvin-Wave Theory

The kinetic equation is divergent and the divergence takes the form $\Psi = \frac{1}{\kappa}\int k^2 n_{\mathbf{k}}\,\mathrm{d}k$

- Strongest divergence is when two wavenumbers vanish and singularity $\propto \Psi^2$
- Assume this occurs, and consider wave turbulence on slowly varying curved vortex line due to two large scale modes

Four-wave kinetic equation

L'vov, Nazarenko, *Low Temp. Phys.*, **36**, 785, (2010)

$$\frac{\partial n_{\mathbf{k}}}{\partial t} = \frac{\epsilon^8 \pi}{12}\int \left\{ |V_{\mathbf{k}}^{1,2,3}|^2 n_1 n_2 n_3 n_{\mathbf{k}} \left[\frac{1}{n_{\mathbf{k}}} - \frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_3}\right] \delta_{1,2,3}^{\mathbf{k}} \delta\left(\omega_{1,2,3}^{\mathbf{k}}\right) + 3|V_1^{\mathbf{k},2,3}|^2 n_1 n_2 n_3 n_{\mathbf{k}} \left[\frac{1}{n_1} - \frac{1}{n_{\mathbf{k}}} - \frac{1}{n_2} - \frac{1}{n_3}\right] \delta_{\mathbf{k},2,3}^{1} \delta\left(\omega_{\mathbf{k},2,3}^{1}\right) \right\} \mathrm{d}\mathbf{k}_1\, \mathrm{d}\mathbf{k}_2\, \mathrm{d}\mathbf{k}_3$$

Interaction Kernel: $V_{\mathbf{k}}^{1,2,3} \propto \Psi \mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3$

New constant energy flux Kolmogorov-Zakharov solution

$$E_{\mathbf{k}} = \omega_{\mathbf{k}} n_{\mathbf{k}} = C_{LN}\,\kappa\,\Lambda\,\epsilon^{1/3}\,\Psi^{-2/3}\,k^{-5/3}$$ L'vov-Nazarenko spectrum

- L'vov-Nazarenko spectrum corresponds to local wave interactions

Effective local four-wave process on background of two large scale modes

Local/Nonlocal Debate

The proof of nonlocality has not been universally accepted by the community

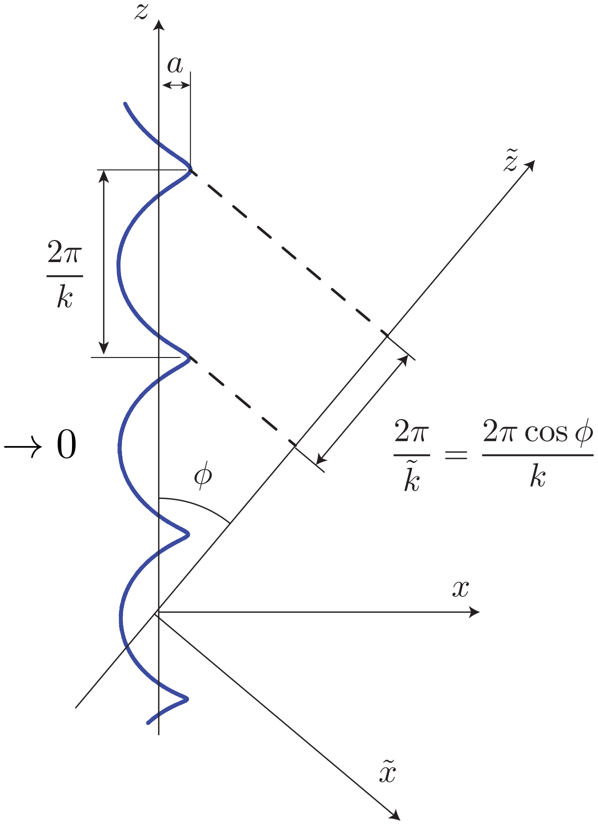
Criticism – geometric symmetries

- Nonlocal limit $\mathbf{k}_2 \to 0$ implies interaction is proportional to local tilt: $\tilde{W} \propto \mathbf{k}_2$
- Biot-Savart equation possesses tilt symmetry

Reorient line to remove local tilt contribution

- Nonlocal interactions must cancel and so $\tilde{W} \propto \mathbf{k}_2^2$ as $\mathbf{k}_2 \to 0$

Kozik-Svisuntov spectum is now local



Counter criticism

- Tilt symmetry does not prevent $\tilde{W} \propto \mathbf{k}_2$ asymptotic
- Tilt symmetry can only be applied in local frame of reference
- Globally, linear asymptotics still occur

Tilt symmetry arguments are irrelevant

1. Lebedev, L'vov, *J. Low Temp. Phys.*, **161**, 548,(2010)
2. Kozik, Svistunov, *J. Low Temp. Phys.*, **161**, 603, (2010)
3. Lebedev, L'vov, Nazarenko, *J. Low Temp. Phys.*, **161**, 606, (2010)
4. Kozik, Svistunov, *Phys. Rev. B*, **82**, 140510, (2010)
5. Sonin, *Phys. Rev. B*, **85**, 104516, (2012)
6. Lvov, Nazarenko, *Phys. Rev. B*, **86**, 226501, (2012)
7. Sonin, *Phys. Rev.* B, **86**, 226502, (2012)

Summary III

Derivation of the kinetic wave equation

- Multi-scale expansion, random phase and amplitude approximation, weak nonlinear and large box limits
- Studied steady state Kolmogorov-Zakharov solutions
- Energy spectrum solution leads to nonlocal wave interactions

Nonlocal Theory

- Local four-wave interactions derived from nonlocal six-wave interactions
- Four-wave interactions on slowly varying curved vortex line
- Slowly varying curved vortex line from two wave modes $\to 0$
- Proof of nonlocality and nonlocal theory not universally accepted

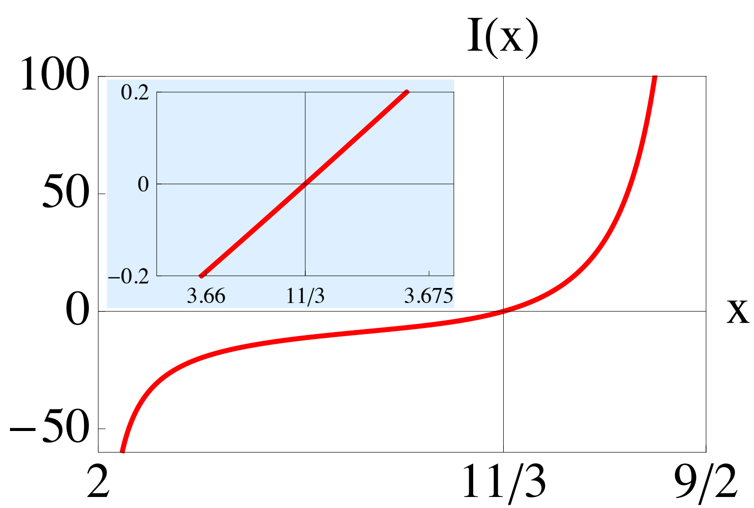
IV. Numerical Simulations and Experiments

Identification of the Local/Nonlocal spectrum

- Exponents of spectra are close, however spectrum prefactors should be different
- We can compute the expected prefactor of the spectrum from the four-wave kinetic equation
- Numerically compute the collision integral as a function of spectrum exponent

Four-wave collision Integral

$$I(x) = \int (q_1 q_2 q_3)^{2-x} \left(1 - q_1^y - q_2^y - q_3^y\right)\left(1 - q_1^x - q_2^x - q_3^x\right) \times \delta\left(1 - q_1^2 - q_2^2 - q_3^2\right) \delta\left(1 - \mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3\right) d\mathbf{q}_1 d\mathbf{q}_2 d\mathbf{q}_3$$



Boué *et al.*, *Phys. Rev. B*, **84**, 064516, (2011)

- Collision integral is convergent for $2 < x < 9/2$
- Vanishes on energy spectrum exponent $x = 11/3$

$$E_{\mathbf{k}} = C_{LN}\, \kappa\, \Lambda\, \epsilon^{1/3}\, \Psi^{-2/3}\, k^{-5/3}$$

Prefactor formula

$$C_{LN} = (128\pi)^{1/3} \left(\left. \frac{dI(x)}{dx} \right|_{x=11/3} \right)^{-1/3} = 0.304$$

Local Nonlinear Equation

Equation of motion in the nonlocal limit

JL, L'vov, Nazarenko, Rudenko, *Phys. Rev. B*, **81**, 104526, (2010)

- Hamiltonian simplifies in the nonlocal limit:

$$\tilde{W}^{1,2,3}_{4,5,6} \underset{\mathbf{k}_2,\mathbf{k}_3\to 0}{=} \frac{3}{4\pi\kappa}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4\mathbf{k}_5\mathbf{k}_6$$

Logarithmic contributions go to unity (20,000 terms go to 1)

- Results in the physical space representation of equation of motion:

$$i\frac{\partial a}{\partial t} = -\frac{\kappa}{4\pi}\frac{\partial}{\partial z}\left[\left(\Lambda - \frac{1}{4}\left|\frac{\partial a}{\partial z}\right|^4\right)\frac{\partial a}{\partial z}\right]$$

Nonlocal Kelvin wave interaction limit of the Biot-Savart equation

Local Nonlinear Equation

The simplicity of the local nonlinear equation implies easy numerical computation

Numerical Setup

$$i\frac{\partial a}{\partial t} = -\frac{\kappa}{4\pi}\frac{\partial}{\partial z}\left[\left(\Lambda - \frac{1}{4}\left|\frac{\partial a}{\partial z}\right|^4\right)\frac{\partial a}{\partial z}\right]$$

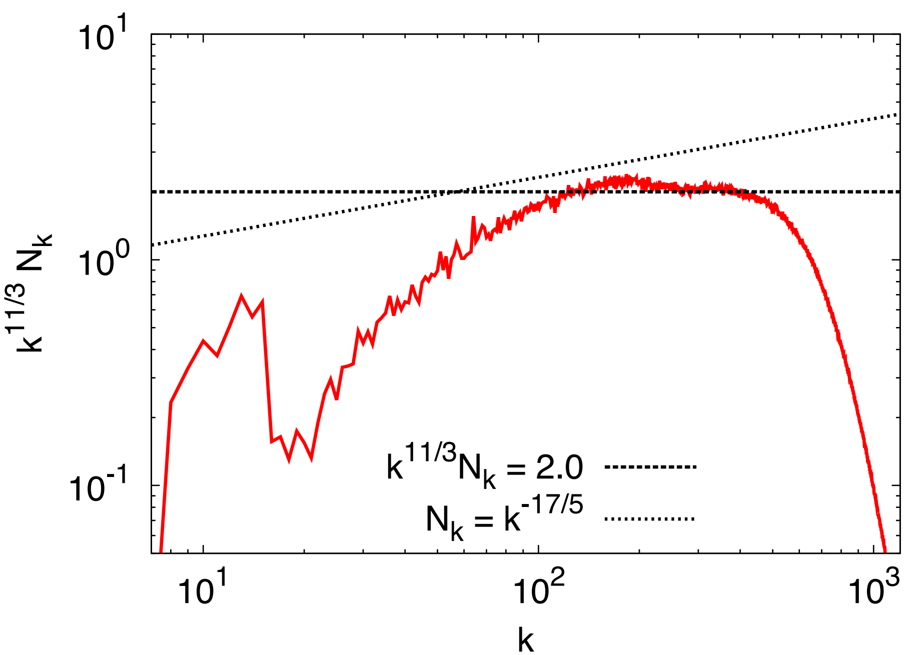
Boué *et al.*, *Phys. Rev. B*, **84**, 064516, (2011)

- Single periodic vortex line
- Include spectral forcing at large scales
- Additional dissipative terms (hyper- viscosity and friction) at both ends of Fourier space to prevent bottlenecks and to create non-equilibrium stationary state
- Observation of the L'vov-Nazarenko spectrum
- Numerical evaluation of Ψ and ϵ
- Prediction of prefactor

$$C_{num} = 0.347$$

- Within 14% of analytical result:

$$C_{LN} = 0.304$$

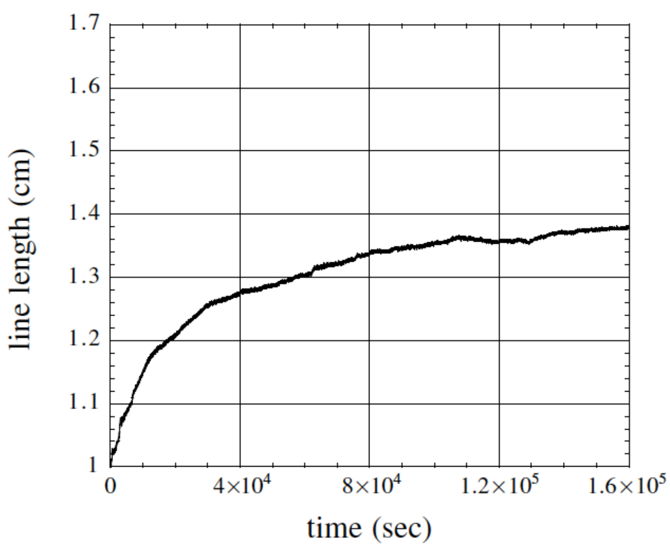


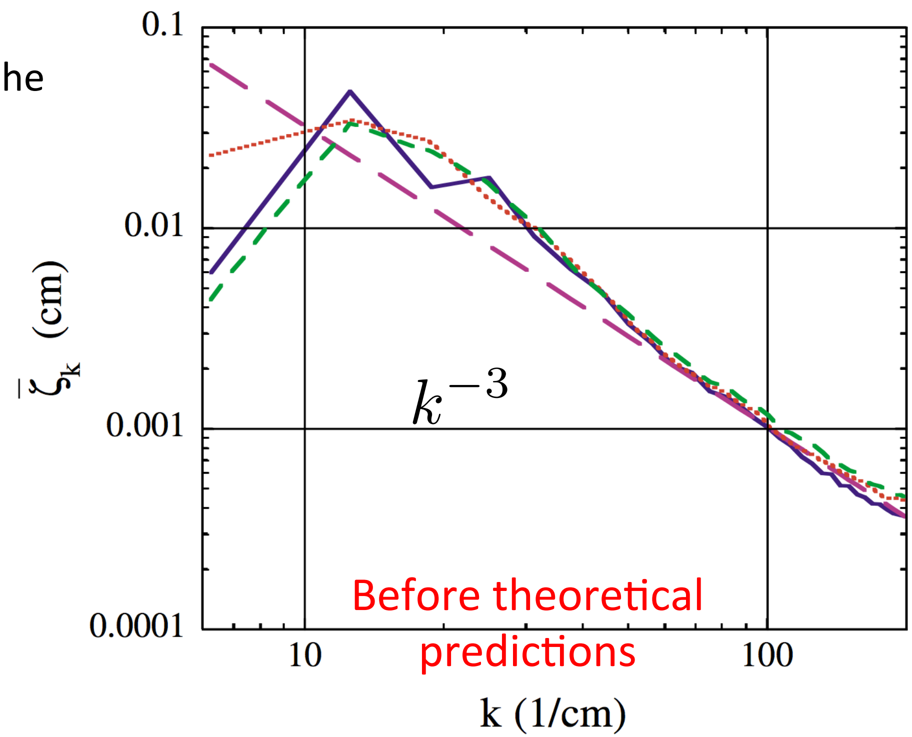
Biot-Savart Simulation I

Vortex filament model (Biot-Savart equation)

- Oscillate end of vortex line at a specific Kelvin wave frequency
- Observed power-law wave action spectrum behaviour
- Numerical data could agree with any of the theoretical predictions

$$\dot{\mathbf{s}} = \frac{\kappa}{4\pi} \oint_{\mathcal{L}} \frac{\mathbf{r} - \mathbf{s}}{|\mathbf{r} - \mathbf{s}|^3} \times \mathrm{d}\mathbf{r}$$





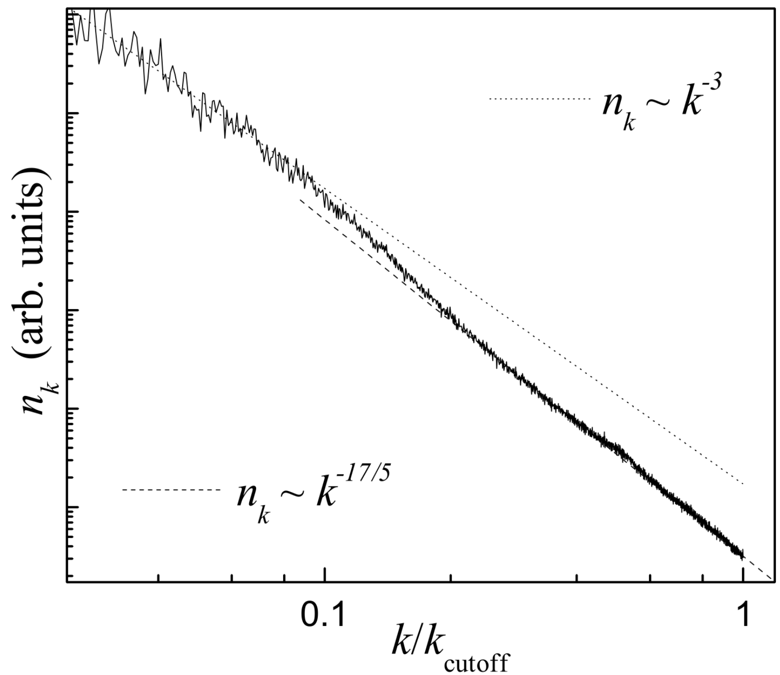
Vinen, Tsubota, Mitani, *Phys. Rev. Lett.*, **91**, 13501, (2003)

Biot-Savart Simulation II

- Simulation of the Biot-Savart equation in the Hamiltonian representation

$$i\kappa\frac{\partial a}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*} \qquad \mathcal{H} = \frac{\kappa^2}{4\pi}\int \frac{1 + \mathrm{Re}\left[a'^*(z_1)a'(z_2)\right]}{\sqrt{(z_1 - z_2)^2 + |a(z_1) - a(z_2)|^2}}\mathrm{d}z_1\mathrm{d}z_2$$

- Scale-separation scheme to approximate far field contributions
- Increases efficiency of the computation from $N^2 \to N \ln N$
- Decaying simulation with an Initial condition of $n_{\mathbf{k}} \propto k^{-3}$
- $n_{\mathbf{k}} \propto k^{-17/5}$ scaling observed at high wavenumbers
- Finite capacity spectrum so transient scalings in decaying setup should agree with theory



Kozik & Svistunov, *Phys. Rev. Lett.*, **94**, 025301, (2005)

Biot-Savart Simulation III

- Vortex filament model without any far field approximations

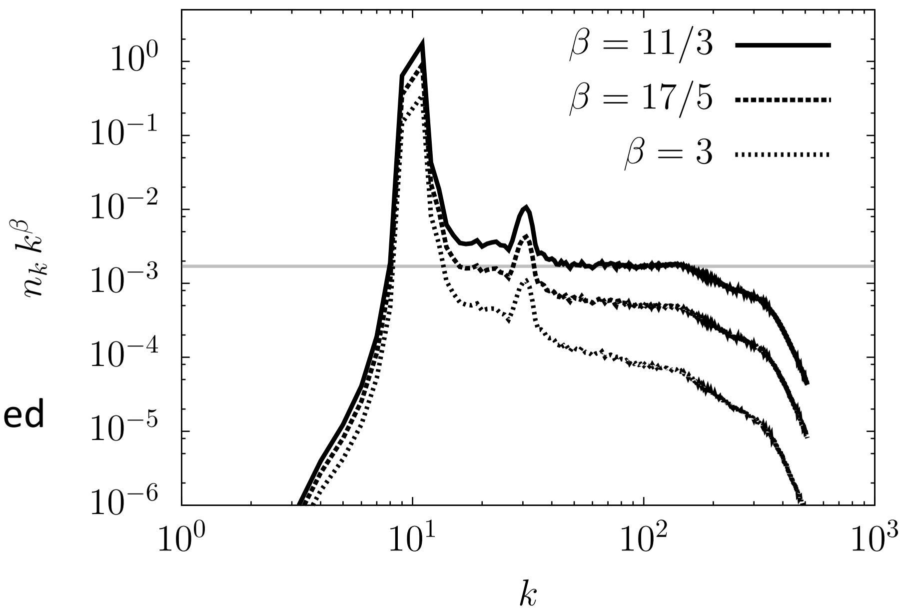
$$\dot{\mathbf{s}} = \frac{\kappa}{4\pi} \oint_{\mathcal{L}} \frac{\mathbf{r} - \mathbf{s}}{|\mathbf{r} - \mathbf{s}|^3} \times d\mathbf{r}$$

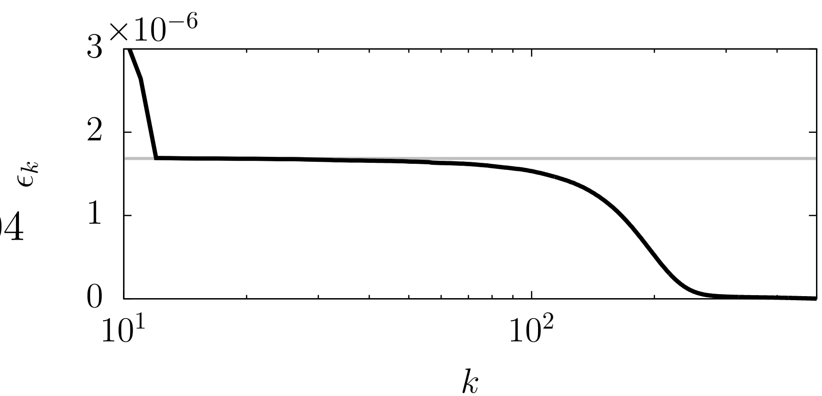
- Additive forcing and (exponential) dissipation filter at each time step
- Non-equilibrium stationary state reached from rest
- Evaluation of energy flux ϵ and Ψ
- L'vov-Nazarenko scaling observed
- Estimation of spectrum prefactor

$$C_{num} = 0.318$$

- Within 5% of analytical result $C_{LN} = 0.304$
- Assuming agreement to Kozik-Svistunov spectrum we get

$$C_{num} = 8.7 \times 10^{-3}$$





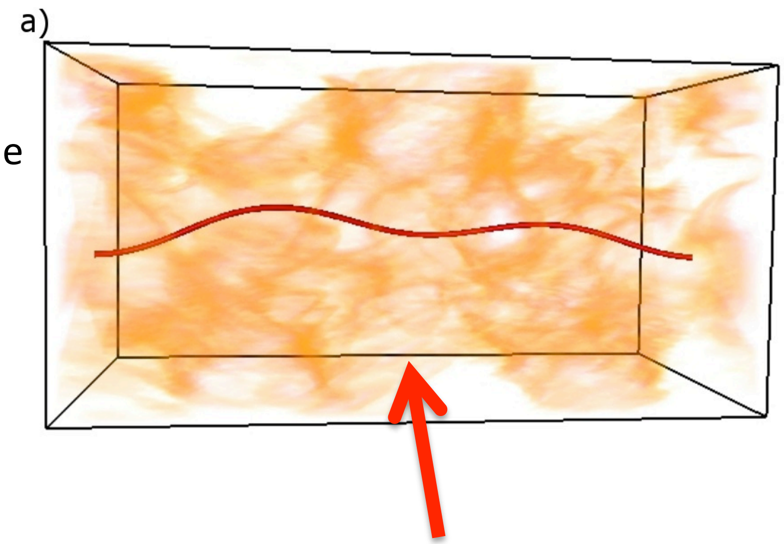
Baggaley & JL. *Phys. Rev. B*, **94**, 025301, (2014)

Gross-Pitaevskii Simulation

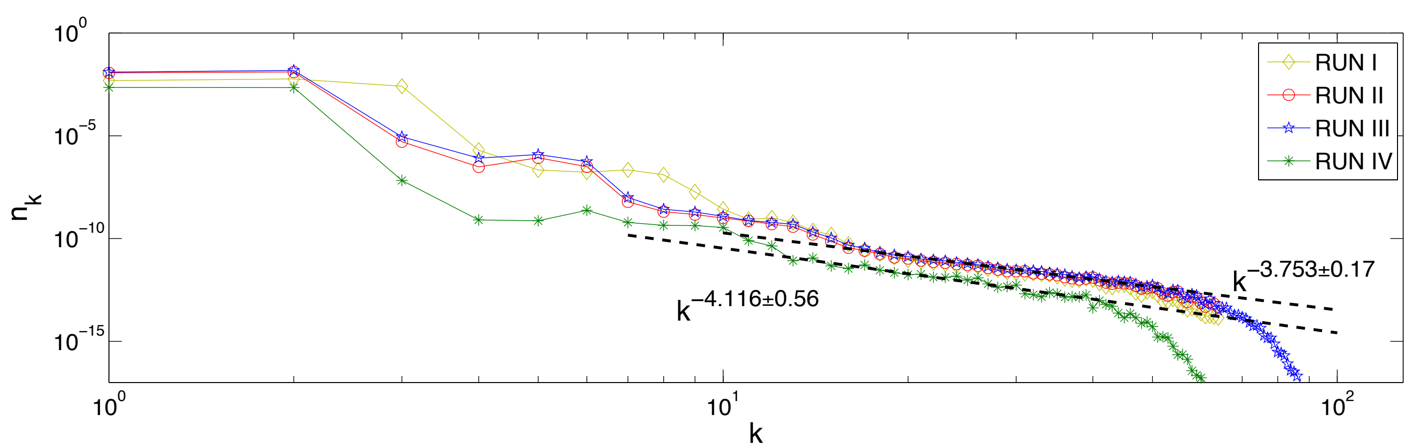
$$i\dot{\Psi} = -\nabla^2\Psi + \Psi\,|\Psi|^2$$

- Decaying simulation of an initial large-scale distribution of Kelvin waves on GP vortex
- Vortex core accurately tracked
- Wave action spectrum constructed from position of vortex core
- Nonlocal Kelvin wave prediction of L'vov-Nazarenko within error bars

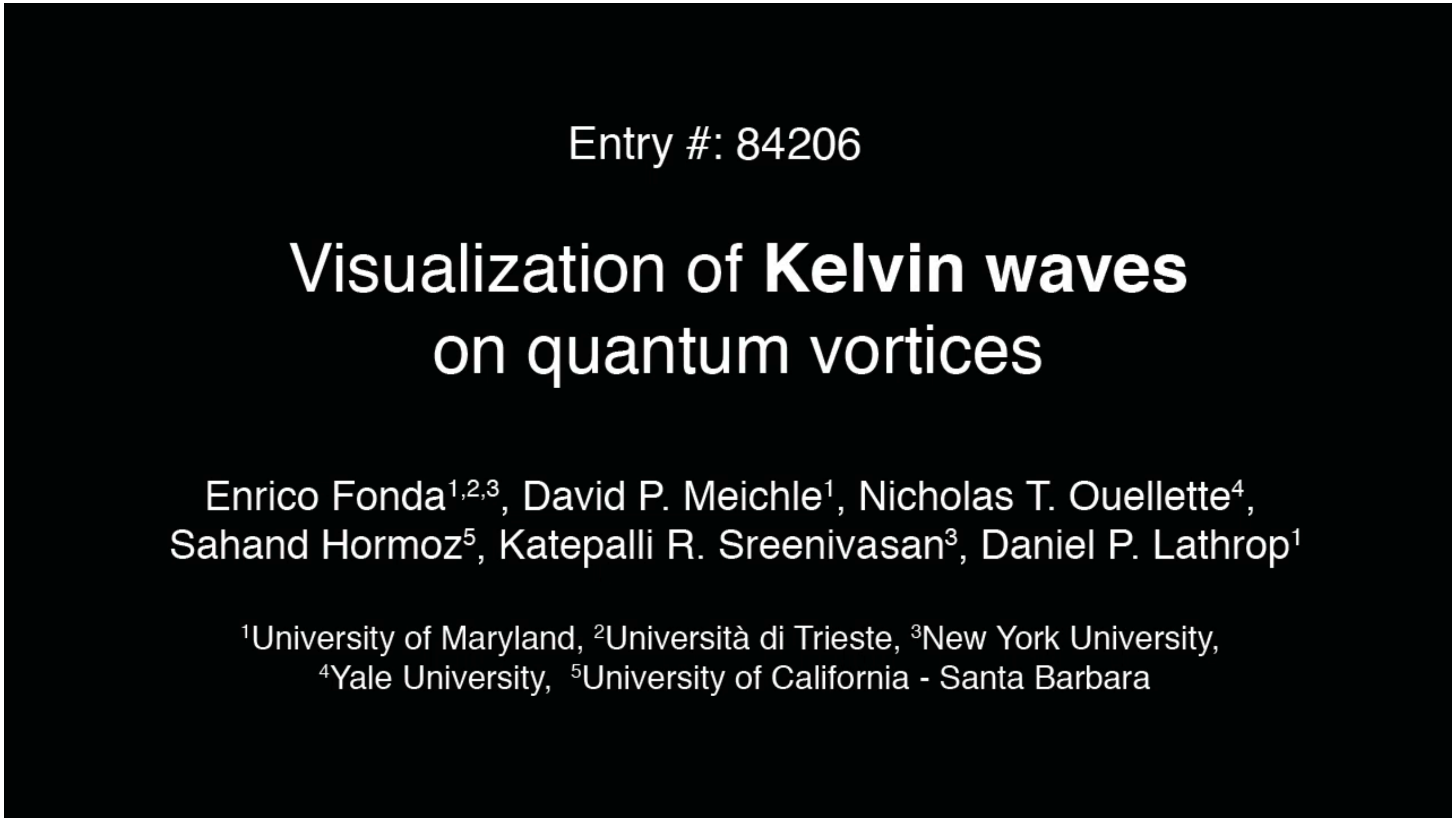
Krstulović, *Phys. Rev. E*, **86**, 055301, (2012)



Cloud of excited phonons



Experimental Observation of Kelvin Waves



Conclusions

Kelvin waves play a key role in the small scale energy transfer in ST

- Permit energy to reach extremely small scales for phonon emission in the zero temperature limit

Considered a Hamiltonian formulism for Kelvin waves

- Theoretical treatment applied to an idealized Kelvin wave setup

Wave turbulence theory of the Kelvin wave problem

- Leading order integrability (LIA), non-resonant four-wave interactions
- Extremely complex final six-wave interaction coefficient
- Nonlocality of the Kolmogorov-Zakharov energy spectrum
- On going debate about nonlocality proof

Kelvin waves in numerical simulations and experiments

- Local and nonlocal energy spectra are almost indistinguishable
- Computation of spectrum prefactor as an alternative way to identify spectra
- Latest numerical simulations seem to agree with nonlocal theory