Laboratoire de Mécanique des Fluides et d'Acoustique

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Wave turbulence in a rotating channel

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Introduction

- rotation: perhaps the simplest anisotropising influence on turbulence;

- most theoretical studies of rotating turbulence have assumed homogeneous and unbounded;

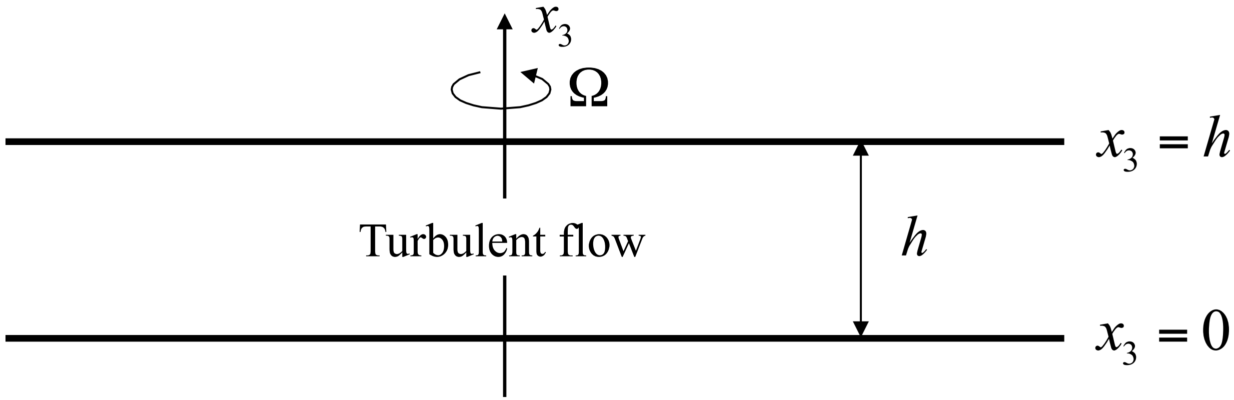
- the wave-turbulence approach has been applied to unbounded rotating turbulence (Galtier 2003, Bellet et al. 2006): energy transfer between resonant triads;

- however, turbulence is never really unconfined and homogeneous;

- there is a considerable literature on wave turbulence confined in all three dimensions (Nazarenko 2011, chapter 10): discrete modes.

It is of interest to study the intermediate case of confinement in just one direction: modes remain continuous, but turbulence is affected by confinement.

Formulation of the problem



- decaying rotating turbulence confined between solid walls
- zero mean flow
- homogeneous and isotropic in x_1, x_2, but not x_3
- small initial Rossby number: $\varepsilon = u'/2\Omega h$

Nondimensionalisation $\begin{cases} \mathbf{x} : h & t : (2\Omega)^{-1} \\ \mathbf{u} : 2\Omega h & p : 4\rho\Omega^2 h^2 \end{cases}$

Formulation of the problem

$$\frac{\partial u_i}{\partial t} + \underbrace{\varepsilon_{i3j} u_j}_{\substack{\text{Coriolis} \\ \text{term}}} + \underbrace{\frac{\partial}{\partial x_j}\left(u_i u_j\right)}_{\substack{\text{Nonlinear} \\ \text{term}}} = -\frac{\partial p}{\partial x_i} + \underbrace{E\frac{\partial^2 u_i}{\partial x_j \partial x_j}}_{\substack{\text{Viscous} \\ \text{term}}}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$u_i = 0 \qquad x_3 = 0, 1$$

$E = \nu / \left(2\Omega h^2\right)$ Ekman number

$u_i = O(\varepsilon) \qquad t = 0$ initial Rossby number

- small Rossby number $\Rightarrow$ weak nonlinearity
- small Ekman number, otherwise turbulence killed by viscosity before nonlinearity intervenes

Modal basis set

Dropping the nonlinear and viscous terms ⇒ inertial waveguide modes:

$$u_i = W_i^{(n)}(x_3;\mathbf{k})\exp\{ik_1x_1 + ik_2x_2 - i\omega_n(k)t\}$$

$$p = \Phi^{(n)}(x_3;\mathbf{k})\exp\{ik_1x_1 + ik_2x_2 - i\omega_n(k)t\}$$

Modes parametrised by:

$\mathbf{k} = (k_1, k_2)$ 2D wave vector

$-\infty < n < \infty$ mode order (integer):

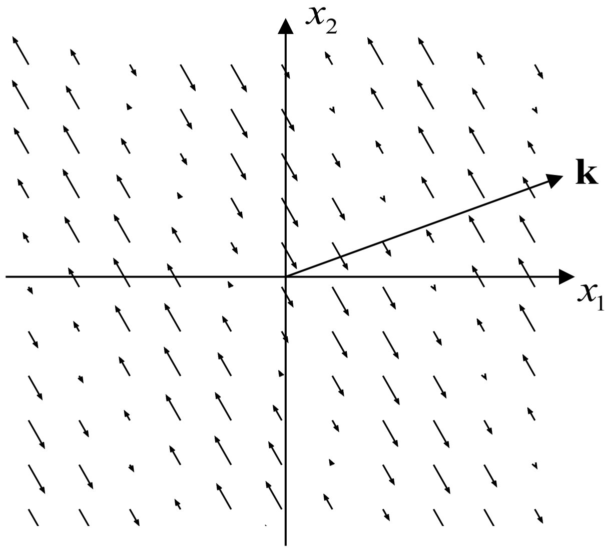
$$W_1^{(n)}, W_2^{(n)}, \Phi^{(n)} \propto \cos n\pi x_3 \qquad W_3^{(n)} \propto \sin n\pi x_3$$

Mode frequency: $$\omega_n(k) = \frac{n\pi}{\left(k^2 + n^2\pi^2\right)^{1/2}}$$

"Discretisation of the x_3 wave vector component"

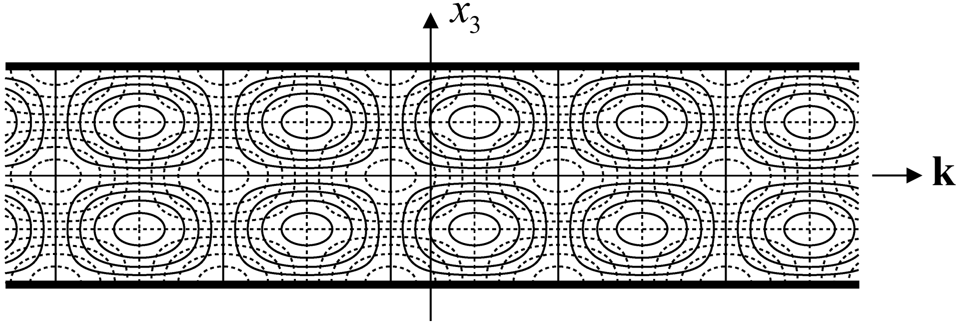
Modal basis set

Visualisations of two particular modes:



$n = 0$

2D, steady



$n = 2,\, k = 4$

Streamlines and contours of out-of-plane velocity component: the pattern moves to the right.

Modal basis set

The modal velocity fields form a complete set for solenoidal vector fields for which $u_3 = 0$ at the walls:

$$u_i(\mathbf{x},t) =$$

$$\sum_{n=-\infty}^{\infty} \int a_n(\mathbf{k},t) W_i^{(n)}(x_3;\mathbf{k}) \exp\left\{ ik_1x_1 + ik_2x_2 - i\omega_n(k)t \right\} d^2\mathbf{k}$$

$a_n(\mathbf{k},t)$ mode amplitude

- the modal basis is orthogonal
- taking the inner product of the momentum equation with the modal basis gives an evolution equation for the mode amplitudes …

Amplitude equation

$$\frac{\partial a_n}{\partial t} + \underbrace{E\left(k^2 + n^2\pi^2\right)a_n}_{\text{Volumetric viscous damping}} =$$

$$\underbrace{\sum_{n_p,\, n_q=-\infty}^{\infty} \int e^{i\left(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\mathbf{k}+\mathbf{p}|)\right)t} N_{nn_pn_q}\left(\mathbf{k},\mathbf{p}\right) a^*_{n_p}\left(\mathbf{p}\right) a^*_{n_q}\left(-\mathbf{k}-\mathbf{p}\right) d^2\mathbf{p}}_{\text{Nonlinearity}} +$$

$$\underbrace{\frac{1}{4\pi^2} E e^{i\omega_n(k)t} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{\partial u_i}{\partial x_3} W_i^{(n)^*}\right]_0^1 \exp\left\{-ik_1x_1 - ik_2x_2\right\} dx_1 dx_2}_{\text{Wall viscous damping}}$$

- * = complex conjugation
- amplitude evolution due to viscosity and nonlinearity
- two types of viscous damping
- three-mode nonlinear interaction $\begin{cases} \mathbf{k}+\mathbf{p}+\mathbf{q} = 0 \\ n \pm n_p \pm n_q = 0 \end{cases}$

Treatment of the wall viscous term: closure of the amplitude equation

The above amplitude equation is exact, but the wall viscous term ⇒ it is not closed. Assuming small Rossby and Ekman numbers from here on, linearised boundary-layer analysis can be used to express the wall term:

$$\frac{\partial a_n}{\partial t} + \underbrace{\Delta_n}_{\text{Viscous damping}} a_n = \underbrace{\sum_{n_p,\, n_q=-\infty}^{\infty} \int e^{i\left(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\mathbf{k}+\mathbf{p}|)\right)t} N_{n n_p n_q}(\mathbf{k},\mathbf{p})\, a^*_{n_p}(\mathbf{p})\, a^*_{n_q}(-\mathbf{k}-\mathbf{p})\, d^2\mathbf{p}}_{\text{Nonlinearity}}$$

$$\Delta_n = \underbrace{E^{1/2} D_n(k)}_{\text{Wall damping}} + \underbrace{E\left(k^2+n^2\pi^2\right)}_{\text{Volumetric damping}}$$

Wall damping due to Ekman pumping by the boundary layers.

The spectral matrix

Spectra are used to represent the energy distribution of 3D homogeneous turbulence. The equivalent here is a spectral matrix:

$$\overline{a_n^*(\mathbf{k},t)\,a_m(\mathbf{k}',t)} = A_{nm}(k,t)\,\delta(\mathbf{k}-\mathbf{k}')$$

Interpretation:

$$\frac{1}{2}\int_0^1 \overline{u_i u_i}\,dx_3 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\int A_{nn}(k,t)\,d^2\mathbf{k} \qquad \text{mean kinetic energy}$$

$A_{nn}(k)$ = energy distribution over modes (real and positive)

$A_{nm}(k) \quad n \neq m$ = correlations between different mode orders

$$\frac{1}{2}\int A_{nn}(k,t)\,d^2\mathbf{k} = \int_0^{\infty} \underbrace{E_n(k,t)}_{\text{Energy spectrum}}\,dk \qquad E_n = \pi k A_{nn}$$

Spectral evolution

The amplitude equation $\Rightarrow$

$$\frac{\partial A_{nm}(k)}{\partial t} + \underbrace{\left(\Delta_n^*(k) + \Delta_m(k)\right) A_{nm}(k)}_{\text{Viscosity}} = \underbrace{\tau_{nm}(k) + \tau_{mn}^*(k)}_{\text{Nonlinearity}}$$

$$\tau_{nm}(k) =$$

$$\sum_{n_p,\, n_q=-\infty}^{\infty} \int e^{-i\left(\omega_n(k) + \omega_{n_p}(p) + \omega_{n_q}(|\mathbf{k}+\mathbf{p}|)\right)t} N_{nn_pn_q}^*(\mathbf{k},\mathbf{p})\, \Theta_{mn_pn_q}(\mathbf{k},\mathbf{p})\, d^2\mathbf{p}$$

$\tau_{nm}(k)$ = spectral transfer matrix

$\Theta_{mn_pn_q}(\mathbf{k},\mathbf{p})$ = third-order spectral moments

- usual closure problem
- small Rossby and Ekman numbers $\Rightarrow$ slow spectral evolution

2D/wave decomposition

At this point we would like to exploit the small Rossby number using the wave turbulence approach to closure. However, there is a problem:

$\boxed{n = 0 \text{ modes have zero frequency} \Rightarrow \text{they are nondispersive}}$

A precondition of wave-turbulence theory being dispersion, it cannot be used for $n = 0$ modes $\Rightarrow$

$$u_i = \underbrace{U_i}_{2D} + \underbrace{v_i}_{\text{wave}} \qquad \text{2D/wave decomposition}$$

$$U_i = \int a_0(\mathbf{k},t) W_i^{(0)}(\mathbf{k}) \exp\{ik_1x_1 + ik_2x_2\} d^2\mathbf{k}$$

- $\mathbf{U}$ independent of x_3 with $U_3 = 0$, hence the name, 2D
- $\mathbf{U}$ slowly varying, whereas $\mathbf{v}$ oscillates over time spans of $O(1)$

Evolution of the 2D component

$$\frac{\partial U_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta}\left(U_\alpha U_\beta\right) = -\frac{\partial \Pi}{\partial x_\alpha} + E\,\frac{\partial^2 U_\alpha}{\partial x_\beta \partial x_\beta} - \underbrace{\left(2E\right)^{1/2} U_\alpha}_{\text{Wall friction}} - \underbrace{\frac{\partial}{\partial x_\beta}\left(\int_0^1 \mathrm{v}_\alpha \mathrm{v}_\beta \; dx_3\right)}_{\text{Forcing by the wave component}}$$

- α, β are indices which run over the values $1, 2$ and to which the summation convention applies
- both nonlinear terms are of $O\left(\varepsilon^2\right)$, but the oscillations of the wave-forcing term ⇒ it has little cumulative effect over the long time spans needed for evolution of $\mathbf{U}$ ⇒

$$\frac{\partial U_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta}\left(U_\alpha U_\beta\right) = -\frac{\partial \Pi}{\partial x_\alpha} + E\,\frac{\partial^2 U_\alpha}{\partial x_\beta \partial x_\beta} - \underbrace{\left(2E\right)^{1/2} U_\alpha}_{\text{Wall friction}}$$

- evolves as a classical 2D flow with wall friction
- initial time scale for evolution of $O\left(\varepsilon^{-1}\right)$
- 2D inverse energy cascade ⇒ dominated by small k at the large εt necessary for evolution of the wave component

Results of wave-turbulence analysis: energy equation of the wave component

Applying wave-turbulence theory to determine $\tau_{nn}(k)\,(n \neq 0) \Rightarrow$ the energy equation of the wave component is

$$\frac{\partial A_{nn}(k)}{\partial t} + 2\Re\left(\Delta_n(k)\right) A_{nn}(k) = \qquad n \neq 0$$

$$\sum_{n_p, n_q \neq 0} \fint_{C_{nn_pn_q}(\mathbf{k})} \frac{A_{n_pn_p}(p)\left(\eta_{nn_pn_q}(\mathbf{k},\mathbf{p}) A_{nn}(k) + \lambda_{nn_pn_q}(\mathbf{k},\mathbf{p}) A_{n_qn_q}(|\mathbf{k}+\mathbf{p}|)\right)}{\Gamma_{n_pn_q}(\mathbf{k},\mathbf{p})} |d\mathbf{p}|$$

$C_{nn_pn_q}(\mathbf{k})$ = resonance curve : $\omega_n(k) + \omega_{n_p}(p) + \omega_{n_q}(|\mathbf{k}+\mathbf{p}|) = 0$

- *remarkably, the above system of equations is closed (the* $n_p, n_q = 0$ *contributions to the right-hand side have cancelled out)*
- only the diagonal elements of A_{nm} (energy distributions) appear
- the right-hand side describes energy transfer between resonant triads of $n \neq 0$ modes
- the second term on the left describes viscous dissipation

Results of wave-turbulence analysis: energy equation of the wave component

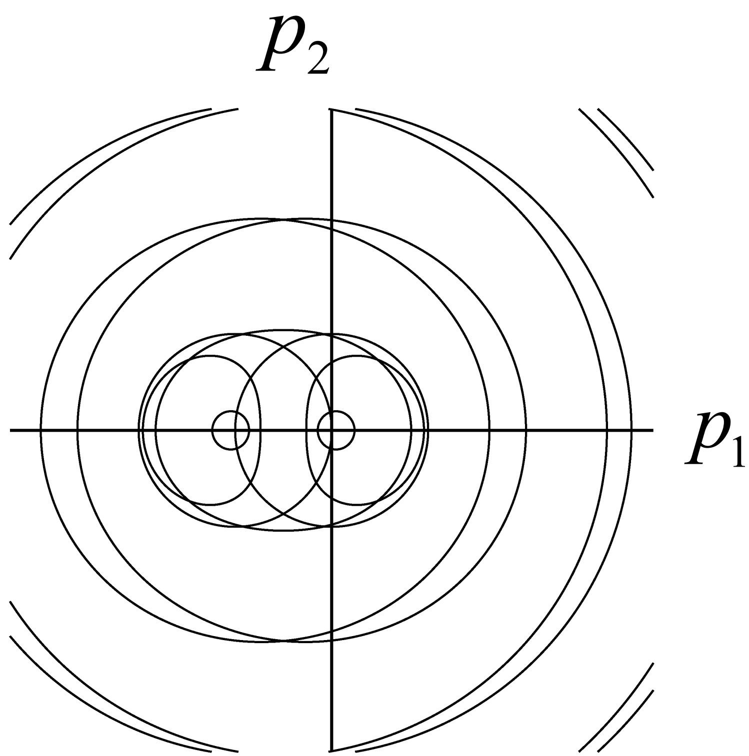
- the energy distributions of the wave component evolve on a time scale of $O\left(\varepsilon^{-2}\right)$
- the total *wave* energy is conserved by resonant triad interactions
- the above equation preserves positivity of A_{nn} (realisability)

$$\eta_{nn_pn_q}\left(\mathbf{k},\mathbf{p}\right)=8\Re\left(N^*_{nn_pn_q}\left(\mathbf{k},\mathbf{p}\right)N_{n_qn_pn}\left(-\mathbf{k}-\mathbf{p},\mathbf{p}\right)\right)$$

$$\lambda_{nn_pn_q}\left(\mathbf{k},\mathbf{p}\right)=4\left|N_{nn_pn_q}\left(\mathbf{k},\mathbf{p}\right)\right|^2$$

$$\Gamma_{n_pn_q}\left(\mathbf{k},\mathbf{p}\right)=\left|\frac{n_p\mathbf{p}}{\left(p^2+n_p^2\pi^2\right)^{3/2}}+\frac{n_q\left(\mathbf{k}+\mathbf{p}\right)}{\left(\left|\mathbf{k}+\mathbf{p}\right|^2+n_q^2\pi^2\right)^{3/2}}\right|$$

Results of wave-turbulence analysis: sample resonance curves



Resonance curves:

$$C_{nn_pn_q}(\mathbf{k})$$

$$n = 2, k_1 = 3, k_2 = 0$$

all n_p, n_q such that

$$n \pm n_p \pm n_q = 0$$

Results of wave-turbulence analysis: wave-mode correlations

For the off-diagonal elements of A_{nm} (correlations):

$$\frac{\partial A_{nm}(k)}{\partial t} + \left(\Xi_n^*(k) + \Xi_m(k)\right) A_{nm}(k) = 0 \qquad n \neq m, \quad n, m \neq 0$$

$$\Xi_n(k) = \Delta_n(k) - \frac{1}{2} \sum_{n_p, n_q = -\infty}^{\infty} \oint_{C_{nn_pn_q}(\mathbf{k})} \frac{\eta_{nn_pn_q}(\mathbf{k},\mathbf{p}) A_{n_pn_p}(p)}{\Gamma_{n_pn_q}(\mathbf{k},\mathbf{p})} |d\mathbf{p}|$$

$$- \frac{1}{2} i \sum_{n_p, n_q = -\infty}^{\infty} \int \frac{\eta_{nn_pn_q}(\mathbf{k},\mathbf{p}) A_{n_pn_p}(p)}{\omega_n(k) + \omega_{n_p}(p) + \omega_{n_q}(|\mathbf{k}+\mathbf{p}|)} d^2\mathbf{p}$$

- the sign of $\Re\left(\Xi_n^*(k) + \Xi_m(k)\right)$ controls the growth or decay of A_{nm}
- the surface integral should be interpreted as a Cauchy principal value

Results of wave-turbulence analysis: wave-mode correlations

$$\Re\left(\Xi_n\left(k\right)\right) = \Re\left(\Delta_n\left(k\right)\right) - \frac{1}{2}\sum_{n_p, n_q = -\infty}^{\infty} \fint_{C_{nn_pn_q}(\mathbf{k})} \frac{\eta_{nn_pn_q}\left(\mathbf{k},\mathbf{p}\right) A_{n_pn_p}\left(p\right)}{\Gamma_{n_pn_q}\left(\mathbf{k},\mathbf{p}\right)} \left|d\mathbf{p}\right|$$

The $n_p = 0$ contribution is

$$-\frac{1}{2}\sum_{n_q = \pm n} \fint_{C_{n0n_q}(\mathbf{k})} \frac{\eta_{n0n_q}\left(\mathbf{k},\mathbf{p}\right) A_{00}\left(p\right)}{\Gamma_{0n_q}\left(\mathbf{k},\mathbf{p}\right)} \left|d\mathbf{p}\right|$$

Because the 2D energy has inverse cascaded to small wavenumbers at the large εt considered here, the integral is dominated by small p. Asymptotic evaluation leads to

$$\Re\left(\Xi_n\left(k,t\right)\right) \sim \Re\left(\Delta_n\left(k\right)\right) + 2\left|n\pi\right|^{-1} k\left(k^2 + n^2\pi^2\right)^{3/2} \int_0^\infty p^{-1} E_0\left(p\right) dp$$

- positivity $\Rightarrow$ decorrelation of $n \neq 0$ modes

Results of wave-turbulence analysis: wave-mode correlations

In the absence of p^{-1}, the integral

$$\int_0^\infty p^{-1} E_0(p)\,dp$$

would be the 2D energy, $O\left(\varepsilon^2\right)$. Small p means that

$$\Re\left(\Xi_n(k,t)\right) >> \varepsilon^2$$

$\Rightarrow$ decorrelation of $n \neq 0$ modes on a time scale asymptotically small compared with that, $O\left(\varepsilon^{-2}\right)$, necessary for energy transfer between such modes.

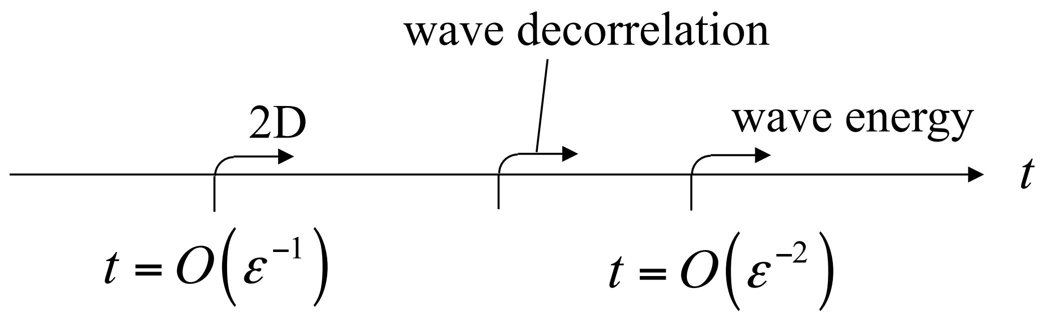
- the presence of the 2D component induces relatively rapid, but energy conserving, decorrelation of the wave modes
- this is presumably due to elastic (in the usual sense of energy conserving) random scattering of the waves by 2D vortices

Conclusions

☞ representation of the flow using inertial waveguide modes
☞ derivation of the modal amplitude equations
☞ spectral matrix: diagonal (energy) and off-diagonal (correlations) elements
☞ 2D/wave decomposition
☞ 2D component: classical 2D flow, initial evolution $t = O\left(\varepsilon^{-1}\right)$
☞ application of wave-turbulence analysis to the wave component
☞ the wave energy distributions evolve independently of the 2D component on a time scale $O\left(\varepsilon^{-2}\right)$ due to transfer of energy between resonant triads of modes
☞ the wave mode correlations are strongly affected by the 2D component, leading to decorrelation on a time scale $o\left(\varepsilon^{-2}\right)$
☞ this can be interpreted as due to elastic random scattering by the widely separated 2D vortices which make up the 2D component at this stage of its evolution

Conclusions

Summary of flow evolution at small Rossby numbers:



☞ the time scale $t = O\left(\varepsilon^{-1}\right)$ is the usual large-eddy turnover time

☞ $t = O\left(\varepsilon^{-2}\right)$ is asymptotically larger, reflecting suppression of nonlinear energy transfers by rotation away from resonance

It remains to numerically implement the wave-energy equations and compare the results with DNS.