Relative dispersion of tracers in turbulent flows

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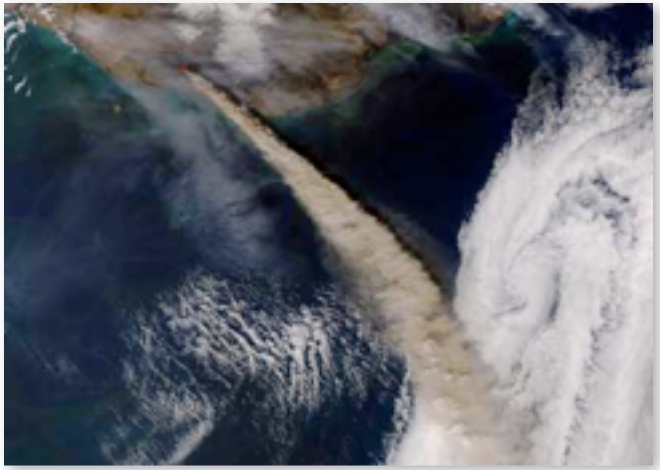
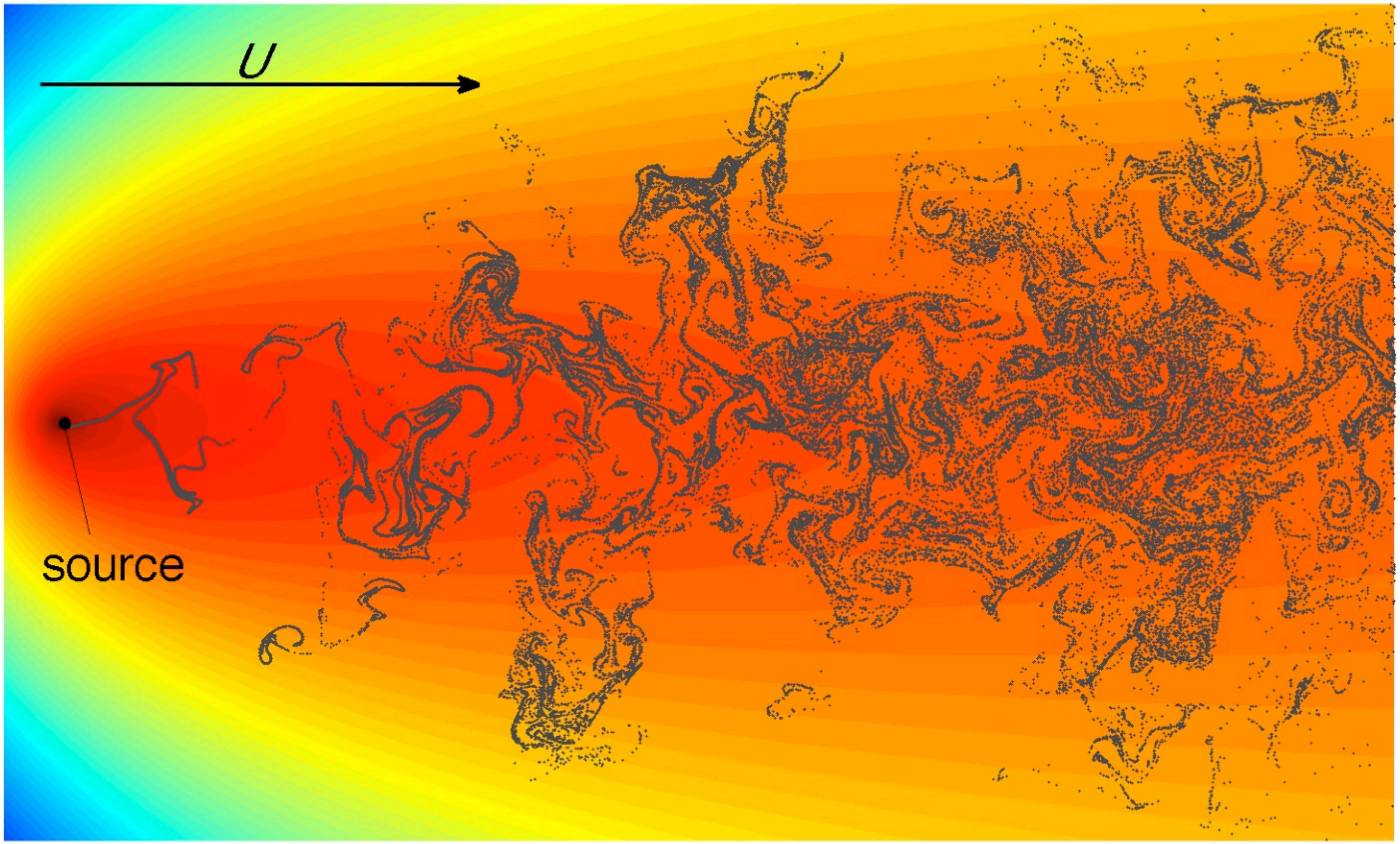
Fluctuations in atmospheric transport



- Fluctuations are important for risk assessments
- **Models/Observations:** space and/or time averages

Mean vs. meandering plumes

- Averaged concentration is well described by eddy diffusivity



- PDFs have tails rather far from Gaussian
- Spatial correlations relates to relative motion of tracers

Atmospheric diffusion

- Concentration field: passive scalar $\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = \kappa \nabla^2 \theta$
- **Batchelor scale:** $\ell_\mathrm{B} = \eta \sqrt{\kappa/\nu}$

  $\eta = \varepsilon^{-1/4} \nu^{3/4}$ Kolmogorov viscous dissipative scale

  ν fluid kinematic viscosity ε kinetic energy dissipation rate

  ozone in air $\kappa \approx 0.14\,\mathrm{cm}^2\,\mathrm{s}^{-1} \Rightarrow \ell_\mathrm{B} \approx 0.8\,\eta \approx 0.8\,\mathrm{mm}$

  1µm aerosol $\kappa \approx 2\,.\,10^{-7} \mathrm{cm}^2\,\mathrm{s}^{-1} \Rightarrow \ell_\mathrm{B} \approx 10^{-3}\,\eta \approx 1\,\mu\mathrm{m}$

- Above ℓ_B, advection dominates ⇒ **Turbulent diffusion** (Taylor 1921)

  Lagrangian tracer: $\dot{\boldsymbol{x}}(t) = \boldsymbol{u}(\boldsymbol{x}(t), t) + \sqrt{2\kappa}\,\boldsymbol{\eta}(t)$

  $\theta \approx$ PDF of the position

$$\langle |\boldsymbol{x}(t) - \boldsymbol{x}(0)|^2 \rangle = \int_0^t \int_0^t \langle \boldsymbol{u}(\boldsymbol{x}(s), s) \cdot \boldsymbol{u}(\boldsymbol{x}(s'), s') \rangle \,\mathrm{d}s\,\mathrm{d}s' + 2\kappa\, t \simeq 2(T_\mathrm{L} u_\mathrm{rms}^2 + \kappa)\, t$$

$$\Rightarrow \quad \partial_t \langle \theta \rangle = -\nabla \cdot \langle \boldsymbol{u}\, \theta \rangle + \kappa \nabla^2 \langle \theta \rangle \approx (\kappa_\mathrm{eff} + \kappa)\, \nabla^2 \langle \theta \rangle$$

Fluctuations and relative dispersion

- Tracers = characteristics of the advection equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{u}(\boldsymbol{x}(t),t) \quad \Rightarrow \quad \theta(\boldsymbol{x}(t),t) = \theta_0(\boldsymbol{x}(0))$$

- Spatial correlations of the concentration

$$\langle\theta(\boldsymbol{x}+\boldsymbol{r},t)\,\theta(\boldsymbol{x},t)\rangle = \iint \langle\theta_0(\boldsymbol{x}_1^0)\,\theta_0(\boldsymbol{x}_2^0)\rangle\, p_2(\boldsymbol{x}+\boldsymbol{r},\boldsymbol{x},t\,|\,\boldsymbol{x}_1^0,\boldsymbol{x}_2^0,0)\,\mathrm{d}\boldsymbol{x}_1^0\mathrm{d}\boldsymbol{x}_2^0$$

$p_2(\boldsymbol{x}_1,\boldsymbol{x}_2,t\,|\,\boldsymbol{x}_1^0,\boldsymbol{x}_2^0,0)$ = joint transition probability density of two tracers $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$

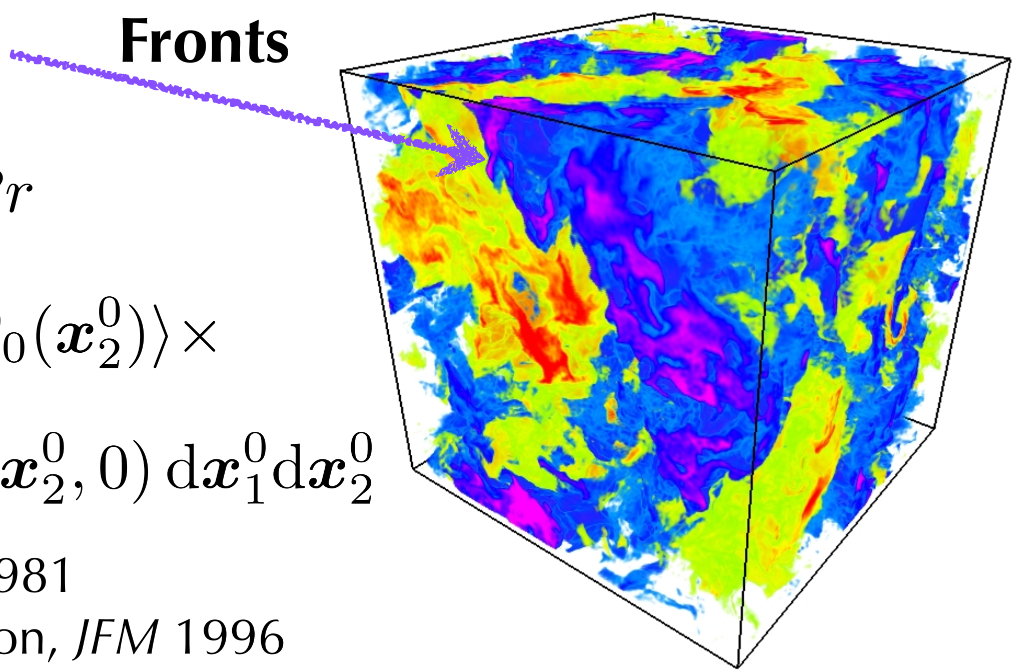
- Scalar dissipation anomaly

$$\varepsilon_\theta = -\kappa\langle(\nabla\theta)^2\rangle \to const$$

when $\kappa,\nu\to 0$ with fixed Pr

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\theta(\boldsymbol{x},t)^2\rangle = \iint \langle\theta_0(\boldsymbol{x}_1^0)\theta_0(\boldsymbol{x}_2^0)\rangle\times$$

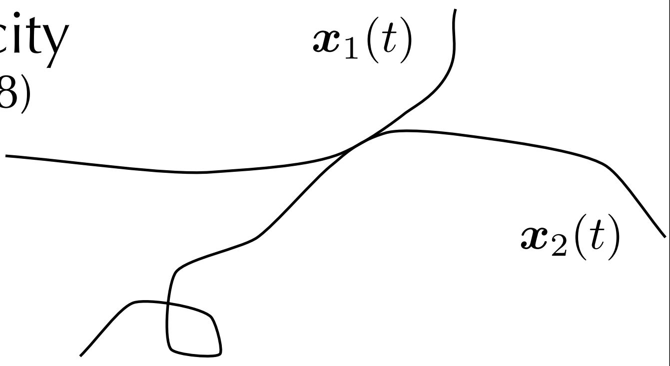
$$\partial_t p_2(\boldsymbol{x},\boldsymbol{x},t|\boldsymbol{x}_1^0,\boldsymbol{x}_2^0,0)\,\mathrm{d}\boldsymbol{x}_1^0\mathrm{d}\boldsymbol{x}_2^0$$



Larchevêque & Lesieur, *J. Méc.* 1981
Nelkin & Kerr, *PoF* 1981 ; Thomson, *JFM* 1996

Turbulent dissipative anomaly

- Generalized flows and spontaneous stochasticity
  (Bernard *et al.*, J. Stat. Phys. 1998; Eyink, *Physica* D 2008)



$$|\boldsymbol{u}(\boldsymbol{x},t) - \boldsymbol{u}(\boldsymbol{x}',t')| \sim |\boldsymbol{x} - \boldsymbol{x}'|^h$$

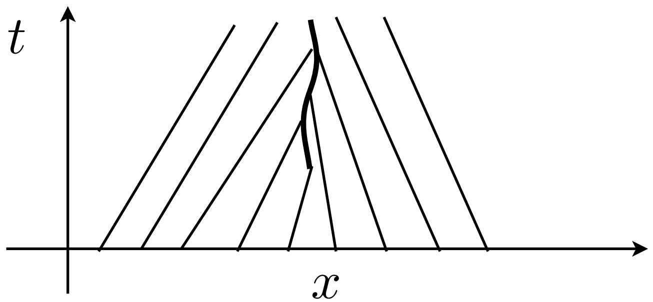
$h < 1$ ⇒ not Lipschitz ⇒ non-uniqueness

- Onsager's conjecture: $h < 1/3$ in order to dissipate energy
  (Duchon & Robert, *Nonlinearity* 2000)

"Local 4/5 law": $\varepsilon(\boldsymbol{x},t) = -\frac{3}{4}\lim_{r\to 0}\frac{\langle \delta_r u^{\|} |\delta_r \boldsymbol{u}|^2\rangle_{\mathrm{ang}}}{r}$

⇒ close relation between energy dissipation in the limit $Re \to \infty$ and singular behaviors in particle separation

- Recently understood in the case of inviscid Burgers equation
  (Eyink & Drivas, *arXiv* 2014)



Backward-in-time trajectories of entropy solutions are Markovian

Pair dispersion

- Statistics of the two-point motion $\boldsymbol{R}(t) = \boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)$
  $\langle\,\cdot\,\rangle_{r_0}$ conditioned on a fixed initial distance $|\boldsymbol{R}(0)| = r_0$

- **Batchelor regime:**
  ballistic separation at small times

$$\left\langle |\boldsymbol{R}(t) - \boldsymbol{R}(0)|^2 \right\rangle_{r_0} \propto (\varepsilon\, r_0)^{2/3}\; t^2$$

for $t \ll \tau_{r_0} \sim \varepsilon^{-1/3} r_0^{2/3}$ turnover time

Batchelor, *Proc. Camb. Phil. Soc.* 1952

- **Richardson–Obukhov law:**
  explosive separation at large times

$$\left\langle |\boldsymbol{R}(t)|^2 \right\rangle_{r_0} \sim g\,\varepsilon\, t^3$$

for $\tau_{r_0} \ll t \ll T_{\mathrm{L}}$

Richardson, *Proc. Roy. Soc. Lond.* 1926
Obukhov, *Izv. Akad. Nauk SSSR* 1941

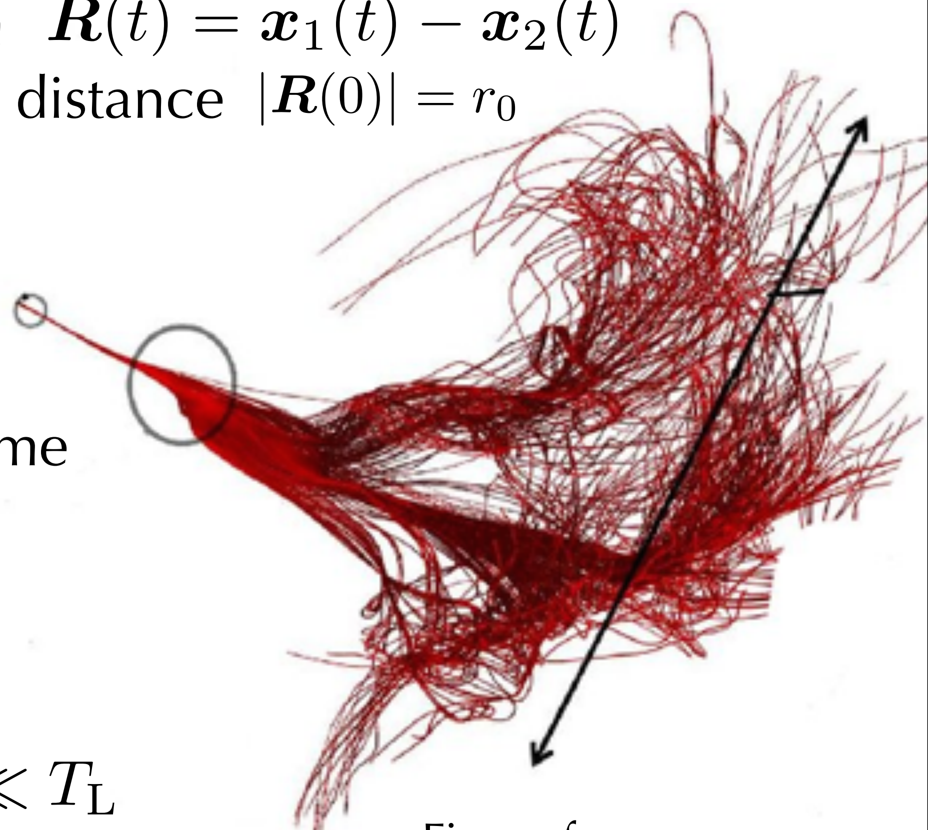


Figure from Scatamacchia et al., *PRL* 2013

Difficult to observe numerically and experimentally because of the large temporal scale separation that is required: $\tau_\eta \ll \tau_{r_0} \ll t \ll T_{\mathrm{L}}$

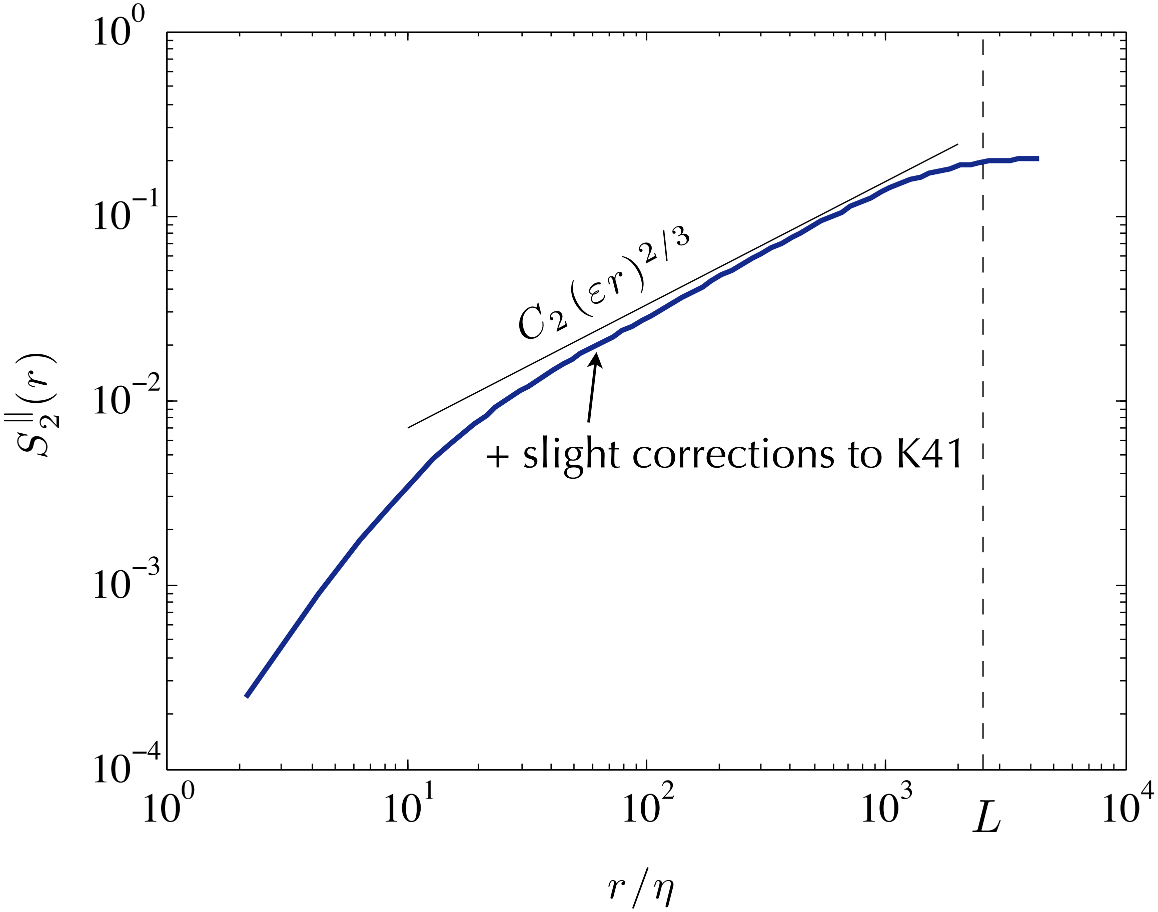
Review by Salazar & Collins
Ann. Rev. Fluid Mech. 2009

$\Rightarrow$ sub-leading terms? Mechanisms?

Numerics

LaTu: MPI pseudo-spectral solver (Homann et al. 2007)

R_λ	ν	η	τ_η	L	T_L	N^3
730	10^{-5}	$7.2 \cdot 10^{-4}$	0.05	1.85	9.6	4096^3



Incompressible NS + large-scale forcing

+ 10^7 Lagrangian trajectories

65 536 processes on BlueGene/P

JÜLICH FORSCHUNGSZENTRUM

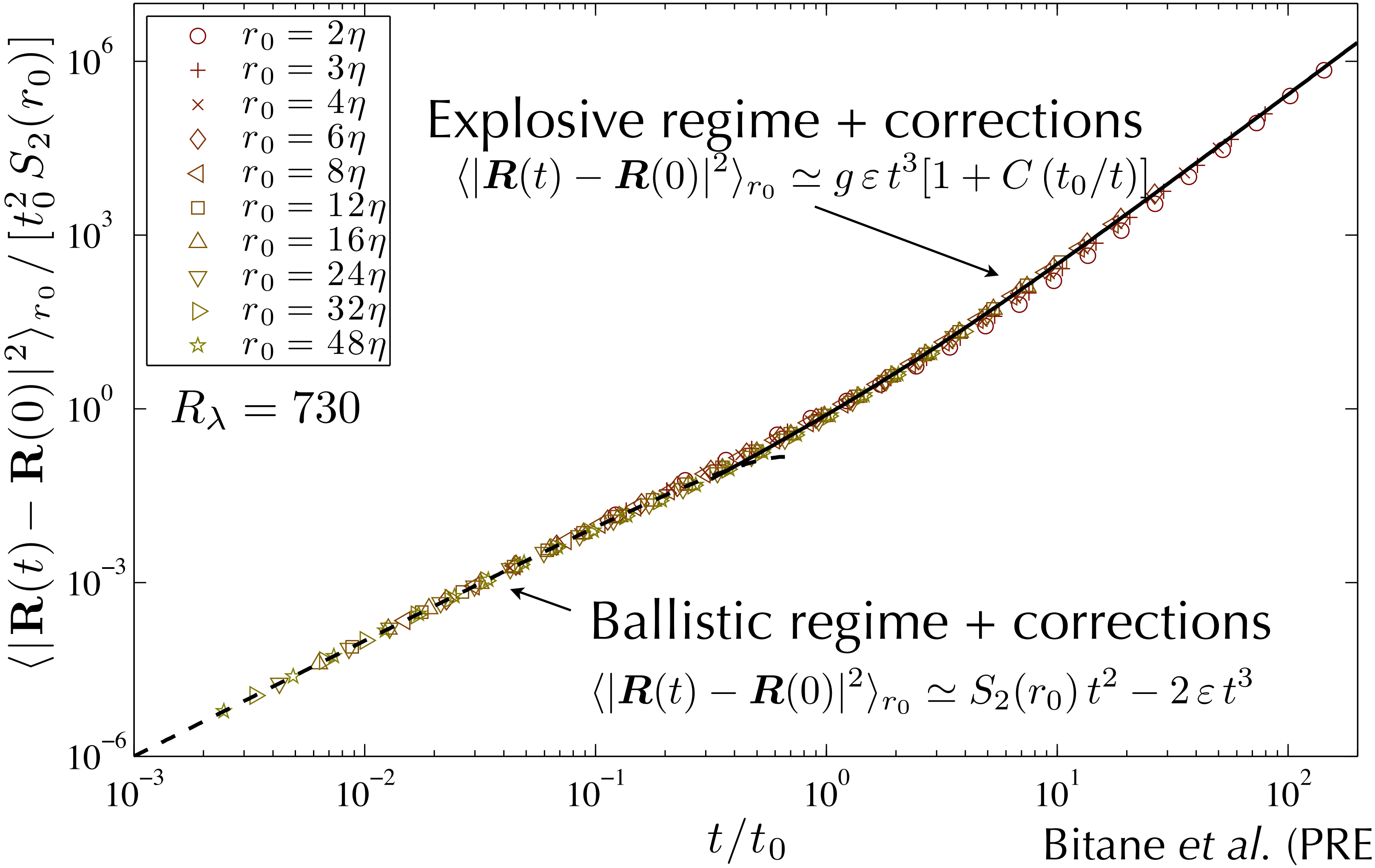
Transition Ballistic/Explosive

Taylor expansion: $\left\langle |\boldsymbol{R}(t) - \boldsymbol{R}(0)|^2 \right\rangle_{r_0} = t^2 S_2(r_0) + t^3 \left\langle \delta \boldsymbol{u} \cdot \delta \mathrm{D}_t \boldsymbol{u} \right\rangle + \mathrm{O}(t^4)$

$S_2(r_0) = \left\langle |\delta \boldsymbol{u}|^2 \right\rangle$

$\langle \delta \boldsymbol{u} \cdot \delta \mathrm{D}_t \boldsymbol{u} \rangle = 2 \frac{\mathrm{d}}{\mathrm{d}t} \langle |\delta \boldsymbol{u}|^2 \rangle = -2\epsilon$

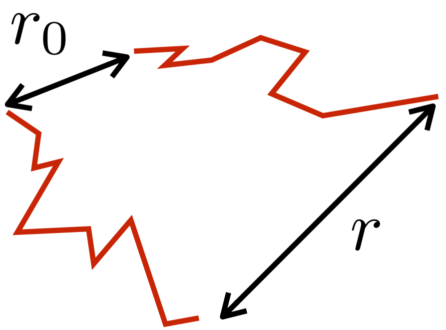
Crossover: $t_0 = \frac{S_2(r_0)}{2\varepsilon}$



Bitane *et al.* (PRE 2013)

Richardson diffusion

Assumption: velocity difference is **uncorrelated** ⇒ separation diffuses



Transition probability $p_2(r,t|r_0,0)$

$$\partial_t p_2 = \nabla\cdot(K(r)\nabla p_2)$$

+ K41(Obukhov) $K(r)\sim\varepsilon^{1/3}r^{4/3}$

$$\Rightarrow \quad p_2(r,t|r_0,0)\propto \frac{r^2}{t^{9/2}}\,\mathrm{e}^{-C\,r^{2/3}/(\varepsilon\,t)}$$ and $$\left\langle|\boldsymbol{R}(t)|^2\right\rangle_{r_0}\sim g\,\varepsilon\,t^3$$

Explosive growth: limiting distribution independent of initial separation r_0

Formalized for the Kraichnan model (Gaussian, δ-correlated velocities)

see Falkovich, Gawedzki, Vergassola, *Rev. Mod. Phys.* 2001

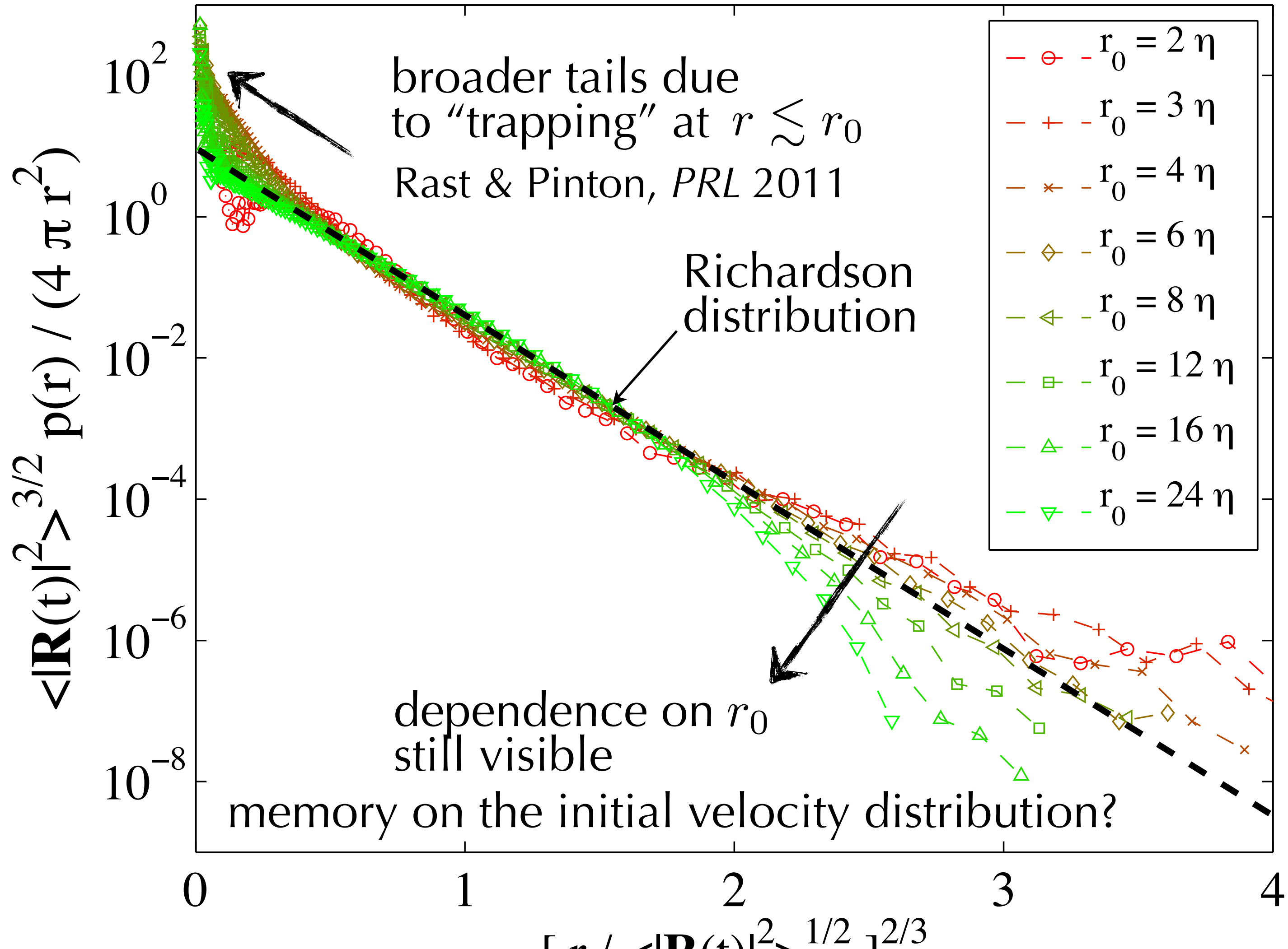
Shortcoming: velocity difference get uncorrelated on timescales $\mathrm{O}(t)$

Phenomenology ⇒ correlation time $\tau_r\sim r^{2/3}$ $+r^2\sim t^3$ $\Rightarrow$ $\tau_r\sim t$

Distribution of distances

Comparison to Richardson distribution $p_2(r,t|r_0,0) \propto \frac{r^2}{t^{9/2}}\,\mathrm{e}^{-C\,r^{2/3}/(\varepsilon\,t)}$

From the numerics:



Markov models

Assumption: acceleration differences are **short correlated**

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{A} = \delta \mathrm{D}_t \boldsymbol{u} \longleftarrow$$ components correlated over a time $\mathrm{O}(\tau_\eta)$

Central-Limit Theorem: $$\boldsymbol{A} \overset{\text{law}}{\equiv} \sqrt{\tau_\eta}\, \mathbb{A}(\boldsymbol{R}, \boldsymbol{V}) \circ \boldsymbol{\eta}(t)$$ when $t \gg \tau_\eta$

with $\mathbb{A}^\mathsf{T}\mathbb{A} = \langle \delta \mathrm{D}_t \boldsymbol{u} \otimes \delta \mathrm{D}_t \boldsymbol{u} \,|\, \delta \boldsymbol{u} \rangle$ correlations of acceleration differences conditioned on $\delta \boldsymbol{u}$

General form:
$$\begin{cases} \mathrm{d}\boldsymbol{R} = \boldsymbol{V}\,\mathrm{d}t \\ \mathrm{d}\boldsymbol{V} = \boldsymbol{a}(\boldsymbol{R}, \boldsymbol{V}, t)\mathrm{d}t + \mathbb{B}(\boldsymbol{R}, \boldsymbol{V}, t)\,\mathrm{d}\boldsymbol{W} \end{cases}$$

Kurbanmuradov & Sabelfeld (1995); Sawford (2001)

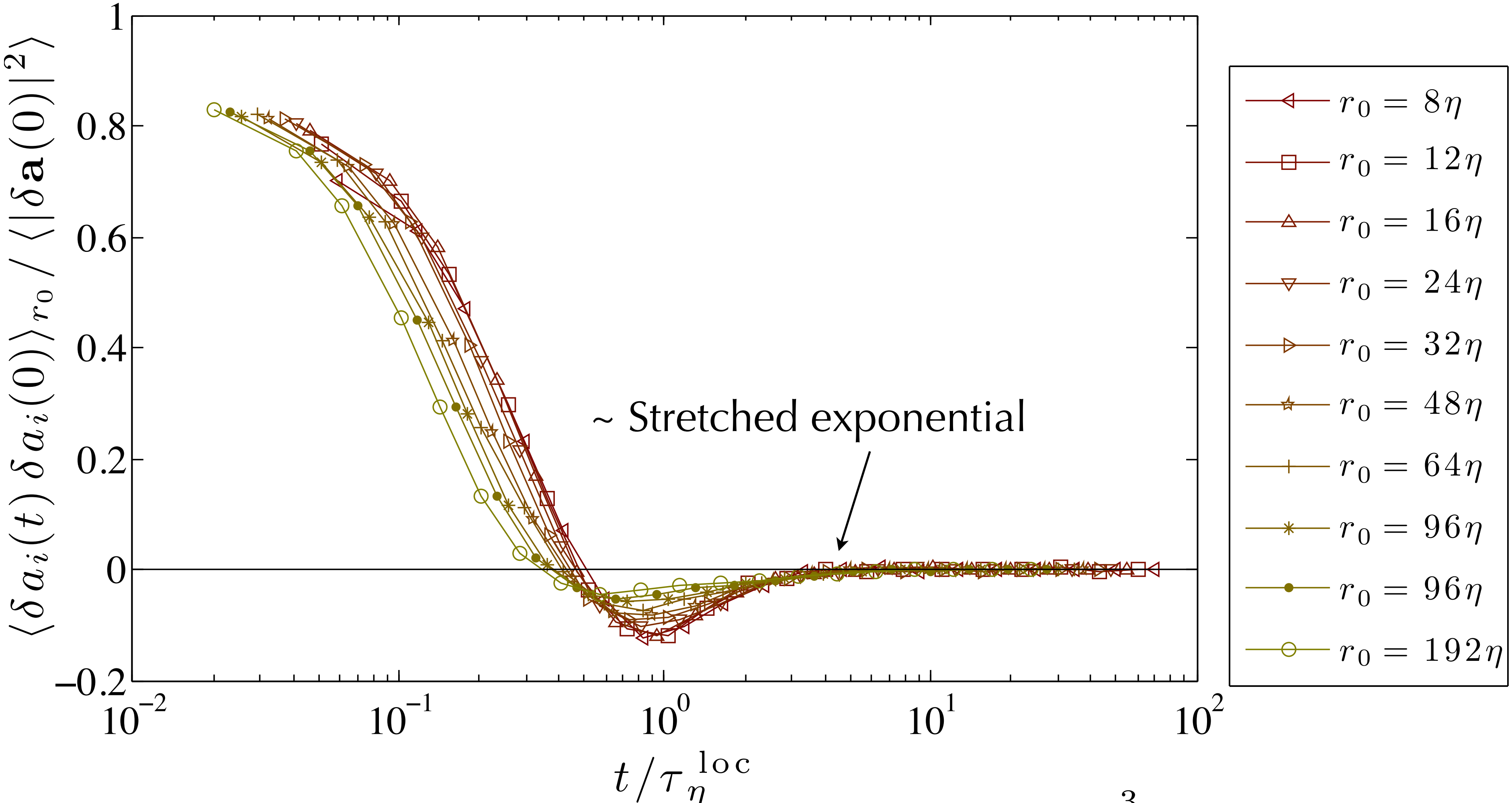
$\Rightarrow$ Fokker–Planck equation for $p(\boldsymbol{r}, \boldsymbol{v}, t | \boldsymbol{r}_0, \boldsymbol{v}_0, 0)$

$$\partial_t p + \partial_{r_i}(v_i\, p) + \partial_{v_i}(a_i\, p) = \frac{1}{2}\partial_{v_i}\partial_{v_j}[B_{ik}B_{jk}\, p]$$

Admissibility condition: "well-mixing"

Consistency with Eulerian statistics $p_E(\boldsymbol{r}, \boldsymbol{v})$ is a stationary solution associated to an initial uniform distribution in space (Thomson 1991)

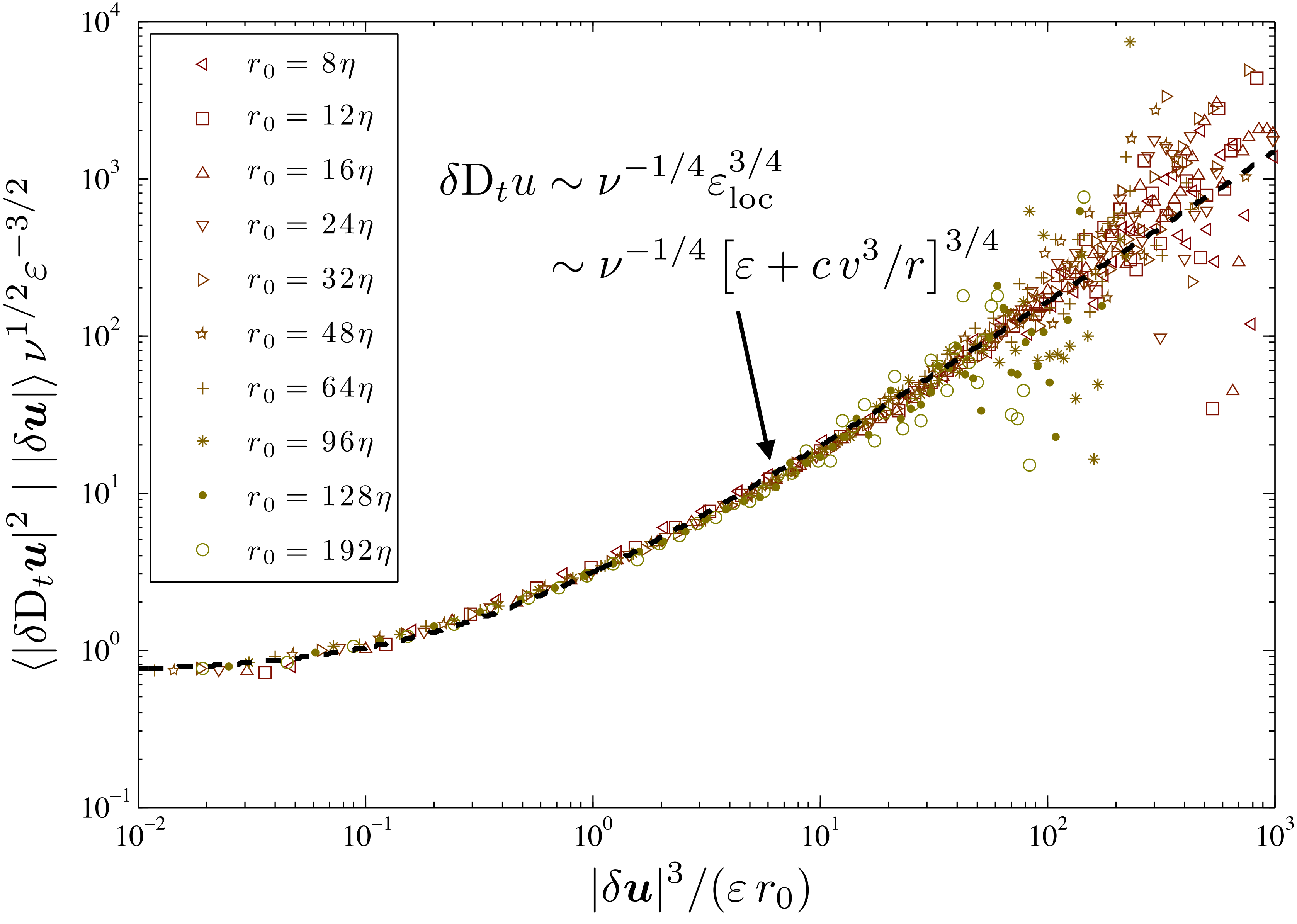
Time-correlation of acceleration



Local "energy transfer rate" $\Rightarrow \quad \varepsilon \mapsto \varepsilon_{\rm loc} = \varepsilon + c\,\dfrac{v^3}{r}$

$$\tau_\eta^{\rm loc} \sim (\nu/\varepsilon_{\rm loc})^{1/2}$$

Conditional acceleration variance



1D illustrative model

Dimensional analysis and data suggest: $\begin{cases} \tau_\eta^{\mathrm{loc}} \sim (\nu/\varepsilon_{\mathrm{loc}})^{1/2} \\ \delta \mathrm{D}_t u \sim \nu^{-1/4} \varepsilon_{\mathrm{loc}}^{3/4} \end{cases}$

Noise amplitude: $\sqrt{\tau_\eta}\, \delta \mathrm{D}_t u \sim \varepsilon_{\mathrm{loc}}^{1/2}$ independent of the viscosity ν

One-dimensional version:

$$\begin{cases} \mathrm{d}R &= V\,\mathrm{d}t \\ \mathrm{d}V &= \left[\epsilon + c\,\dfrac{|V|^3}{R}\right]^{1/2} \circ \mathrm{d}W \end{cases}$$

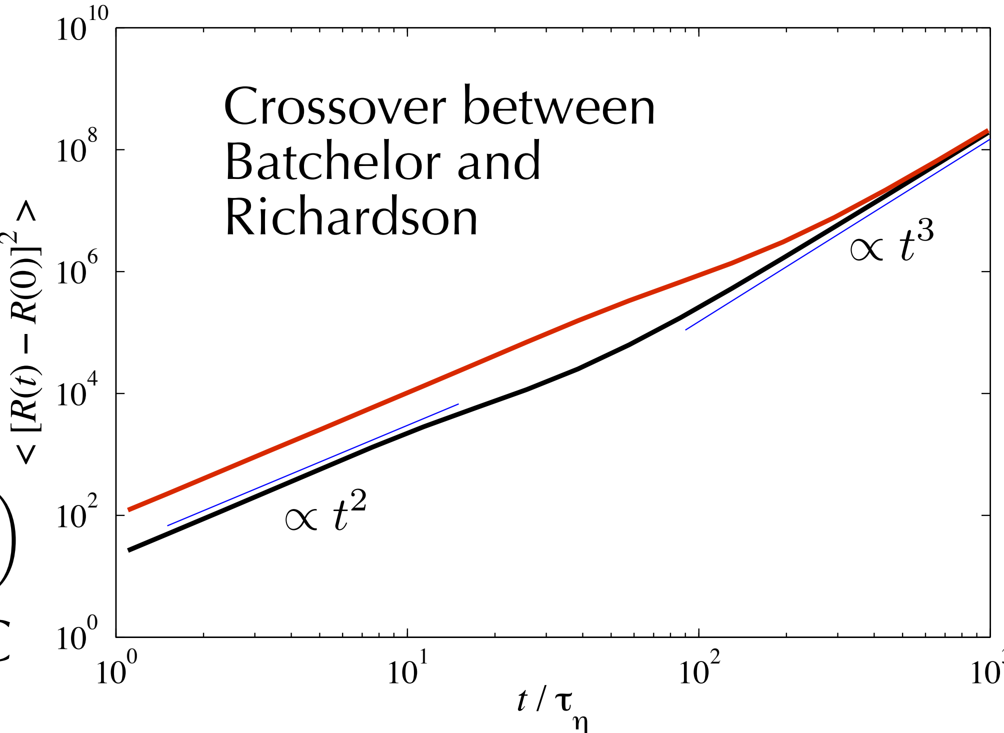
Drift $\propto \dfrac{|V|V}{R}$

not enough to prevent collapse:
$R \to 0$ at a finite time
$\Rightarrow$ cutoff at small scales $R = \eta$

Asymptotic "explosive" scaling solutions

$$p(r, v, t\,|\,r_0, v_0, 0) \to \frac{1}{t^2}\Psi\left(\frac{r}{t^{3/2}}, \frac{v}{t^{1/2}}\right)$$

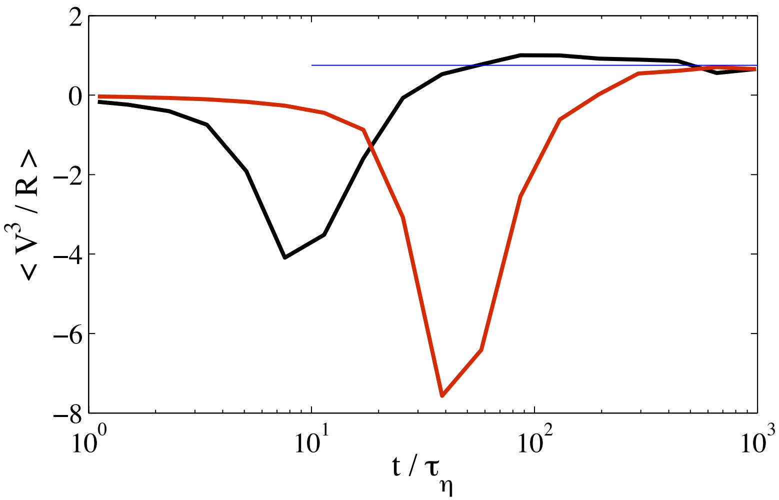
incompatible with small-scale BC

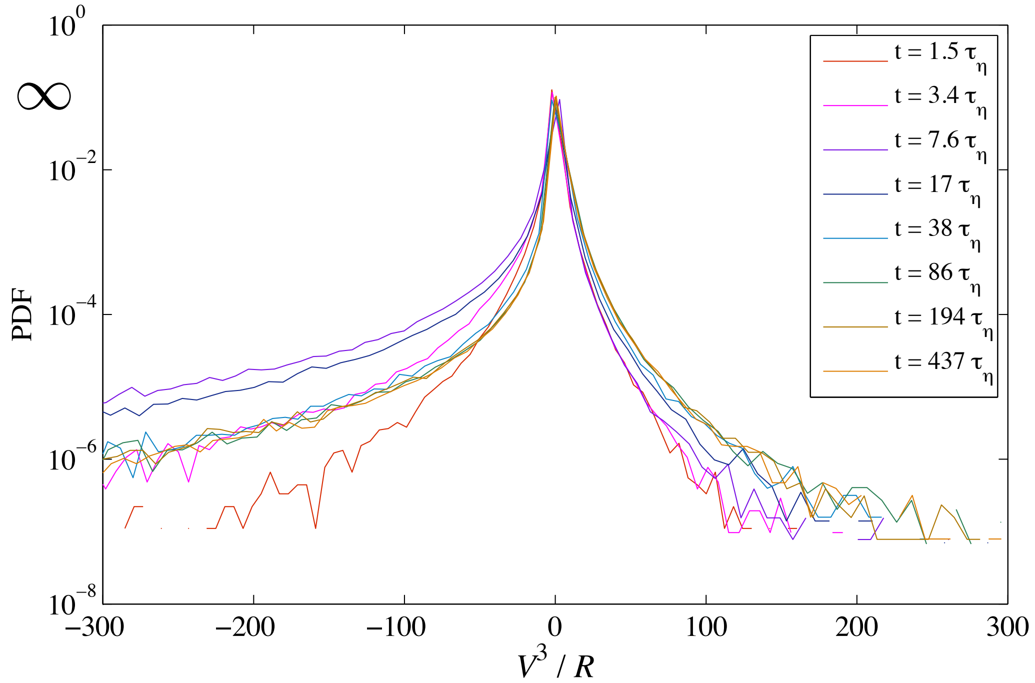


Phenomenology

$$\begin{cases} \mathrm{d}R &= V\,\mathrm{d}t \\ \mathrm{d}V &= \left[\epsilon + c\,\dfrac{|V|^3}{R}\right]^{1/2} \circ \mathrm{d}W \end{cases}$$

$|V|^3/R \to$ stationary when $t \to \infty$



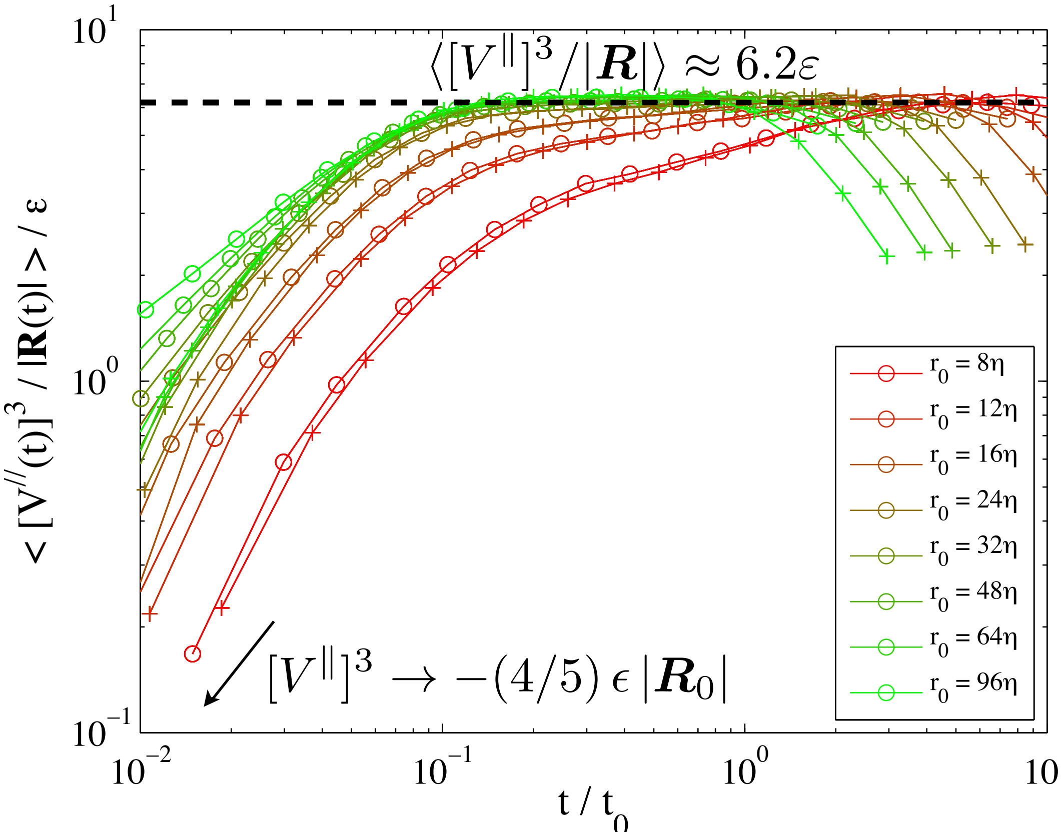


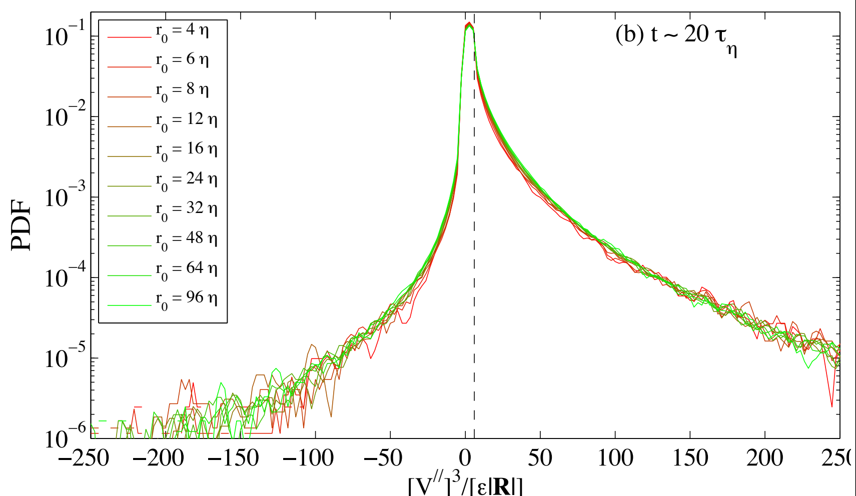
$$V(t) = \int \left[\epsilon + c\,\frac{|V(s)|^3}{R(s)}\right]^{1/2} \circ \mathrm{d}W_s \to \text{ diffusion at large times}$$

However: distribution of $|V|^3/R$ depends on the small-scale cutoff

Same mechanisms in turbulence?

The "local transfer rate" $[V^{\|}]^3/|\boldsymbol{R}|$ becomes stationary along Lagrangian pairs





Richardson explosive separation equivalent to the diffusion of velocity differences?? Independent of scaling solutions?

Compatibility with Eulerian intermittency? **Well-mixing?**

"Eulerian statistics" from the model

$$\begin{cases} \mathrm{d}R &= V\,\mathrm{d}t \\ \mathrm{d}V &= \left[\epsilon + c\,\dfrac{|V|^3}{R}\right]^{1/2} \circ \mathrm{d}W \end{cases}$$

Stationary solutions?

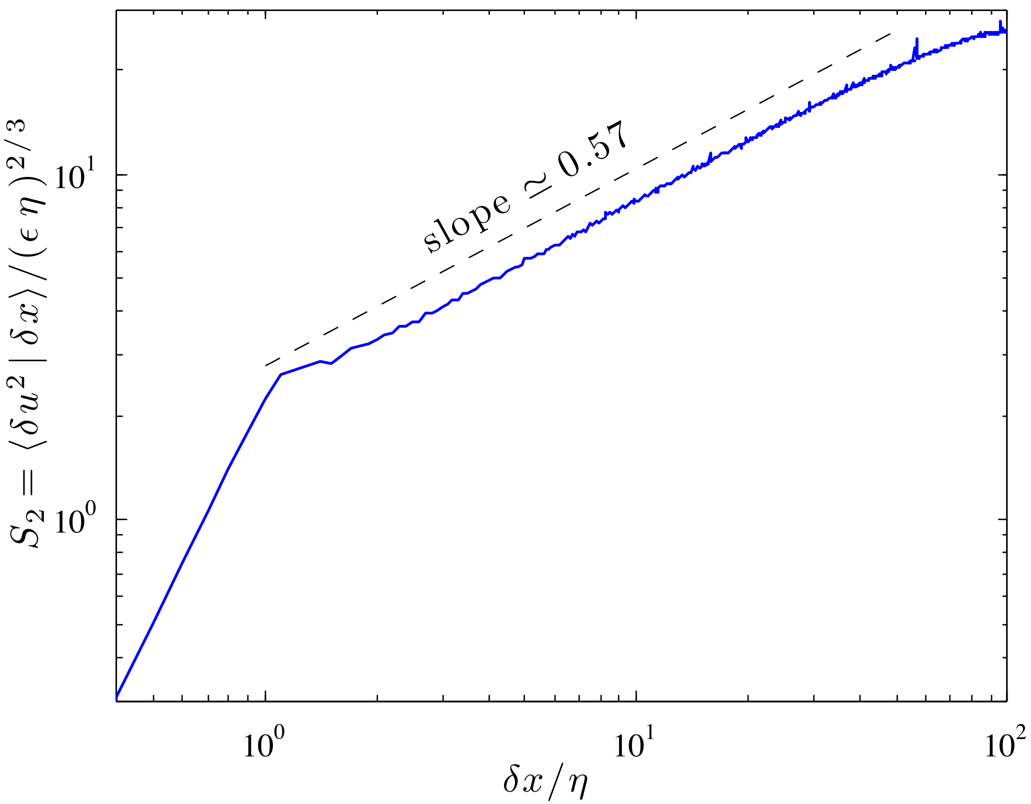
Well mixing?

requires imposing boundary conditions at large scales $R = L$

$p(V|R) \approx$ Eulerian statistics

Spatial scaling is incompatible with the observed time behavior of separation

Non-universal exponent that depends on the choice of the small-scale boundary



Limits of Markov modeling

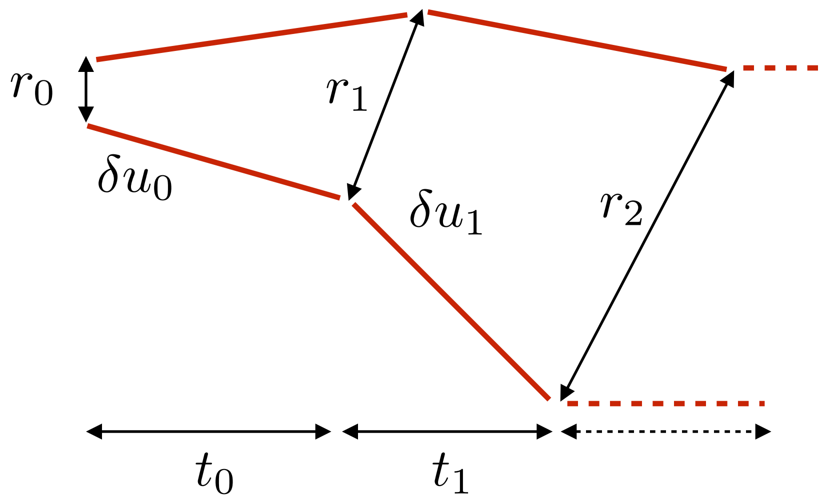
- Is acceleration really short-time correlated?
  - ⇒ OK for components but not amplitude (Mordant *et al.*, *PRL* 2004)
  - ⇒ Stretched exponential correlations (non-mixing process)

- Is the asymptotic diffusion of velocities the mechanism explaining Richardson's scaling $R \sim t^{3/2}$?
  - ⇒ Is it compatible with the observed intermittent behaviors? e.g. for exit times (Boffetta & Sokolov, *PRL* 2002)
  - ⇒ Are finite-Re effects solely responsible for lack of scaling? (Scatamacchia *et al.*, *PRL* 2013)

- Is turbulent relative dispersion really a Markov process?
  - ⇒ Relation to Lévy walks / waiting times approaches (Shlesinger *et al.*, *PRL* 1987, Rast & Pinton, *PRL* 2011)
  - ⇒ Some deviations might be due to memory effects (Eyink & Benveniste, *PRE* 2013)

A piecewise ballistic approach

- Ballistic regime is key in the convergence to the explosive behavior
- Build a simple model that reproduces some essential mechanisms



NB: Non-Markovian with respect to the continuous time

Continuous-time random walk

$$\vec{r}_n \mapsto \vec{r}_{n+1} = \vec{r}_n + \Delta t_n \delta\vec{u}_n$$

$$t_n \mapsto t_{n+1} = t_n + \Delta t_n$$

with Δr_n and Δt_n random variables that depends upon r_n and δu_n

The δu_n's are independent

$$\begin{cases} \alpha_n = \delta u_n^{\|} / |\delta\vec{u}_n| \\ \beta_n = |\delta\vec{u}_n|^3 / (\varepsilon r_n) \end{cases} \qquad \Delta t_n = \frac{|\delta\vec{u}_n|^2}{\varepsilon}$$

$$\begin{cases} r_{n+1} = r_n \sqrt{1 + 2\alpha_n\beta_n + \beta_n^2} \\ t_{n+1} = t_n + \varepsilon^{-1/3} \beta_n^{2/3} r_n^{2/3} \end{cases}$$

- K41 version: statistics of α_n and β_n independent of r_n

$\Rightarrow$ can be easily extended to intermittent statistics assuming $|\delta\vec{u}_n| \sim r_n^{h_n}$

Another scaling?

$$\begin{cases} r_{n+1} = r_n \sqrt{1 + 2\alpha_n \beta_n + \beta_n^2} \\ t_{n+1} = t_n + \varepsilon^{-1/3} \beta_n^{2/3} r_n^{2/3} \end{cases}$$

⟵ Multiplicative process in R

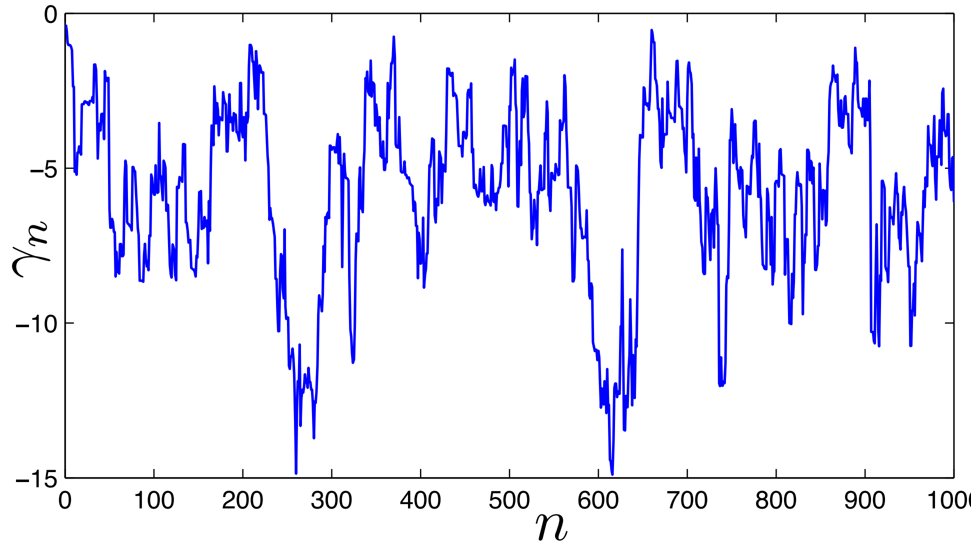
⟵ Additive process in t

Change of variables: $\gamma_n = \ln \dfrac{r_n}{r_0} - \dfrac{3}{2} \ln \dfrac{t_n}{\bar{t}_0}$ $\qquad \bar{t}_0 = \varepsilon^{-1/3} r_0^{2/3}$

$$\gamma_{n+1} = \frac{3}{2} \ln \frac{\left(1 + 2\alpha_n \beta_n + \beta_n^2\right)^{1/3}}{\beta_n^{2/3} + \mathrm{e}^{-\frac{2}{3}\gamma_n}}$$

The γ_n's are becoming **stationary**

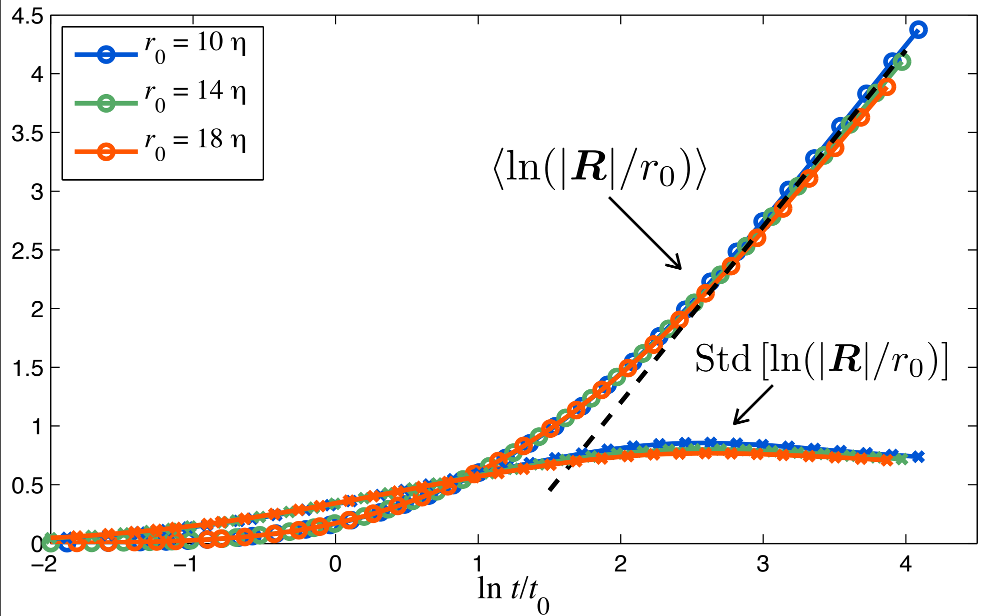
large negative excusions (tracers approaching each other): 1D random walk with positive drift

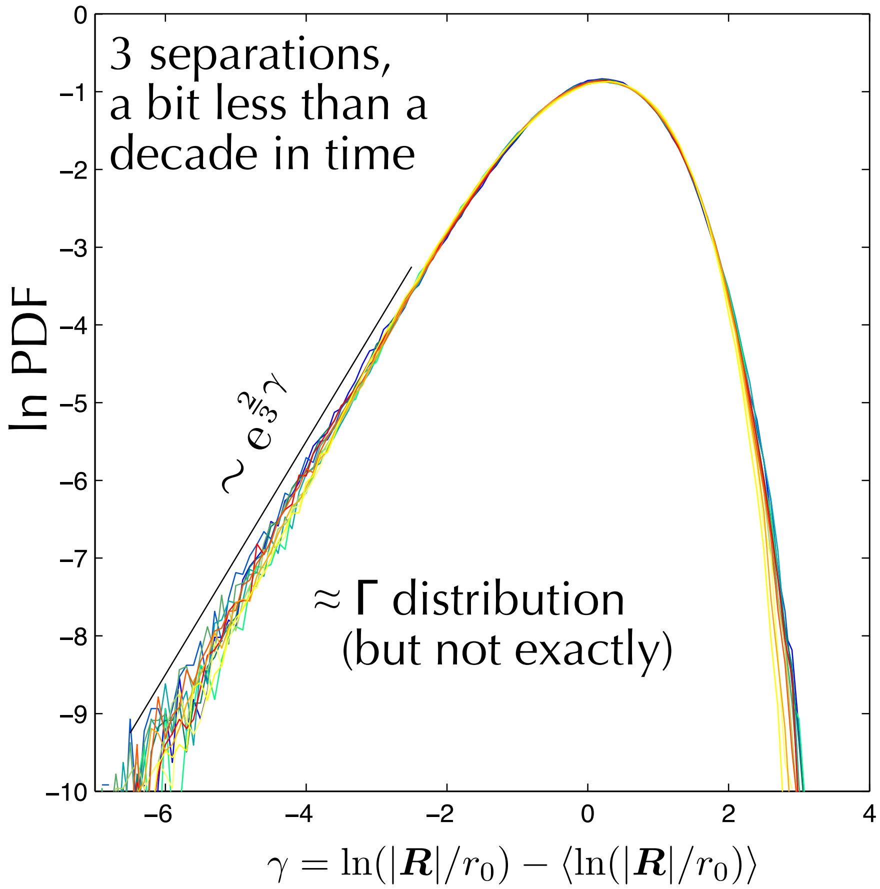


This suggests: $\left\langle \ln \dfrac{|\boldsymbol{R}|}{r_0} \right\rangle \simeq \dfrac{3}{2} \ln \dfrac{t}{\bar{t}_0} + \langle \gamma \rangle$ $\qquad \mathrm{Var}\left[\ln \dfrac{|\boldsymbol{R}|}{r_0} \right] \simeq \mathrm{Var}\left[\gamma\right] = \mathrm{const}$

$$\mathrm{PDF}(\ln |\boldsymbol{R}|) \to \Psi[\, \ln |\boldsymbol{R}| - \langle \ln |\boldsymbol{R}| \rangle \,]$$

$\ln (R/r_0)$ a multiplicative process?





- Numerical results are compatible with the piecewise ballistic scenario.

- Extension to account for intermittency.
- Interpret time-irreversibility of pair separation